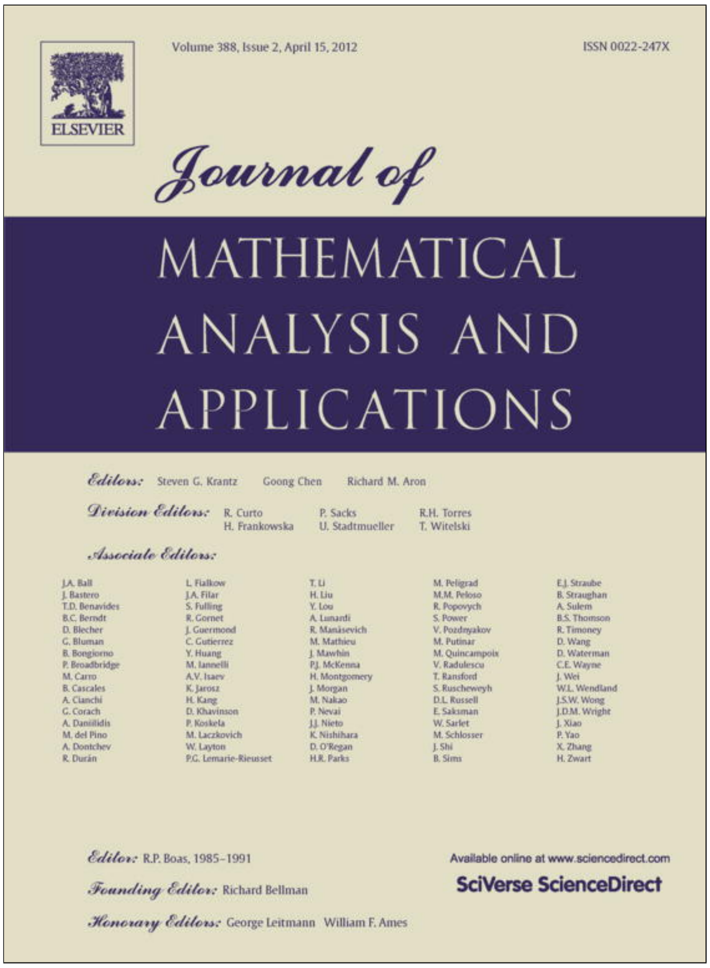
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Periodic solutions for second order singular damped differential equations ☆

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ABSTRACT

We study the existence of positive periodic solutions for second order singular damped differential equations by combining the analysis of the sign of Green's functions for the linear damped equation, together with a nonlinear alternative principle of Leray–Schauder. Recent results in the literature are generalized and significantly improved.

1. Introduction

In this paper, we study the existence of positive T-periodic solutions for the following singular damped differential equation

$$x'' + h(t)x' + a(t)x = f\big(t, x, x'\big), \tag{1.1}$$

where $h, a \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and the nonlinearity $f \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times (0, \infty) \times \mathbb{R}, \mathbb{R})$. In particular, the nonlinearity may have a repulsive singularity at $x = 0$, which means that

$$\lim_{x\to 0^+} f(t, x, y) = +\infty, \quad \text{uniformly in } (t, y) \in \mathbb{R}^2.$$

Electrostatic or gravitational forces are the most important examples of singular interactions.

During the last two decades, the study of the existence of periodic solutions for singular differential equations has attracted the attention of many researchers [3,5,6,21,25,29,30,33]. Some strong force conditions introduced by Gordon [14] are standard in the related earlier works [10,17,30,31,33]. Compared with the case of a strong singularity, the study of the existence of periodic solutions under the presence of a weak singularity is more recent, but has also attracted many researchers [6,7,9,12,25,28]. Some classical tools have been used to study singular differential equations in the literature, including the method of upper and lower solutions [2,25], degree theory [30,31,33], some fixed point theorems in cones

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for completely continuous operators [11,26,27], Schauder's fixed point theorem [6,12,28] and a nonlinear Leray–Schauder alternative principle [7,8,20].

