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The Lazer–Solimini equation with state-dependent delay

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ABSTRACT

Sufficient criteria are established for the existence of T-periodic solutions of a family of Lazer–Solimini equations with state-dependent delay. The method of proof relies on a combination of Leray–Schauder degree and a priori bounds.

1. Introduction

Singular nonlinearities arise naturally in physical models when considering gravitational or electromagnetic forces. In 1987, Lazer and Solimini [1] proposed the equations

$$x'' \pm \frac{1}{x^\alpha} = p(t) \tag{1}$$

as a toy model for the study of scalar ODEs with singular nonlinearity and periodic dependence on time. This work has become a hallmark in the area, and since its publication a wide variety of topological and variational methods have been systematically employed in the study of different extensions and variants of (1) (see the recent reviews [2,3]).

When speaking about gravitational forces, the introduction of relativistic effects makes sense. One of the known consequences of Special Relativity is that state-dependent delays come into play [4,5]. Motivated by this reflection, we propose in this note the study of an analogue of Lazer–Solimini equations with state-dependent delay

$$x'' + g[x](t) = p(t), \tag{2}$$

$$x'' - g[x](t) = -p(t), \tag{3}$$

where $p \in C(\mathbb{R} \setminus T\mathbb{Z})$ and

$$g[x](t) \equiv g(x(t - \tau(t, x(t)))),$$

being $\tau : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ a nonnegative continuous function which is T-periodic in the first variable. Finally, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which verifies the standing hypothesis

(H1) $\lim_{x\to+\infty} g(x) = 0, \ \lim_{x\to0^+} g(x) = +\infty.$

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In the classical terminology, it is said that (2) has an *attractive* singularity, whereas (3) has a *repulsive* singularity. Often, we simply speak about the attractive and the repulsive case. In this latter case, a minus sign in the forcing p has been added for convenience.

Needless to say, delayed systems have been the focus of attention of many researchers as a response of its many applications. In particular, state-dependent delays plays a key role in a variety of biological and mechanical models (see the review [6] and the bibliography therein). Although second order scalar ODEs with delays have been considered in some relevant recent papers (see for instance [7–10] only to cite a few of them), up to our knowledge the inclusion of singularities is not adequately covered by the existing references.

In order to explain our main results, let us fix $\bar{p} := \frac{1}{T}\int_0^T p(t)dt$ the mean value of p. After integration over a whole period, it becomes apparent that $\bar{p} > 0$ is a necessary condition for existence of T-periodic solution of both (2) and (3). In the case without delay, Lazer and Solimini proved that $\bar{p} > 0$ is also sufficient for the attractive case, whereas in the repulsive case a counterexample can be found proving that additional conditions are required (for instance the so-called *strong force condition*) for existence of T-periodic solution. Our aim is to provide a complementary sufficient condition which is valid also for the equations with state-dependent delay.

Theorem 1. *Assume that g satisfies* (H1) *and*

(H2) $g(x) > \bar{p} > 0$ *for every* $x \le T\|p^+\|_1$.

Then (2) *(resp.* (3)*) has at least one positive T-periodic solution.*

For the attractive case, we can prove a different result.

Theorem 2. *Assume that g satisfies* (H1) *and* $\bar{p} > 0$. *If* $p(t)$ *is bounded above and*

(H3) $\limsup_{x\to 0^+} \tau(t,x) < \frac{\min\{v\in\mathbb{R}^+ : g(v)=\|p^+\|_\infty\}}{\|p^+\|_1}$ *uniformly in* t,

then (2) *has at least one positive T-periodic solution.*

Up to out knowledge, Theorem 1 is new even for the equation without delay. On the other hand Theorem 2 is a generalization of the classical result by Lazer–Solimini, which is recovered by taking $\tau(t,x) \equiv 0$. Clearly, condition (H2) is related with the strength of the singularity and is valid for any delay. On the other hand, (H3) is related with the behavior of the delay near the singularity. However, it could have some interest from the point of view of Physics, since in Special Relativity the expected delay should be proportional to the distance of the particle to the singularity, that is of the type $\tau(t,x) = x$, which trivially satisfies (H3).

