Z. angew. Math. Phys. 51 (2000) 333–345
0044-2275/00/030333–13 \$ 1.50+0.20/0
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Zeitschrift für angewandte Mathematik und Physik ZAMP

Periodic motions of forced infinite lattices with nearest neighbor interaction*

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Abstract. It is proved the existence of an infinite number of periodic solutions of a infinite lattice of particles with a periodic perturbation and nearest neighbor interaction between particles, by using a priori bounds and topological degree together with a limiting argument. We consider a Toda lattice and a singular repulsive lattice as main situations. The question of order between particles is also discussed.

Mathematics Subject Classification (1991). 82C20, 82C22.

Keywords. Lattice, Toda interaction, singular forces, a priori bounds, Brouwer degree.

1. Introduction

We are concerned with the existence of T-periodic solutions of the infinite system of non-autonomous differential equations

$$x_i'' + cx_i' = g_{i-1}(x_i - x_{i-1}) - g_i(x_{i+1} - x_i) + h_i(t), \qquad i \in \mathbb{Z} \tag{1}$$

where $c \geq 0$ and h_i are continuous T-periodic functions.

This system describes the motion of a 1-dimensional lattice of particles of unitary mass periodically perturbed and each interacting with its neighbors by restoring forces depending on the distance between particles and a possible viscous friction. If $\overline{h}_i$ denote the mean value of h_i, our aim is to prove existence and multiplicity of T-periodic solutions under the main assumption that $\overline{h}_i = 0$ for every i.

In this paper, we are interested in two main situations. Section 2 is devoted to the study a lattice of Toda type, whose classical model presents a exponentially decreasing nonlinearity [10]. Section 3 considers a lattice with singular repulsive forces between particles.

In the latter years, several papers have appeared concerning the autonomous conservative case [1, 2, 3, 4, 9] of system (1). In these works, the variational

*Research supported by D.G.E.S. PB95-1203, M.E.C., Spain.

structure of the problem allows to consider periodic solutions as critical points of a suitable functional, so classical variational techniques as the mountain pass Theorem are available. In contrast with this type of “free” systems, in our case a forcing term appears determining much of the motion. Also, the variational structure is lost due to the (possible) linear friction term, whence a completely different method of proof of topological type is required.

The strategy of proof is made of two main steps: first a related finite system of particles is studied by means of classical tools from topological degree. The method is inspired from [12]. Also, a similar approach in order to get a priori bounds of the T-periodic solutions can be found in [13], where a similar type of nonlinearity is considered for a nonlinear equation of arbitrary order. Second, a simple limiting argument is applied to the finite system, by using the a priori estimates deduced in the previous step. We stress that the obtained estimates are independent of the number of particles of the finite system, and this fact may be of interest by itself for applications. This limiting argument from a finite system has a remarkable analogy with [3], and as it was noted there, this could be very interesting for numerical applications.

2. Lattices of Toda type

In the early 1970's, Toda considered a 1-dimensional lattice in which the force between neighbor particles is an exponentially decreasing function of their distance. It is known that the classical Toda lattice is a free explicitly integrable system. If $g_i(s) = e^{-s}$, system (1) is a forced Toda lattice. We are going to consider a larger set of “admissible” restoring forces.

Definition. *System* (1) *is said* of Toda type *if for all* i, $g_i : \mathbb{R} \to \mathbb{R}^+$ *is a continuous function such that*

$$\lim_{x \to -\infty} g_i(x) = +\infty, \qquad \lim_{x \to +\infty} g_i(x) = 0. \tag{2}$$

2.2. Main result

We are going to look for T-periodic solutions of (1) on the configuration space

$$\mathcal{H} = \{x = \{x_i\}_{i \in \mathbb{Z}} \in C^2(\mathbb{R})^{\mathbb{Z}} : \int_0^T x_0(t)dt = 0\}$$

By a T-periodic solution we understand a solution $x \in \mathcal{H}$ such that $x_i(0) = x_i(T), x_i'(0) = x_i'(T)$ for each $i \in \mathbb{Z}$. Note that if x is a solution of (1), then also is $x + C$ for every constant C, so we assume $\int_0^T x_0(t)dt = 0$ as a normalization condition.