However, the singular differential equation (1.1), in which there is the damping term and the nonlinearity is dependent of the derivative, has not attracted much attention in the literature. There are not so many existence results for (1.1) even when the nonlinearity is independent of the derivative. Several existence results can be found in [22,32]. In this paper, we try to fill this gap and establish the existence of positive T-periodic solutions of (1.1) using a nonlinear alternative of Leray–Schauder, which has been used in [7,8,20]. We would like to emphasize that the inclusion of the derivative dependence in the nonlinearity implies a new technical difficulty concerning the right choice of the function space and associated norm. Our new results generalize in several aspects some recent results contained in [6,7,20,26,29]. Our main motivation is to obtain new existence results for positive T-periodic solutions of the following differential equations

$$x'' + h(t)x' + a(t)x = \big(e(t) + \kappa(t)\big|x'\big|^{\gamma}\big)\left(\frac{b(t)}{x^{\alpha}} + \mu c(t)x^{\beta}\right), \tag{1.2}$$

where $a, b, c, e, h, \kappa \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, $\alpha, \beta > 0$ and $\mu > 0$ is a parameter. The reason for considering Eq. (1.2) is mainly academic, since it unifies many of the particular examples in the quoted literature and illustrates how our method is applicable to equations depending on the derivative. From a point of view of modelling, singular forces arise in classical mechanics when electromagnetic or gravitational forces are involved, and in many of such models the inclusion of a damping term modelling the loss of energy by friction is reasonable.

The rest of this paper is organized as follows. In Section 2, we present a survey on some known results concerning the sign of Green's function of the linear damped equation

$$x'' + h(t)x' + a(t)x = 0, \tag{1.3}$$

associated to periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{1.4}$$

We present two classes of functions h, a to guarantee that (1.3)–(1.4) has a positive or non-negative Green's function. These two classes have been studied in [19] and [4], respectively. The proofs of the results in [4] are based on the so-called "L^p-criterion" developed by Torres in [26] for the Green's function of the Hill equation

$$x'' + a(t)x = 0. \tag{1.5}$$

During the last several years, the "L^p-criterion" for (1.5) has become a standard assumption in the searching for periodic solutions of second order nonlinear regular and singular differential equation

$$x'' + a(t)x = f(t, x). \tag{1.6}$$

See, for example, [6,7,12,20,28]. Here we note that the Green's function of (1.3) with separated boundary conditions has also been considered in [1,13].

In Section 3, by employing a nonlinear alternative principle of Leray–Schauder, we prove the main existence results for (1.1) under the positiveness of the Green's function associated with (1.3)–(1.4). Applications of the new results to (1.2) are also given. The results are applicable to the case of a strong singularity as well as the case of a weak singularity. We are mainly motivated by the recent papers [7,8,20], in which periodic singular problem (1.6) has been studied. We have generalized those results in [8,20] and improved those in [22].

From now on, let us denote by p^* and p_* the essential supremum and infimum of a given function $p \in L^1[0, T]$, if they exist. Also, we write $p \succ 0$ if $p \geqslant 0$ for almost every $t \in [0, T]$ and it is positive in a set of positive measure. The usual L^p-norm is denoted by $\|\cdot\|_p$. The conjugate exponent of p is denoted by q: $1/p + 1/q = 1$.

2. Sign of Green's function

We say that (1.3)–(1.4) is nonresonant when its unique T-periodic solution is the trivial one. When (1.3)–(1.4) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + h(t)x' + a(t)x = l(t) \tag{2.1}$$

admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G(t, s)l(s)\,ds,$$

where $G(t, s)$ is the Green's function of problem (1.3)–(1.4).

In next section, we will assume that the following standing hypothesis is satisfied:

(A) The Green's function $G(t, s)$, associated with (1.3)–(1.4), is positive for all $(t, s) \in [0, T] \times [0, T]$.

The rest part of this section is devoted to present some sufficient conditions, which guarantee that (A) is satisfied. We have two classes.

2.1. Class 1: the general case $a, h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$

Definition 2.1. We say that (1.3) admits the anti-maximum principle if (2.1) has a unique T-periodic solution for any $l \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$ and the unique T-periodic solution x_l of (2.1) satisfies $x_l(t) > 0$ for all t if $l \succ 0$.

