Bull. London Math. Soc. 44 (2012) 47–54
doi:10.1112/blms/bdr060

Bounded and homoclinic-like solutions of a second-order singular differential equation

Denis Bonheure and Pedro J. Torres

Abstract

We study the existence of positive solutions for the model scalar second-order boundary value problem

$$\begin{cases} -u'' + c(x)u' + a(x)u = \dfrac{b(x)}{u(x)^p}, & x \in \mathbb{R}, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

where a, b, c are locally bounded coefficients and $p > 0$.

1. *Introduction*

This note is devoted to the study of the existence of a decaying nontrivial positive solution for the model equation

$$-u'' + c(x)u' + a(x)u = \frac{b(x)}{u(x)^p}, \tag{1}$$

where $a, b, c \in C(\mathbb{R}; \mathbb{R})$, that is, the existence of a positive function u that solves (1) for every $x \in \mathbb{R}$ as well as the boundary conditions

$$\lim_{|x|\to\infty} u(x) = 0. \tag{2}$$

When such a solution satisfies in addition

$$\lim_{|x|\to\infty} u'(x) = 0,$$

then it is usually called a homoclinic solution or a pulse, although here, 0 is not a stationary solution of equation (1).

Equation (1) is a particular case of a more general class of Sturm equations of the type

$$-(P(x)u')' + Q(x)u = R(x)f(u), \tag{3}$$

where P is a strictly positive absolutely continuous function.

Such equations, even in the case $P \equiv 1$, where they are referred to as of Schrödinger or Klein–Gordon type, appear in many scientific areas, including quantum field theory, gas dynamics, fluid mechanics and chemistry. For instance, the study of pulse propagation is of primary importance in nonlinear optics and plasma physics, where homoclinic solutions of the corresponding model equations are sometimes referred to as bound states or bright solitons. A typical example is the study of the ability of layered media to support the propagation of electromagnetic guided waves.

Observe that equation (1) with constant c arises also when studying traveling wave fronts for parabolic reaction–diffusion equations with a singular local reaction term.

Received 28 September 2010; revised 17 March 2011; published online 24 August 2011.

2010 *Mathematics Subject Classification* 34C37 (primary), 34B40 (secondary).

P.J.T. is partially supported by D.G.I. MTM2008-02502, M.E.C., Spain.

One can study (1) with $c = 0$ via a variational procedure; see, for example, the surveys [**3**, **4**, **12**]. Actually, one could also treat the case where $c \in L^1(\mathbb{R})$ in the same way with only minor changes. Indeed, in this case, the self-adjoint form of the equation is nondegenerate. In the case where c is nonintegrable, we prefer to tackle the problem via a topological approach as the variational formulation of the problem would require working with weighted Sobolev spaces with unbounded or vanishing weight functions. Without being insurmountable (see, for instance, [**6**] for related problems), this difficulty makes the variational approach more delicate than the topological one.

Our method of proof combines the method of upper and lower solutions [**11**, **13**] with some fixed point theorems in cones, which are well-known consequences of fixed-point index theory [**8**]. This is not the first time that such types of results, extensively used in equations on compact intervals, have been employed for problems defined on the whole real line. However, the point of view of the few recent relevant papers is quite different from the one we employ here. In [**2**, **7**], generalizations to Fréchet spaces of fixed point theorems of cone-compressing and cone-condensing type are proved and then employed. On the other hand, [**16**] needs to use a weighted norm in $BC(0, +\infty)$.

Our main result is as follows.

Theorem 1. *Let us assume that*
(A) $a \in C(\mathbb{R}; \mathbb{R})$ *and there exists* $\underline{a} > 0$ *such that* $a(x) \geqslant \underline{a}$ *for all* x;
(B) $b \in C(\mathbb{R}; \mathbb{R})$ *is a nonzero nonnegative function such that* $b/a \in L^\infty(\mathbb{R})$*;*
(C) $c \in C(\mathbb{R}; \mathbb{R})$ *is such that* $c/a \in L^\infty(\mathbb{R})$.
Then there exists a unique positive solution $u \in BC(\mathbb{R})$ *of equation* (1)*. Moreover, if* b *satisfies*

$$\lim_{|x|\to\infty} b(x)/a(x) = 0,$$

then u *satisfies* $\lim_{|x|\to\infty} u(x) = 0$.