Theorem 1. *Let consider a system of Toda type such that $\overline{h}_i = 0$ for all $i \in \mathbb{Z}$. Then, for any $K \in \mathbb{R}^+$ there exists a T-periodic solution $x \in \mathcal{H}$ of* (1) *such that*

$$\int_0^T g_i(x_{i+1}(t) - x_i(t))dt = KT, \qquad \forall i \in \mathbb{Z}. \tag{3}$$

We remark that condition (3) assures the existence of an infinite number of essentially different T-periodic solutions.

2.2. A related finite system

Let consider the finite system of $2n+1$ equations

$$\begin{cases} x''_{-n} + cx'_{-n} = -g_{-n}(x_{-n+1} - x_{-n}) + h_{-n}(t) + K \\ x''_i + cx'_i = g_{i-1}(x_i - x_{i-1}) - g_i(x_{i+1} - x_i) + h_i(t), \quad i = -n+1, \dots, n-1 \\ x''_n + cx'_n = g_{n-1}(x_n - x_{n-1}) + h_n(t) - K \end{cases} \tag{4}$$

where $K > 0$ and $\overline{h}_i = 0$.

By making the change of variables

$$\begin{cases} y(t) = x_0(t) \\ d_i(t) = x_{i+1}(t) - x_i(t), \qquad i = -n, \dots, n-1 \end{cases} \tag{5}$$

the equivalent system is

$$y'' + cy' = g_{-1}(d_{-1}) - g_0(d_0) + h_0(t) \tag{6}$$

$$\begin{cases} d''_{-n} + cd'_{-n} = 2g_{-n}(d_{-n}) - g_{-n+1}(d_{-n+1}) + \widehat{h}_{-n}(t) - K \\ d''_i + cd'_i = 2g_{i-1}(d_{i-1}) - g_i(d_i) - g_{i-2}(d_{i-2}) + \widehat{h}_i(t), \quad {\scriptstyle i=-n+1,\dots,n-2} \\ d''_{n-1} + cd'_{n-1} = 2g_{n-1}(d_{n-1}) - g_{n-2}(d_{n-2}) + \widehat{h}_{n-1}(t) - K \end{cases} \tag{7}$$

where $\widehat{h}_i = h_{i+1} - h_i$. First, we are going to study the sub-system (7). Let us consider the following homotopy

$$\begin{cases} d''_{-n} + cd'_{-n} = 2g_{-n}(d_{-n}) - g_{-n+1}(d_{-n+1}) + \lambda\widehat{h}_{-n}(t) - K \\ d''_i + cd'_i = 2g_{i-1}(d_{i-1}) - g_i(d_i) - g_{i-2}(d_{i-2}) + \lambda\widehat{h}_i(t), \quad {\scriptstyle i=-n+1,\dots,n-2} \\ d''_{n-1} + cd'_{n-1} = 2g_{n-1}(d_{n-1}) - g_{n-2}(d_{n-2}) + \lambda\widehat{h}_{n-1}(t) - K \end{cases} \tag{8}$$

with $\lambda \in [0,1]$.

Lemma 1. *There exists $N = (N_{-n}, \dots, N_{n-1}) \in (\mathbb{R}^+)^{2n}$ such that*

$$\|d_i\|_\infty < N_i, \qquad {\scriptstyle i=-n,\dots,n-1}$$

for any T-periodic solution $d = (d_i)$ of (8).

Proof. An integration of (8) over a period gives

$$2\int_0^T g_{-n}(d_{-n}(t))dt - \int_0^T g_{-n+1}(d_{-n+1}(t))dt = KT$$

$$2\int_0^T g_{i-1}(d_{i-1}(t))dt - \int_0^T g_i(d_i(t))dt - \int_0^T g_{i-2}(d_{i-2}(t))dt = 0, {\scriptstyle i=-n+1,\dots,n-2}$$

$$2\int_0^T g_{n-1}(d_{n-1}(t))dt - \int_0^T g_{n-2}(d_{n-2}(t))dt = KT.$$

If $\int_0^T g_i(d_i(t))dt$ are seen like unknowns of a linear system of $2n$ equations, it is easy to verify that

$$\int_0^T g_i(d_i(t))dt = KT, \qquad {\scriptstyle i=-n,\dots,n-1} \tag{9}$$

is the unique solution.