Theorem 2.2. *Assume that* (1.3) *admits the anti-maximum principle. Then, the Green's function $G(t,s)$, associated with* (1.3)–(1.4)*, is non-negative for all $(t,s) \in [0,T] \times [0,T]$.*

Proof. For the case $h \equiv 0$, this result is proved in [35, Theorem 4.1]. This proof remains invariant in our more general setting. □

For the positiveness of Green's function, we can use the following result.

Theorem 2.3. *Let us assume that the distance between two consecutive zeroes of a non-trivial solution of* (1.3) *is always strictly greater than T. Then, the Green's function $G(t,s)$ does not vanish.*

Proof. For the case $h \equiv 0$, this result is proved in [26, Theorem 2.1] (see also [35, Lemma 4.13]). As in the latter result, this proof remains invariant in our more general setting. □

By combining Theorems 2.2 and 2.3, one has abstract conditions for property (A) to hold.

Next we recall one explicit criterion proved by Halk and Torres in [19] that (1.3) admits the anti-maximum principle. To do this, let us define the functions

$$\sigma(h)(t) = \exp\left(\int_0^t h(s)\,ds\right),$$

and

$$\sigma_1(h)(t) = \sigma(h)(T)\int_0^t \sigma(h)(s)\,ds + \int_t^T \sigma(h)(s)\,ds.$$

Lemma 2.4. *(See [19, Theorem 2.2].) Assume that $a \not\equiv 0$ and the following two inequalities are satisfied*

$$\int_0^T a(s)\sigma(h)(s)\sigma_1(-h)(s)\,ds \geqslant 0, \tag{2.2}$$

and

$$\sup_{0\leqslant t\leqslant T}\left\{\int_t^{t+T} \sigma(-h)(s)\,ds \int_t^{t+T} \big[a(s)\big]_+\sigma(h)(s)\,ds\right\} \leqslant 4, \tag{2.3}$$

where $[a(s)]_+ = \max\{a(s), 0\}$. Then the anti-maximum principle for (1.3) *holds.*

The results in [19] can be exploited to derive a lower bound for the distance among zeroes of solutions of (1.3) as well.

Lemma 2.5. *Assume that $a \not\equiv 0$ and* (2.3) *holds. Then, the distance between two consecutive zeroes of a non-trivial solution of* (1.3) *is always strictly greater than T.*

Proof. By contradiction, let us assume that x is a non-trivial solution of (1.3) with two consecutive zeroes at $c < b$ such that $b - c \leqslant T$. Then $x(c) = x(b) = 0$ and, without loss of generality, we can assume that $c \in [0,T]$ and $x(t) > 0$ for $t \in\,]c,b[$. By a direct application of [19, Lemma 3.4], one has

$$4 < \int_c^b \sigma(-h)(s)\,ds \int_c^b \big[a(s)\big]_+\sigma(h)(s)\,ds.$$

Note that all the integrated functions are non-negative. Since $b - c \leqslant T$, we have

$$4 < \int_c^{c+T} \sigma(-h)(s)\,ds \int_c^{c+T} \big[a(s)\big]_+ \sigma(h)(s)\,ds,$$

which is a contradiction with (2.3). □

As a consequence of the two previous lemmas, we have explicit conditions for property (A).

Corollary 2.6. *Assume that $a \not\equiv 0$ and (2.2)–(2.3) hold. Then the Green's function $G(t,s)$, associated with (1.3)–(1.4), is positive for all $(t,s) \in [0,T] \times [0,T]$.*

2.2. Class 2: special case $\int_0^T a(t)\sigma(h)(t)\,dt > 0$ and $\bar{h} = 0$

Next we consider one special case, that is

$$h \in \tilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z}) := \big\{h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}):\ \bar{h} = 0\big\}.$$

In this case, one criterion has been developed by Cabada and Cid in [4].

To describe these, given an exponent $q \in [1,\infty]$, the best constant in the Sobolev inequality

$$C\|u\|_{q,[0,1]} \leqslant \|u'\|_{2,[0,1]} \quad \text{for all } u \in H_0^1(0,1)$$

is denoted by $\mathbf{M}(q)$. The explicit formula for $\mathbf{M}(q)$ is known. That is,

$$\mathbf{M}(q) = \begin{cases} (\frac{2\pi}{q})^{1/2}(\frac{2}{q+2})^{1/2-1/q}\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)} & \text{for } 1 \leqslant q < \infty,\\ 2 & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler. In particular, $\mathbf{M}(2) = \pi$, $\mathbf{M}(\infty) = 2$. See [34].