For further references, we fix the following assumption:
(B_0) $b \in C(\mathbb{R}; \mathbb{R})$ is a nonzero nonnegative function such that

$$\lim_{|x|\to\infty} b(x)/a(x) = 0.$$

2. *The linear equation*

Let us consider the linear equation

$$Lu := -u'' + c(x)u' + a(x)u = 0. \tag{4}$$

We first observe that under assumptions (A) and (C), the equation is disconjugate at $\pm\infty$ and presents a dichotomy; see [**9**] for precise definitions. These facts are the basis of our approach as they ensure the existence of a nice Green kernel to represent L^{-1} in an integral form.

Lemma 1. *Assume that* a *satisfies* (A) *and* c *fulfills* (C)*. Then the linear equation* (4) *possesses a positive* (*increasing*) *solution* u_1 *and a linearly independent positive* (*decreasing*) *solution* u_2 *such that*

$$\lim_{x\to-\infty} u_1(x) = \lim_{x\to+\infty} u_2(x) = 0$$

and

$$\lim_{x\to+\infty} u_1(x) = \lim_{x\to-\infty} u_2(x) = +\infty.$$

Moreover, u_1, u_2 *can be chosen in such a way that*

$$u_1'(0)u_2(0) - u_1(0)u_2'(0) = 1. \tag{5}$$

Proof. The proof follows from classical arguments of the theory of linear second-order equations. The assumptions imply that the origin is a saddle point, hence u_2 or u_1 is taken as a solution of the stable or unstable manifold, respectively. Condition (A) also implies that a given solution cannot have positive maxima or negative minima. A direct consequence is the disconjugacy of the equation and the monotone behavior of u_1, u_2. Condition (5) is easily obtained, multiplying u_1, u_2 by a suitable constant. □

For every right-hand side h such that h/a is bounded, the nonhomogeneous equation

$$-u'' + c(x)u' + a(x)u = h(x)$$

has a unique bounded solution (one may argue, for instance, against the classical theory of lower and upper solutions). This solution can be computed by variation in constants leading to the Green function for bounded solutions

$$G(x,s) = \begin{cases} u_1(x)u_2(s)e^{-\int_0^s c(\tau)d\tau}, & -\infty < x \leqslant s < +\infty \\ u_2(x)u_1(s)e^{-\int_0^s c(\tau)d\tau}, & -\infty < s \leqslant x < +\infty \end{cases} \tag{6}$$

where u_1, u_2 are the solutions described in Lemma 1. Note that by Lemma 1, u_1, u_2 intersect at a unique point x_0, so that we can define a function $p \in BC(\mathbb{R})$ by

$$p(x) = \begin{cases} \dfrac{1}{u_2(x)}, & x \leqslant x_0, \\ \dfrac{1}{u_1(x)}, & x > x_0. \end{cases} \tag{7}$$

It follows from the monotonicity of u_1 and u_2 that $p(x) = 1/\max(u_1(x), u_2(x))$. We conclude this section by collecting some properties of the Green function. Most of these have been stated in [**14**] for $c = 0$. We provide a short proof in order to keep our paper self-contained.

Proposition 1. *Assume that a and c satisfy* (A) *and* (B), *respectively. Then one has*
(P1) $G(x,s) > 0$ *for every* $(x,s) \in \mathbb{R}^2$;
(P2) $G(x,s) \leqslant G(s,s)$ *for every* $(x,s) \in \mathbb{R}^2$;
(P3) *for any nonempty compact subset* $P \subset \mathbb{R}$,

$$G(x,s) \geqslant m_1(P)p(s)G(s,s), \quad \text{for all } (x,s) \in P \times \mathbb{R},$$

where

$$m_1(P) = \min\{u_1(\inf P), u_2(\sup P)\}; \tag{8}$$

(P4) $G(s,s)p(s) \geqslant G(x,s)p(x)$ *for every* $(x,s) \in \mathbb{R}^2$.

Proof. Properties (P1) and (P2) are trivial because of the positivity and monotonicity of u_1 and u_2.

We prove (P3) for $(x,s) \in P \times \mathbb{R}$ with $x \leqslant s$ since the remaining possibility is analogous. Using the fact that u_2 is a positive nondecreasing function and that the function p satisfies

$$p(x) \leqslant \frac{1}{u_2(x)}, \quad x \in \mathbb{R}, \tag{9}$$

we have

$$G(x,s) \geqslant u_2(\inf P)u_1(s)e^{-\int_0^s c(\tau)\,d\tau} \geqslant m_1(P)\frac{G(s,s)}{u_2(s)} \geqslant m_1(P)p(s)G(s,s).$$

Finally, we consider (P4). Again, we only consider the case $x \leqslant s$. Write

$$G(s,s)p(s) = G(x,s)p(x)\frac{u_1(s)}{u_1(x)}\frac{p(s)}{p(x)}.$$

If $p(s) = 1/u_2(s)$, then $p(x) = 1/u_2(x)$ too and the conclusion follows from the monotonicity of u_1 and u_2. If $p(s) = 1/u_1(s)$, then either $p(x) = 1/u_1(x)$ or $p(x) = 1/u_2(x)$. In the first case, the conclusion is obvious, while in the second case, one uses the fact that $u_2(x) \geqslant u_1(x)$. □

Finally, the following lemma will be useful.