Other interesting remark concerns the regularity of the involved coefficients. By revising the proofs, one realizes that Theorem 1 remains true for the case $p \in L^1(\mathbb{R}\setminus T\mathbb{Z})$, of course by considering the solutions in the Caratheodory sense. On the other hand, an analogue of Theorem 2 for a purely L^1-Caratheodory ambient is an open problem even for the equation without delay, a fact yet noticed in [11].

From now on, we consider the Banach spaces $X = C^1(\mathbb{R}\setminus T\mathbb{Z})$ endowed with the usual C^1-norm and $Z = L^1(\mathbb{R}\setminus T\mathbb{Z})$ with the L^1-norm. Given $f \in Z, f^+ = \max\{f, 0\}$ denotes the positive part and $f^- = \max\{-f, 0\}$ the negative part of f. The rest of the paper is organized as follows. In Section 2, we present some a priori bounds for the solutions of a convenient homotopic equation. Then in Section 3 the main results are proved by using a well-known continuation theorem of Capietto–Mawhin–Zanolin [12].

2. A priori bounds

In this section we prove some lemmas that will be used in the proof of our main results. Let us consider the following homotopic equations

$$x'' + g_\lambda[x](t) = p_\lambda(t). \tag{4}$$

$$x'' - g_\lambda[x](t) = -p_\lambda(t) \tag{5}$$

where $g_\lambda[x](t) := g(x(t-\lambda\tau(t,x(t))))$, $p_\lambda(t) = (1-\lambda)\bar{p} + \lambda p(t)$ and $\lambda \in [0, 1]$.

From now on, The following lemma, which is due to Lazer–Solimini [1], will be useful.

Lemma 1 ([1])**.** *Let $x \in X$ be a T-periodic function such that $x'' \in Z$. Then*

$$\|x'\|_\infty \le \|(x'')^\pm\|_1.$$

By using this lemma, we can find a uniform bound for x' when x is a T-periodic solution of (4) or (5).

Lemma 2. *If x is a T-periodic solution of* (4) *or* (5) *and g satisfies* (H1) *then*

$$\|x'\|_\infty \le \|p^+\|_1.$$

Proof. Let $x \in X$ be a solution of (4), then by Lemma 1 we have

$$\begin{aligned}\|x'\|_\infty &< \|(x'')^+\|_1 = \|\,(p_\lambda(t) - g_\lambda[x](t))^+\,\|_1 \\ &\le \|p_\lambda^+\|_1 \le \|p^+\|_1.\end{aligned}$$

The proof for a solution $x \in X$ of (5) is analogous, taking into account that $\|x'\|_\infty \le \|(x'')^-\|_1 = \|(-x'')^+\|_1$. □

The next step is to find an upper bound for T-periodic solutions of Eqs. (4)–(5).

Lemma 3. *Assume that g satisfies (H1) and $\bar{p} > 0$. Then there exists a positive constant M not depending on $\lambda \in [0, 1]$ such that*

$$x(t) < M \quad \text{for all } t,$$

for every T-periodic solution $x(t)$ of (4) or (5).

Proof. Let x by a T-periodic solution of (4) or (5). Integrating on both sides of the equation we get

$$\begin{aligned}\int_0^T g_\lambda[x](t)dt &= \int_0^T ((1-\lambda)\bar{p} + \lambda p(t))dt \\ &= T\bar{p}.\end{aligned}$$