Theorem 2.7. *(See [4, Theorem 5.1].) Assume that $h \in \tilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z})$ and $\int_0^T a(t)\sigma(h)(t)\,dt > 0$. Suppose further that there exists $1 \leqslant p \leqslant \infty$ such that*

$$\big(B(T)\big)^{1+1/q}\|\mathcal{A}_+\|_{p,T} < (\leqslant)\, \mathbf{M}^2(2q),$$

where

$$B(T) = \int_0^T \sigma(-h)(t)\,dt,$$

and

$$\mathcal{A}_+(t) = a_+(t)\big(\sigma(h)(t)\big)^{2-1/p}.$$

Then the Green's function $G(t,s)$, associated with (1.3)–(1.4), is positive (non-negative) for all $(t,s) \in [0,T] \times [0,T]$.

Remark 2.8. From the proof of [4, Theorem 5.1], we know that the Green's function of (1.3)–(1.4) can be written as

$$G(t,s) = \tilde{G}\big(B(t), B(s)\big)\sigma(h)(s), \quad \text{for all } (t,s) \in [0,T] \times [0,T],$$

where $\tilde{G}(r,s)$ is the Green's function related to the linear equation

$$x''(r) + a\big(B^{-1}(r)\big)\sigma(2h)\big(B^{-1}(r)\big)x(r) = l\big(B^{-1}(r)\big)\sigma(2h)\big(B^{-1}(r)\big), \quad r \in \big[0, B(T)\big]$$

with the periodic boundary condition

$$x(0) = x\big(B(T)\big), \qquad x'(0) = x'\big(B(T)\big).$$

Example 2.9. In the case $h(t) \equiv 0$, $a(t) \equiv k^2$ with $0 < k \leqslant \pi/T$, the Green's function has the form

$$G(t,s) = \begin{cases} \frac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leqslant s \leqslant t \leqslant T, \\ \frac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

See [11,26]. Using Remark 2.8, for the case that

$$h \in \tilde{\mathbb{C}}\big(\mathbb{R}/B^{-1}(T)\mathbb{Z}\big), \quad a(t)\sigma(2h)(t) = k^2, \quad k > 0,$$

one may easily see that the Green's function of (1.3)–(1.4) has the form

$$G(t,s) = \sigma(h)(s) \begin{cases} \frac{\sin k(B(t)-B(s)) + \sin k(T-B(t)+B(s))}{2k(1-\cos kT)}, & 0 \leqslant s \leqslant t \leqslant T, \\ \frac{\sin k(B(s)-B(t)) + \sin k(T-B(s)+B(t))}{2k(1-\cos kT)}, & 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

3. Main results

In this section, we state and prove the new existence results for (1.1). The proof is based on the following nonlinear alternative of Leray–Schauder, which can be found in [15] or [16, pp. 120–130] and has been used by Meehan and O'Regan in [23,24]. From now on, property (A) is fixed as standing hypothesis.

Lemma 3.1. *Assume Ω is an open subset of a convex set K in a normed linear space X and $p \in \Omega$. Let $T : \overline{\Omega} \to K$ be a compact and continuous map. Then one of the following two conclusions holds:*

(I) *T has at least one fixed point in $\overline{\Omega}$.*
(II) *There exists $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda T x + (1-\lambda)p$.*

Theorem 3.2. *Suppose that* (1.3) *satisfies* (A) *and*

$$\int_0^T a(t)\sigma(h)(t)\,dt > 0. \tag{3.1}$$

Furthermore, assume that there exists a constant $r > 0$ such that:

(H_1) *There exists a continuous function $\phi_r \succ 0$ such that $f(t,x,y) \geqslant \phi_r(t)$ for all $(t,x,y) \in [0,T] \times (0,r] \times (-\infty,\infty)$.*
(H_2) *There exist continuous, non-negative functions $g(\cdot)$, $h(\cdot)$ and $\varrho(\cdot)$ such that*

$$0 \leqslant f(t,x,y) \leqslant \big(g(x) + h(x)\big)\varrho\big(|y|\big), \quad \text{for all } (t,x,y) \in [0,T] \times (0,r] \times \mathbb{R},$$

where $g(\cdot) > 0$ is non-increasing, $h(\cdot)/g(\cdot)$ is non-decreasing in $(0,r]$ and $\varrho(\cdot)$ is non-decreasing in $(0,\infty)$.
(H_3) *The following inequality holds*

$$\frac{r}{g(\iota r)\{1 + \frac{h(r)}{g(r)}\}\varrho(Lr)} > \omega^*,$$

where

$$\omega(t) = \int_0^T G(t,s)\,ds, \qquad L = \frac{2\int_0^T a(t)\sigma(h)(t)\,dt}{\min_{0 \leqslant t \leqslant T} \sigma(h)(t)},$$

and

$$\iota = m/M, \qquad m = \min_{0 \leqslant s,t \leqslant T} G(t,s), \qquad M = \max_{0 \leqslant s,t \leqslant T} G(t,s).$$

Then (1.1) *has at least one positive T-periodic solution x with $0 < \|x\| \leqslant r$.*

Proof. Since (H_3) holds, we can choose $n_0 \in \{1,2,\ldots\}$ such that $\frac{1}{n_0} < \iota r$ and

$$\omega^* g(\iota r)\left\{1 + \frac{h(r)}{g(r)}\right\}\varrho(Lr) + \frac{1}{n_0} < r.$$

Let $N_0 = \{n_0, n_0+1, \ldots\}$. Consider the family of equations

$$x'' + h(t)x' + a(t)x = \lambda f_n\big(t, x(t), x'(t)\big) + \frac{a(t)}{n}, \tag{3.2}$$

where $\lambda \in [0,1]$, $n \in N_0$ and

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{if } x \geqslant 1/n, \\ f(t,1/n,y) & \text{if } x \leqslant 1/n. \end{cases}$$

A T-periodic solution of (3.2) is just a fixed point of the operator equation

$$x = \lambda T_n x + (1-\lambda)p, \tag{3.3}$$

where $p = 1/n$ and T_n is a continuous and completely continuous operator defined by

$$(T_n x)(t) = \int_0^T G(t,s) f_n\big(s, x(s), x'(s)\big)\, ds + \frac{1}{n},$$

where we used the fact

$$\int_0^T G(t,s)a(s)\, ds \equiv 1.$$

First we claim that any fixed point x of (3.3) for any $\lambda \in [0,1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that x is a fixed point of (3.3) for some $\lambda \in [0,1]$ such that $\|x\| = r$. Note that

$$\begin{aligned} x(t) - \frac{1}{n} &= \lambda \int_0^T G(t,s) f_n\big(s, x(s), x'(s)\big)\, ds \\ &\geqslant \lambda m \int_0^T f_n\big(s, x(s), x'(s)\big)\, ds \\ &= \iota M \lambda \int_0^T f_n\big(s, x(s), x'(s)\big)\, ds \\ &\geqslant \iota \max_{t \in [0,T]} \left\{ \lambda \int_0^T G(t,s) f_n\big(s, x(s), x'(s)\big)\, ds \right\} \\ &= \iota \left\| x - \frac{1}{n} \right\|. \end{aligned}$$

By the choice of n_0, $\frac{1}{n} \leqslant \frac{1}{n_0} < \iota r$. Hence, for all $t \in [0,T]$, we have

$$x(t) \geqslant \iota \left\| x - \frac{1}{n} \right\| + \frac{1}{n} \geqslant \iota \left(\|x\| - \frac{1}{n} \right) + \frac{1}{n} \geqslant \iota r.$$

Next we claim that

$$\|x'\| \leqslant Lr \tag{3.4}$$

for any T-periodic solution $x(t)$ of Eq. (3.2). Note that (3.2) is equivalent to

$$\big(\sigma(h)(t)x'\big)' + a(t)\sigma(h)(t)x = \sigma(h)(t)\left(\lambda f_n\big(t, x(t), x'(t)\big) + \frac{a(t)}{n}\right). \tag{3.5}$$