Lemma 2. *Assume that a, b and c satisfy conditions* (A), (B_0) *and* (C), respectively. *Then the equation*

$$-u'' + c(x)u' + a(x)u = b(x), \tag{10}$$

has a unique bounded positive solution that can be written as $u(x) = \int_{\mathbb{R}} G(x,s)b(s)\,ds$. Moreover, we have $u(\pm\infty) = 0$.

Proof. It is clear that $\int_{\mathbb{R}} G(x,s)b(s)\,ds$ is a positive solution of (10). The boundedness is justified next. In the self-adjoint form, the homogeneous linear equation (4) reads

$$-[P(x)u']' + a(x)P(x)u = 0, \tag{11}$$

where $P(x) = e^{-\int_0^x c(\tau)\,d\tau}$. Note that, by the choice of u_1, the function $P(x)u_1'(x)$ is positive and, moreover, it is strictly increasing by means of the latter equation. Therefore, it has a nonnegative limit at $-\infty$;

$$L_1 := \lim_{x\to-\infty} P(x)u_1'(x) \geqslant 0.$$

An analogous argument gives

$$L_2 := \lim_{x\to+\infty} P(x)u_2'(x) \leqslant 0.$$

Now, by integrating (11) over $]-\infty, x[$ and $]x, +\infty[$, respectively, one gets

$$P(x)u_1'(x) - L_1 = \int_{-\infty}^{x} a(s)P(s)u_1(s)\,ds, \quad L_2 - P(x)u_2'(x) = \int_x^{+\infty} a(s)P(s)u_2(s)\,ds.$$

Then, by combining Liouville's formula and (5),

$$\begin{aligned}\int_{\mathbb{R}} G(x,s)a(s)\,ds &= u_2(x)\int_{-\infty}^{x} a(s)p(s)u_1(s)\,ds + u_1(x)\int_x^{+\infty} a(s)p(s)u_2(s)\,ds\\ &= [u_2(x)u_1'(x) - u_1(x)u_2'(x)]P(x) - L_1u_2(x) + L_2u_1(x)\\ &\leqslant [u_2(x)u_1'(x) - u_1(x)u_2'(x)]P(x) = 1.\end{aligned}$$

With this in mind,

$$\int_{\mathbb{R}} G(x,s)b(s)\,ds = \int_{\mathbb{R}} G(x,s)\frac{b(s)}{a(s)}a(s)\,ds \leqslant \left\|\frac{b}{a}\right\|_\infty \int_{\mathbb{R}} G(x,s)a(s)\,ds \leqslant \left\|\frac{b}{a}\right\|_\infty,$$

so the boundedness is proved.

Note also that uniqueness is straightforward; on the contrary, the difference of two bounded solutions would be a nontrivial bounded solution of the homogeneous equation (4), which is impossible. Hence, it remains to compute the limits at $\pm\infty$.

Let us prove that $\lim_{x\to+\infty} u(x) = 0$. We separate the argument into two cases. Assume first that u reaches a positive limit L monotonically. Then $\lim_{x\to+\infty} u'(x) = 0$. If $L > 0$, the equation shows that $u''(x) \geqslant \underline{a}L/2$ as $x \to \infty$, which is impossible. Therefore, we conclude that $L = 0$.

If u is not asymptotically monotone, there is a sequence of local maxima $u(t_n) > 0$ with $\lim_{n\to\infty} t_n = +\infty$. From the equation, we infer that

$$a(t_n)u(t_n) \leqslant b(t_n)$$

and, using assumptions (A)–(B_0), we conclude that $u(t_n) \to 0$. □

3. *An auxiliary result with a compactly supported potential*

In this section, we assume the auxiliary condition

($\mathrm{B_C}$) $b \in C(\mathbb{R};\mathbb{R})$ is a nonzero, nonnegative function and has compact support.