Because of the continuity of the involved functions, there is $t_1 \in]0, T[$ such that $g_\lambda[x](t_1) = g(x(t_1 - \lambda\tau(t_1, x(t_1)))) = \bar{p}$. Let us define $t_{0,\lambda} := t_1 - \lambda\tau(t_1, x(t_1))$. By Lemma 2,

$$x(t) - x(t_{0,\lambda}) = \int_{t_{0,\lambda}}^t x'(s)ds \le T\|x'\|_\infty \le T\|p^+\|_1$$

for any $t \in]t_{0,\lambda}, t_{0,\lambda} + T[$. On the other hand, (H1) and $\bar{p} > 0$ implies that the set $\{v \in \mathbb{R}^+ : g(v) = \bar{p}\}$ is bounded, closed and non-empty, so in consequence it has a maximum, call it C^*. Thus,

$$x(t) \le T\|p^+\|_1 + x(t_{0,\lambda}) < T\|p^+\|_1 + C^* + 1 =: M,$$

and it is obvious that this constant does not depend on λ. □

Finally, we look for a lower bound of possible T-periodic solutions.

Lemma 4. *Under the conditions of Theorem 1, there exists $\varepsilon_1 > 0$ not depending on $\lambda \in [0, 1]$ such that*

$$x(t) > \varepsilon_1 \quad \text{for all } t,$$

for every T-periodic solution $x(t)$ of (4) or (5).

Proof. Arguing as in the proof of Lemma 3,

$$x(t) = \int_{t_{0,\lambda}}^t x'(s)ds + x(t_{0,\lambda}),$$

where $g(x(t_{0,\lambda})) = \bar{p}$. By (H1)–(H2), $C_* = \min\{v \in \mathbb{R}^+ : g(v) = \bar{p}\}$ is well-defined and $C_* > T\|p^+\|_1$. Therefore, by applying Lemma 2 once more,

$$x(t) = \int_{t_{0,\lambda}}^t x'(s)ds + x(t_{0,\lambda}) \ge C_* - T\|p^+\|_1 > \frac{C_* - T\|p^+\|_1}{2} =: \varepsilon_1 > 0. \quad \square$$

Lemma 5. *Under the conditions of Theorem 2, there exists $\varepsilon_2 > 0$ not depending on $\lambda \in [0, 1]$ such that*

$$x(t) \ge \varepsilon_2 \quad \text{for all } t,$$

for every T-periodic solution $x(t)$ of (4).

Proof. For a given T-periodic solution $x(t)$ of (4), assume that $x(t_0) = \min_{t\in[0,T]} x(t)$. Then,

$$g_\lambda[x](t_0) \le x''(t_0) + g_\lambda[x](t_0) = p_\lambda(t_0) \le \|p^+\|_\infty.$$

Then, by using the hypothesis (H1),

$$x(t_0 - \lambda\tau(t_0, x(t_0))) \ge D_* := \min\{v \in \mathbb{R}^+ : g(v) = \|p^+\|_\infty\} > 0. \tag{6}$$

On the other hand, if we call $\tilde{\varepsilon} := D_* - \|p^+\|_1 \limsup_{x\to 0^+} \tau(t, x)$, by condition (H3), there exists $\varepsilon > 0$ such that

$$D_* - \|p^+\|_1\tau(t, x) \ge \tilde{\varepsilon} > 0 \quad \text{for all } 0 < x \le \varepsilon. \tag{7}$$

By the Mean Value Theorem and Lemma 2 we have

$$\begin{aligned} x(t_0-\lambda\tau(t_0,x(t_0)))-x(t_0) &\le x'(\zeta)\tau(t_0,x(t_0)) \\ &\le \|x'\|_\infty \tau(t_0,x(t_0)) \\ &\le \|p^+\|_1 \tau(t_0,x(t_0)). \end{aligned}$$

Then, by using (6)

$$\begin{aligned} x(t_0) &\ge x(t_0-\lambda\tau(t_0,x(t_0)))-\|p^+\|_1\tau(t_0,x(t_0)) \\ &\ge D_* - \|p^+\|_1\tau(t_0,x(t_0)) > 0. \end{aligned}$$