Integrating (3.5) from 0 to T, we obtain

$$\int_0^T a(t)\sigma(h)(t)x(t)\, dt = \int_0^T \sigma(h)(t)\left(\lambda f_n\big(t, x(t), x'(t)\big) + \frac{a(t)}{n}\right) dt.$$

By the periodic boundary conditions, $x'(t_0) = 0$ for some $t_0 \in [0, T]$. Therefore

$$\begin{aligned}
\left|\sigma(h)(t)x'(t)\right| &= \left|\int_{t_0}^{t} \left(\sigma(h)(s)x'(s)\right)' ds\right| \\
&= \left|\int_{t_0}^{t} \sigma(h)(s)\left(\lambda f_n\big(s, x(s), x'(s)\big) + \frac{a(s)}{n} - a(s)x(s)\right) ds\right| \\
&\leqslant \int_0^T \sigma(h)(s)\left(\lambda f_n\big(s, x(s), x'(s)\big) + \frac{a(s)}{n} + a(s)x(s)\right) ds \\
&= 2\int_0^T a(s)\sigma(h)(s)x(s)\, ds \\
&\leqslant 2r\int_0^T a(s)\sigma(h)(s)\, ds,
\end{aligned}$$

where we have used the assumption (3.1). Therefore,

$$\Big(\min_{0\leqslant t\leqslant T} \sigma(h)(t)\Big)\left|x'(t)\right| \leqslant 2r\int_0^T a(s)\sigma(h)(s)\, ds,$$

which implies that (3.4) holds.

Thus we have from condition (H_2), for all $t \in [0, T]$,

$$\begin{aligned}
x(t) &= \lambda\int_0^T G(t,s) f_n\big(s, x(s), x'(s)\big)\, ds + \frac{1}{n} \\
&= \lambda\int_0^T G(t,s) f\big(s, x(s), x'(s)\big)\, ds + \frac{1}{n} \\
&\leqslant \int_0^T G(t,s) f\big(s, x(s), x'(s)\big)\, ds + \frac{1}{n} \\
&\leqslant \int_0^T G(t,s) g\big(x(s)\big)\left\{1 + \frac{h(x(s))}{g(x(s))}\right\}\varrho\big(\left|x'(s)\right|\big)\, ds + \frac{1}{n} \\
&\leqslant g(\iota r)\left\{1 + \frac{h(r)}{g(r)}\right\}\varrho(Lr)\int_0^T G(t,s)\, ds + \frac{1}{n} \\
&\leqslant g(\iota r)\left\{1 + \frac{h(r)}{g(r)}\right\}\varrho(Lr)\omega^* + \frac{1}{n_0}.
\end{aligned}$$

Therefore,

$$r = \|x\| \leqslant g(\iota r)\left\{1 + \frac{h(r)}{g(r)}\right\}\varrho(Lr)\omega^* + \frac{1}{n_0}.$$

This is a contradiction to the choice of n_0 and the first claim is proved.

Let be $C_T^1 = \{x\colon\ x, x' \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})\}$ with the norm $\|x\| = \max_{t\in[0,T]} |x(t)|$. Then C_T^1 is a normed linear space (not complete). Define

$$B_r = \left\{x \in C_T^1\colon\ \|x\| < r\right\}.$$

Then B_r is an open subset in C_T^1 with $p = 1/n \in B_r$ since $1/n < r$. Now using Lemma 3.1, we know that

$$x = T_n x$$

has a fixed point, denoted by x_n, in, i.e., equation

$$x'' + h(t)x' + a(t)x = f_n\big(t, x(t), x'(t)\big) + \frac{a(t)}{n} \tag{3.6}$$

has a periodic solution x_n with $\|x_n\| < r$. Since $x_n(t) \geqslant 1/n > 0$ for all $t \in [0, T]$ and x_n is actually a positive periodic solution of (3.6).

Next we claim that $x_n(t)$ have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$\min_{t\in[0,T]} x_n(t) \geqslant \delta \tag{3.7}$$

for all $n \in N_0$.