Of course, ($\mathrm{B_C}$) is stronger than (B_0). This condition will be used as a first step in the proof of Theorem 1.

Let $BC^+(\mathbb{R})$ be the set of positive bounded continuous functions defined on $\mathbb{R}$. We learned from the preceding section that we can look for a positive bounded solution of (1) as a fixed point of the operator $T\colon BC^+(\mathbb{R}) \to BC^+(\mathbb{R})$, which can be written as

$$Tu(x) := \int_{\mathbb{R}} G(x,s)\frac{b(s)}{u(s)^p}\,ds.$$

Under condition ($\mathrm{B_C}$), such an operator is well defined and a fixed point is indeed a positive solution of (1) satisfying (2).

In order to find a fixed point of T, we use the following well-known theorem on cones (see, for instance, [**10**, p. 148] or [**1**]).

Theorem 2. *Let P be a cone in the Banach space B. Assume Ω_1, Ω_2 are open bounded subsets of B with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. If $T\colon P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a continuous and compact map such that one of the following conditions is satisfied:*

(H1) *$\|Tu\| \leqslant \|u\|$ if $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geqslant \|u\|$ if $u \in P \cap \partial\Omega_2$;*

(H2) *$\|Tu\| \geqslant \|u\|$ if $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leqslant \|u\|$ if $u \in P \cap \partial\Omega_2$;*

then, T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Since our operator works on a space of functions defined on the whole real line, the Ascoli–Arzela theorem is not enough to ensure compactness. We use a compactness criterion inspired by [**15**]. The proof is included for the convenience of the reader.

Lemma 3. *Let $\Omega \subset BC(\mathbb{R})$. Let us assume that the functions $u \in \Omega$ are equicontinuous in each compact interval of $\mathbb{R}$ and that, for all $u \in \Omega$, we have*

$$|u(x)| \leqslant \xi(x), \quad \forall x \in \mathbb{R}, \tag{12}$$

where $\xi \in BC(\mathbb{R})$ satisfies

$$\lim_{|x|\to+\infty} \xi(x) = 0. \tag{13}$$

Then Ω is relatively compact.

Proof. Given a sequence $(u_n)_n$ of functions of Ω, we have to prove that there exists a partial sequence that is uniformly convergent to a certain u. Note that the elements of Ω are uniformly bounded by $\|\xi\|_\infty$ and equicontinuous on compact intervals by hypothesis; therefore, the Ascoli–Arzela theorem and a diagonal argument provide a partial sequence (we still denote it by $(u_n)_n$) that is uniformly convergent to a certain u on compact intervals. Of course, u satisfies also (12). Now we have to prove that

$$\forall \varepsilon > 0,\ \exists n_0 \text{ s.t. } n \geqslant n_0 \implies \|u_n - u\|_\infty < \varepsilon.$$

By using (13), fix $k > 0$ such that $\max_{x\in\mathbb{R}\setminus]-k,k[} |\xi(x)| < \varepsilon/2$. On the other hand, by using the uniform convergence on compact intervals, there exists n_0 such that $\max_{x\in[-k,k]} |u_n(x) - u(x)| < \varepsilon/2$ for all $n \geqslant n_0$. Then

$$\|u_n - u\|_\infty \leqslant \max_{x\in[-k,k]} |u_n(x) - u(x)| + \max_{x\in\mathbb{R}\setminus]-k,k[} |u_n(x) - u(x)| < \varepsilon,$$

and the proof is completed. □

Proposition 2. *Let us assume* (A), (B_C) *and* (C). *Then, there exists a positive solution* $u \in BC^+(\mathbb{R})$ *of* (1) *satisfying* (2).

In order to apply Theorem 2, we consider the Banach space $BC(\mathbb{R})$ endowed with the supremum norm and look for an invariant cone for T. Let us consider the set

$$P = \left\{ u \in BC(\mathbb{R}) \,:\, u(x) \geqslant 0,\ \min_{y\in \operatorname{Supp}(b)} u(y) \geqslant m_1 p_0 \|u\| \right\}, \tag{14}$$

where $p_0 = \inf_{\operatorname{Supp}(b)} p(x)$, $p(x)$ being defined by (7), and the constant $m_1 \equiv m_1(\operatorname{Supp}(b))$ is defined by (8). Note that the compactness of $\operatorname{Supp}(b)$ implies that $p_0 > 0$. Also, it is easy to see, by definition, that $m_1 p_0 \leqslant 1$, and hence this cone is non-empty.