By using assumption (2), let $\psi_1^i < \psi_2^i$ be fixed numbers satisfying

$$g_i(x) > K \qquad \forall x < \psi_1^i, \tag{10}$$

$$g_i(x) < K \qquad \forall x > \psi_2^i. \tag{11}$$

If $d_i(t) < \psi_1^i$ for all $t \in [0, T]$, then $g_i(d_i(t)) > K$ and (9) is contradicted only integrating over a period. Thus, there exists $t_1 \in [0, T]$ such that $d_i(t_1) > \psi_1^i$. By an analogous argument, $d_i(t_2) < \psi_2^i$ for some $t_2 \in [0, T]$, and by continuity of the solutions, there exists $\tilde{t} \in [0, T]$ such that

$$\psi_1^i < d_i(\tilde{t}) < \psi_2^i.$$

Now, we multiply each equation of (8) by e^{ct}. Taking into account that $d_i'' e^{ct} + c d_i' e^{ct} = (d_i' e^{ct})'$ and by using (9) we get

$$\|(d_i' e^{ct})'\|_{L^1} \le (4KT + \|\widehat{h}_i\|_{L^1})e^{cT} =: M_i$$

after an integration over a period. Take $d_i(t_*) = \min\{d_i(t) : t \in [0, T]\}$. Then,

$$|d_i'(t)e^{ct}| = |\int_{t_*}^t (d_i'(s)e^{cs})' ds| \le \|(d_i' e^{ct})'\|_{L^1} \le M_i$$

for all $t \in [0, T]$. In consequence, $\|d_i'\|_\infty \le M_i$. Note that M_i only depends on i, K but not on n.

Besides,

$$|d_i(t) - d_i(\tilde{t})| = |\int_{\tilde{t}}^{t} d_i'(s)ds| \leq T\|d_i'\|_\infty \leq TM_i$$

for all $t \in [0,T]$, so $\|d_i\|_\infty \leq TM_i + d_i(\tilde{t}) \leq TM_i + \psi_2^i =: N_i$. □

In the following lemma, the Brouwer degree is denoted by $\deg_B$. For definition and main properties of topological degree we make reference to [8].

Lemma 2. *Let $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a continuous function of components $F = (F_{-n}, \dots, F_{n-1})$ defined by*

$$\begin{aligned}
F_{-n}(d_{-n}, \dots, d_{n-1}) &= 2g_{-n}(d_{-n}) - g_{-n+1}(d_{-n+1}) - K \\
F_i(d_{-n}, \dots, d_{n-1}) &= 2g_{i-1}(d_{i-1}) - g_i(d_i) - g_{i-2}(d_{i-2}), \quad i = -n+1, \dots, n-2 \\
F_{n-1}(d_{-n}, \dots, d_{n-1}) &= 2g_{n-1}(d_{n-1}) - g_{n-2}(d_{n-2}) - K
\end{aligned}$$

for all $(d_{-n}, \dots, d_{n-1}) \in \mathbb{R}^{2n}$. Assume that there exists a compact set $D \subset \mathbb{R}^{2n}$ such that

$$(d_{-n}(t), \dots, d_{n-1}(t)) \in D, \qquad \forall t \in [0,T]$$

for any T-periodic solution of (8)*, $\lambda \in [0,1]$. Then, if*

$$deg_B(F, \Omega, 0) \neq 0$$

for some Ω open bounded set containing D, there exists at least a T-periodic solution of (7)*.*

Proof. It is an immediate consequence of the results in [5]. □

Proposition 1. *Under the previous assumptions, there exists at least a T-periodic solution of system* (4)*.*