Since (H_1) holds, there exists a continuous function $\phi_r \succ 0$ such that $f(t, x, y) \geqslant \phi_r(t)$ for all $(t, x, y) \in [0, T] \times (0, r] \times \mathbb{R}$. Therefore,

$$x_n(t) = \int_0^T G(t,s)\left\{ f_n\big(t, x_n(s), x_n'(s)\big) + \frac{a(s)}{n}\right\} ds$$

$$\geqslant \int_0^T G(t,s)\phi_r(s)\, ds,$$

which means that (3.7) holds with

$$\delta = \min_{0\leqslant t\leqslant T} \int_0^T G(t,s)\phi_r(s)\, ds > 0.$$

Using the similar procedure in the proof of (3.4), we can prove that

$$\|x_n'\| \leqslant Lr \tag{3.8}$$

for all $n \geqslant n_0$.

From Eq. (1.1), $\|x_n\| < r$ and (3.8), it is easy to see that $\{\|x_n''\|\}_{n\in N_0}$ is also uniformly bounded. In consequence, $\{\|x_n'\|\}_{n\in N_0}$ is equicontinuous, and therefore $\{x_n\}_{n\in N_0}$ is a bounded and equi-continuous family in C_T^1. Now the Arzela–Ascoli Theorem guarantees that $\{x_n\}_{n\in N_0}$ has a subsequence, $\{x_{n_k}\}_{k\in\mathbb{N}}$, converging uniformly on $[0, T]$ to a function $x \in X$. From the fact $\|x_n\| < r$ and (3.7), x satisfies $\delta \leqslant x(t) \leqslant r$ for all $t \in [0, T]$. Moreover, x_{n_k} satisfies the integral equation

$$x_{n_k}(t) = \int_0^T G(t,s) f\big(s, x_{n_k}(s), x_{n_k}'(s)\big)\, ds + \frac{1}{n_k}.$$

Letting $k \to \infty$, we arrive at

$$x(t) = \int_0^T G(t,s) f\big(s, x(s), x'(s)\big)\, ds.$$

Therefore, x is a positive T-periodic solution of (1.1) and satisfies $0 < \|x\| \leqslant r$. □

Corollary 3.3. *Let the nonlinearity in* (1.1) *be*

$$f(t, x, y) = \big(1 + |y|^\gamma\big)\big(x^{-\alpha} + \mu x^\beta\big), \tag{3.9}$$

where $\alpha > 0$, $\beta, \gamma \geqslant 0$, $\mu > 0$ *is a positive parameter.*

(i) *If* $\beta + \gamma < 1$, *then* (1.1) *has at least one positive periodic solution for each* $\mu > 0$.
(ii) *If* $\beta + \gamma \geqslant 1$, *then* (1.1) *has at least one positive periodic solution for each* $0 < \mu < \mu_1$, *where* μ_1 *is some positive constant.*

Proof. We will apply Theorem 3.2. To this end, the assumption (H_1) is fulfilled by $\phi_r(t) = r^{-\alpha}$. Take

$$g(x) = x^{-\alpha}, \qquad h(x) = \mu x^{\beta}, \qquad \varrho(x) = 1 + |x|^{\gamma}.$$

Then (H_2) is satisfied and the existence condition (H_3) becomes

$$\mu < \frac{\sigma^{\alpha} r^{1+\alpha} - \omega^* - \omega^* L^{\gamma} r^{\gamma}}{r^{\alpha+\beta}(\omega^* + \omega^* L^{\gamma} r^{\gamma})}$$

for some $r > 0$. So (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{\sigma^{\alpha} r^{1+\alpha} - \omega^* - \omega^* L^{\gamma} r^{\gamma}}{r^{\alpha+\beta}(\omega^* + \omega^* L^{\gamma} r^{\gamma})}.$$

Note that $\mu_1 = \infty$ if $\beta + \gamma < 1$ and $\mu_1 < \infty$ if $\beta + \gamma \geqslant 1$. We have (i) and (ii). □

Corollary 3.4. *Let the nonlinearity in* (1.1) *be*

$$f(t, x, y) = \left(1 + |y|^{\gamma}\right)\left(\frac{1}{x^{\alpha}} - \frac{\mu}{x^{\beta}}\right), \tag{3.10}$$

where $\alpha > \beta > 0$, $\gamma \geqslant 0$ with $\gamma < \alpha + 1$, $\mu > 0$ is a positive parameter. Then there exists a positive constant μ_2 such that (1.1) *has at least one positive T-periodic solution for each $0 \leqslant \mu < \mu_2$.*