Proof of Proposition 2. We define the open bounded sets Ω_1, Ω_2 as the open balls in $BC(\mathbb{R})$ of radius r and R, to be fixed later.

Step 1. We claim that $T(P \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset P$. Take $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Property (P1) of the Green function and the sign of b imply that $Tu(x) \geqslant 0$ for all x. Besides, for all $\tau \in \mathbb{R}$, we have

$$\begin{aligned}\min_{x\in \operatorname{Supp}(b)} Tu(x) &= \min_{x \operatorname{Supp}(b)} \int_{-\infty}^{+\infty} G(x,s)\frac{b(s)}{u(s)^p}\, ds \geqslant m_1 \int_{-\infty}^{+\infty} p(s)G(s,s)\frac{b(s)}{u(s)^p}\, ds \\ &\geqslant m_1 \int_{\operatorname{Supp}(b)} p(\tau)G(\tau,s)\frac{b(s)}{u(s)^p}\, ds = m_1 p_0 (Tu)(\tau),\end{aligned}$$

where we have used (P3) and (P4). Therefore $T(P \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset P$.

Step 2. Compactness. The continuity of T is trivial so that we focus on the compactness property. We will use Lemma 3. Let $(u_n)_n \subset P \cap (\overline{\Omega_2} \setminus \Omega_1)$ be a bounded sequence. Define the sequence $(v_n)_n \subset P$ by $v_n(x) = Tu_n(x)$. We just need to prove that, up to a subsequence, v_n converges uniformly in $\mathbb{R}$. Note that the sequence u_n^p is uniformly bounded, say by M. Therefore, we compute

$$|v_n(x)| = \left| \int_{\mathbb{R}} G(x,s)\frac{b(s)}{u_n(s)^p}\, ds \right| \leqslant \frac{1}{(rm_1p_0)^p} \int_{\mathbb{R}} G(x,s)b(s)\, ds.$$

Now, observe that, by Lemma 2, the function $\int_{\mathbb{R}} G(x,s)b(s)\, ds$ goes to zero at $\pm\infty$. Moveover, the equicontinuity of the v_n sequence is clear as it follows directly from the continuity of the Green function. Hence, we are able to apply the previous compactness criterion.

Step 3. We claim that $\|Tu\| \geqslant \|u\|$, if $u \in P \cap \partial\Omega_1$. For $u \in P \cap \partial\Omega_1$,

$$m_1 p_0 r \leqslant u(x) \leqslant r \quad \text{for all } x \in \operatorname{Supp}(b).$$

If $u \in P \cap \partial\Omega_1$ and r is small enough,

$$\|Tu(x)\| \geqslant \frac{1}{r^p} \sup_{x\in\mathbb{R}} \int_{\operatorname{Supp} b} G(x,s)b(s)\,ds \geqslant r = \|u\|.$$

Step 4. We claim that $\|Tu\| \leqslant \|u\|$, if $u \in P \cap \partial\Omega_2$. For $u \in P \cap \partial\Omega_2$,

$$m_1 p_0 R \leqslant u(x) \leqslant R \quad \text{for all } x \in \operatorname{Supp}(b).$$

Taking R big enough, we obtain

$$\|Tu\| \leqslant \frac{1}{(Rm_1p_0)^p} \sup_{x\in\mathbb{R}} \int_{\operatorname{Supp}(b)} G(x,s)b(s)\,ds \leqslant R = \|u\|.$$

It follows from the previous steps that we can apply Theorem 2, and the proof is complete. □

4. *Proof of Theorem* 1

The existence of a bounded solution is proved by using the theory of upper and lower solutions (see [**11**, **13**] for more details). Let $\underline{b}$ satisfy ($\mathrm{B_C}$) and be such that $\underline{b}(x) \leqslant b(x)$ for every $x \in \mathbb{R}$. By Proposition 2, the equation

$$-u'' + c(x)u' + a(x)u = \frac{\underline{b}(x)}{u^p}$$

has a positive bounded solution $\alpha(x)$. This function is a lower solution of (1). A well-ordered upper solution is easily found as a constant

$$\beta > \max\left\{ \|\alpha\|_\infty, \left(\left\| \frac{b}{a} \right\|_\infty \right)^{1/(1+p)} \right\}.$$

Then the classical theory of upper and lower solutions provides a bounded solution of the original equation (1) between α and β.

Suppose now that ($\mathrm{B_0}$) holds and let us prove the convergence to 0 of the bounded solution u. The argument is similar to that employed in Lemma 2. Let us prove that $\lim_{x\to+\infty} u(x) = 0$; the limit at $-\infty$ follows in the same way. There are two possibilities.