Now, if $x(t_0) \le \varepsilon$, combining (7) with the latter inequality, one gets $x(t_0) \ge \tilde{\varepsilon}$. The proof is finished by taking $\varepsilon_2 = \frac{1}{2}\min\{\varepsilon, \tilde{\varepsilon}\}$. □

3. Proof of main results

Let us define the linear operator

$$\mathcal{L}: D(\mathcal{L}) \subset X \to Z \qquad \mathcal{L}(x) := x'' - x,$$

where $D(\mathcal{L}) := \{x : x \in X,\ x' \text{ is absolutely continuous on } \mathbb{R}\}$, and the Nemitskii operator $\mathcal{N}: X^+ \times [0,1] \to Z, X^+ := \{x \in X : x(t) > 0 \text{ for all } t\} \subset X$ given by

$$\mathcal{N}(x;\lambda) := \begin{cases} p_\lambda(t) - g_\lambda[x](t) - x(t), & \text{in the case of Eq. (4)} \\ -p_\lambda(t) + g_\lambda[x](t) - x(t), & \text{in the case of Eq. (5).} \end{cases}$$

Then, x is a T-periodic solution of Eq. (4) or (5), $\lambda \in [0,1]$ if and only if $x \in D(\mathcal{L})$ is a solution of

$$\mathcal{L}x = \mathcal{N}(x;\lambda), \quad \lambda \in [0,1].$$

In particular, (2)–(3) are equivalent to $\mathcal{L}x = \mathcal{N}(x;1)$. Since $\mathcal{L}$ is invertible, we can write equivalently

$$x - \mathcal{L}^{-1}\mathcal{N}(x;\lambda) = 0. \tag{8}$$

Consequently, finding T-periodic solutions of Eq. (4) or Eq. (5) is equivalent to finding the fixed points of the operator $\mathcal{L}^{-1}\mathcal{N}$ in Ω. With this in mind, we present the proof of Theorem 1.

Proofs of Theorems 1 and 2. Let $\Omega \subset X$ be the open bounded set defined by

$$\Omega = \{x \in X : \varepsilon_1 < x(t) < M \text{ and } \|x'(t)\|_\infty < M_1,\ \forall t \in \mathbb{R}\},$$

where ε_1, M are the positive constants fixed in Lemmas 3 and 4 and $M_1 := \|p^+\|_1$. By the a priori bounds derived in Section 2, (8) has no solutions $(x,\lambda) \in (D(\mathcal{L}) \cap \partial\Omega) \times [0,1]$. Since $\mathcal{L}^{-1}\mathcal{N}(\cdot;\lambda)$ is a compact operator, by the global continuation principle of Leray–Schauder [13, Theorem 14.C], Theorem 1 will be proved if we show that the degree is nonzero for some $\lambda \in [0,1]$.

Let us first consider the case of Eq. (2). Taking $\lambda = 0$, Eq. (4) becomes

$$x'' + g(x(t)) = \bar{p}.$$

Define the function $F: [\varepsilon_1, M] \times [-M_1, M_1] \to \mathbb{R}^2$ given by

$$F(u,v) = (v, \bar{p} - g(u)).$$

By a classical result of Capietto et al. [12, Theorem 1], we can compute the Leray–Schauder degree of $\mathcal{I} - \mathcal{L}^{-1}\mathcal{N}(\cdot;\lambda)$ as the Brouwer degree of F as follows

$$D_{LS}(\mathcal{I} - \mathcal{L}^{-1}\mathcal{N}(\cdot;0), \Omega) = D_B(F,\ [\varepsilon_1, M] \times [-M_1, M_1]).$$

Such degree is easily computed by elementary techniques and shown to be 1. Hence, by the existence property of the degree, there is $x \in D(\mathcal{L}) \cap \Omega$ such that $\mathcal{L}x = \mathcal{N}(x;1)$. Exactly the same proof is valid for Theorem 2. The proof for Eq. (5) is analogous, only changing a sign on the second component of function F, which gives a degree equal to -1. □

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