Proof. Take

$$g(x) = x^{-\alpha}, \qquad h(x) \equiv 0, \qquad \varrho(y) = 1 + |y|^{\gamma}.$$

Then (H_2) is satisfied and the existence condition (H_3) becomes

$$\sigma^{\alpha} r^{1+\alpha} > \left(1 + L^{\gamma} r^{\gamma}\right)\omega^* \tag{3.11}$$

for some $r > 0$. Since $\alpha + 1 > \gamma$, we can choose $r > 0$ large enough such that (3.11) is satisfied. Next we show that (H_1) is satisfied. Let

$$l(x) = x^{-\alpha} - \mu x^{-\beta}, \quad x \in (0, +\infty)$$

and

$$s_1 = \mu^{-\frac{1}{\alpha-\lambda}}, \qquad s_2 = (\alpha/\mu\lambda)^{\frac{1}{\alpha-\lambda}}.$$

Since $\alpha > \beta$, one can easily verify that $s_1 < s_2$ and

$$l(s_1) = 0, \qquad l'(s_2) = 0, \qquad l'(s) < 0, \quad s \in (0, s_2).$$

Therefore, $l(s)$ is decreasing on $(0, s_1) \subset (0, s_2)$. On the other hand, we can choose $\mu > 0$ small enough such that

$$r \in (0, s_1).$$

Thus,

$$\min_{s \in (0,r)} l(s) = l(r) > l(s_1) = 0.$$

This implies that the condition (H_1) is satisfied if we take

$$\phi_r(t) \equiv l(r). \qquad \square$$

Remark 3.5. Corollary 3.4 is interesting because the singularity on the right-hand side combines attractive and repulsive effects. The analysis of such differential equations with mixed singularities is at this moment very incomplete, and few references can be cited [3,18]. Therefore, the results in Corollary 3.4 can be regarded as one contribution to the literature trying to fill partially this gap in the study of singularities of mixed type.

Remark 3.6. It is easy to find results analogous to Corollaries 3.3 and 3.4 for the general equation (1.2) when $b, c, e, \kappa \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $b, e, \kappa \succ 0$. Here we consider the nonlinearities (3.9) and (3.10) only for simplicity.

Finally in this section, we consider one special case of (1.1), that is

$$x'' + h(t)x' + a(t)x = f(t, x), \tag{3.12}$$

in which the nonlinearity does not depend on the derivative. The following result is a direct consequence of Theorem 3.2.

Theorem 3.7. *Suppose that* (1.3) *satisfies* (A)*. Furthermore, assume that there exists a constant* $r > 0$ *such that*:

(H_1') *There exists a continuous function* $\phi_r \succ 0$ *such that* $f(t, x) \geqslant \phi_r(t)$ *for all* $(t, x) \in [0, T] \times (0, r]$.
(H_2') *There exist continuous, non-negative functions* $g(\cdot)$ *and* $h(\cdot)$ *such that*

$$0 \leqslant f(t, x) \leqslant g(x) + h(x), \quad \textit{for all } (t, x) \in [0, T] \times (0, r],$$

where $g(\cdot) > 0$ *is non-increasing,* $h(\cdot)/g(\cdot)$ *is non-decreasing in* $(0, r]$.
(H_3') *The following inequality holds*

$$\frac{r}{g(\sigma r)\{1 + \frac{h(r)}{g(r)}\}} > \omega^*.$$

Then (3.12) *has at least one positive periodic solution x with* $0 < \|x\| \leqslant r$.

Remark 3.8. Even in Theorem 3.7, we have generalized those results contained in [6,8,26], in which only equations without damping term are considered. We emphasize that our results are applicable to the case of a strong singularity as well as the case of a weak singularity. Note that in the proof of Theorem 3.2, the positiveness of Green's function plays an important role. Finally, we remark that the results contained in this paper can be translated to the L^1-Caratheodory framework without significant changes.

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