If $\lim_{x\to+\infty} u(x) = L > 0$ monotonically, we easily reach a contradiction as in Lemma 2.

If there is a sequence of local maxima $u(x_n) > 0$ with $(x_n)_n \to +\infty$, then we infer from the equation that

$$u(x_n)^{1+p} \leqslant \frac{b(x_n)}{a(x_n)}.$$

Passing to the limit, we conclude that $u(x_n) \to 0$ and the proof is done.

It remains to prove the uniqueness. In fact, we can write the equation as

$$u'' - c(x)u' = f(x,u), \tag{15}$$

with $f(x,u) = a(x)u - b(x)/u^p$ uniformly increasing in the second variable, and apply comparison arguments like those employed in [**5**]. The argument is as follows. Assume that there are two positive bounded solutions u_1, u_2 and define the difference $d(x) = u_1(x) - u_2(x)$. Then

$$d'' - c(x)d' = f(x,u_1) - f(x,u_2).$$

Since $f(x,u)$ is increasing, d is a bounded function without positive maxima or negative mimina. Therefore, it must have a limit at $\pm\infty$. A simple limiting argument as those employed before shows that this limit must be zero. Combining this information with the fact that d cannot have positive maxima or negative minima, one deduces that $d(x) \equiv 0$.

Acknowledgement. We are grateful to an anonymous referee for a careful reading of the manuscript and some useful suggestions.

References

1. R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications* (Cambridge University Press, Cambridge, 2001).
2. R. P. Agarwal and D. O'Regan, 'Cone compression and expansion fixed point theorems in Fréchet spaces with applications', *J. Differ. Equations* 171 (2001) 412–429.
3. T. Bartsch and A. Szulkin, 'Hamiltonian systems: periodic and homoclinic solutions by variational methods', *Handbook of differential equations: ordinary differential equations*, vol. II (eds A. Canada, P. Drabek and A. Fonda; Elsevier, Amsterdam, 2005) 77–146.
4. D. Bonheure and L. Sanchez, 'Heteroclinic orbits for some classes of second and fourth order differential equations', *Handbook of differential equations: ordinary differential equations*, vol. III (eds A. Canada, P. Drabek and A. Fonda; Elsevier, Amsterdam, 2006) 103–202.
5. J. Campos and P. J. Torres, 'On the structure of the set of bounded solutions on a periodic Lienard equation', *Proc. Amer. Math. Soc.* 127 (1999) 1453.
6. C. De Coster and M. Willem, 'Density, spectral theory and homoclinics for singular Sturm–Liouville systems', *J. Comp. App. Math.* 52 (1994) 45–70.
7. M. Frigon and D. O'Regan, 'Fixed points of cone-compressing and cone-extending operators in Fréchet spaces', *Bull. London Math. Soc.* 35 (2003) 672–680.
8. A. Granas and J. Dugundji, *Fixed point theory* (Springer, New York, 2003).
9. P. Hartman, *Ordinary differential equations* (John Wiley & Sons Inc., New York, 1964).
10. M. A. Krasnosel'skii, *Positive solutions of operator equations* (Noordhoff, Groningen, 1964).
11. Z. Opial, 'Sur les intégrales bornees de l'equation $u'' = f(t, u, u')$', *Ann. Polonici Math.* 4 (1958) 314–324.
12. P. H. Rabinowitz, 'Variational methods for Hamiltonian systems', *Handbook of dynamical systems*, vol. 1A (eds B. Hasselblatt and A. Katok; Elsevier, Amsterdam, 2002) 1091–1127.
13. K. W. Schrader, 'Boundary value problems for second order ordinary differential equations', *J. Differ. Equations* 3 (1967) 403–413.
14. P. J. Torres, 'Guided waves in a multi-layered optical structure', *Nonlinearity* 19 (2006) 2103–2113.
15. K. Zima, 'Sur l'existence des solutions d'une équation intégro-différentielle', *Ann. Polon. Math.* 27 (1973) 181–187.
16. M. Zima, 'On positive solutions of boundary value problems on the half-line', *J. Math. Anal. Appl.* 259 (2001) 127–136.

Denis Bonheure
Département de Mathématique
U.L.B., CP214 Boulevard du Triomphe
1050 Bruxelles
Belgium

denis.bonheure@ulb.ac.be

Pedro J. Torres
Universidad de Granada
Departamento de Matemática Aplicada
18071 Granada
Spain

ptorres@ugr.es