Proof. Let study the equivalent system $(6)-(7)$. First, we are going to prove the existence of a T-periodic solution of (7). By Lemmas 1 and 2, we only have to prove that

$$\deg_B(F, \Omega, 0) \neq 0$$

where F is defined in Lemma 2 and Ω is an open bounded set of $\mathbb{R}^{2n}$ large enough. To this purpose, we make a convex homotopy between F and $\tilde{F} : (\mathbb{R}^+)^{2n} \longrightarrow \mathbb{R}^{2n}$ defined by

$$\begin{aligned}
\tilde{F}_{-n}(d_{-n}, \dots, d_{n-1}) &= 2\tilde{g}(d_{-n}) - \tilde{g}(d_{-n+1}) - K \\
\tilde{F}_i(d_{-n}, \dots, d_{n-1}) &= 2\tilde{g}(d_{i-1}) - \tilde{g}(d_i) - \tilde{g}(d_{i-2}), \quad i = -n+1, \dots, n-2 \\
\tilde{F}_{n-1}(d_{-n}, \dots, d_{n-1}) &= 2\tilde{g}(d_{n-1}) - \tilde{g}(d_{n-2}) - K
\end{aligned}$$

where $\tilde{g} : \mathbb{R} \to \mathbb{R}^+$ is a continuous function with continuous and negative derivative, satisfying (2) and such that

$$\tilde{g}(x) < g_i(x), \qquad \forall x \in \mathbb{R},\ i = -n, \dots, n-1. \tag{12}$$

It is clear that this choice is possible. Then, the respective Brouwer degrees coincide (maybe with a larger Ω) if we find a priori estimates for the solutions of

$$\lambda F(d_{-n}, \dots, d_{n-1}) + (1-\lambda)\tilde{F}(d_{-n}, \dots, d_{n-1}) = 0, \qquad \lambda \in [0,1],$$

that is,

$$\begin{cases} 2(\lambda g_{-n}(d_{-n}) + (1-\lambda)\tilde{g}(d_{-n})) - (\lambda g_{-n+1}(d_{-n+1}) + (1-\lambda)\tilde{g}(d_{-n+1})) = K \\ 2(\lambda g_{i-1}(d_{i-1}) + (1-\lambda)\tilde{g}(d_{i-1})) - (\lambda g_i(d_i) + (1-\lambda)\tilde{g}(d_i)) - \\ \qquad\qquad -(\lambda g_{i-2}(d_{i-2}) + (1-\lambda)\tilde{g}(d_{i-2})) = 0, \quad i = -n+1, \dots, n-2 \\ 2(\lambda g_{n-1}(d_{n-1}) + (1-\lambda)\tilde{g}(d_{n-1})) - (\lambda g_{n-2}(d_{n-2}) + (1-\lambda)\tilde{g}(d_{n-2})) = K \end{cases}$$

with $\lambda \in [0,1]$. As in the proof of Lemma 1, $\lambda g_i(d_i) + (1-\lambda)\tilde{g}(d_i)$ can be considered as unknowns of a linear system of equations with a unique solution, namely,

$$\lambda g_i(d_i) + (1-\lambda)\tilde{g}(d_i) = K, \quad {\scriptstyle i=-n,\dots,n-1}.$$

From here, by using (12),

$$\tilde{g}(d_i) < K, \quad i = -n, \dots, n-1$$

and as $\tilde{g}$ is strictly decreasing, there exists the inverse $\tilde{g}^{-1}$ and

$$d_i > \tilde{g}^{-1}(K), \quad i = -n, \dots, n-1.$$

On the other hand,

$$g_i(d_i) > K, \quad i = -n, \dots, n-1$$

and by using (11) it follows that $d_i < \psi_2^i$ for all i. In conclusion, we have found a priori bounds for the solutions of the convex homotopy and hence it is proved that

$$\deg_B(F, \Omega, 0) = \deg_B(\tilde{F}, \Omega, 0)$$

for Ω large enough. Finally, we compute this last degree. If we define

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix},$$

it is easy to prove that $\det A \neq 0$. Taking into account that $\tilde{g}$ is strictly decreasing, then the vector field $\tilde{F}$ has the unique zero $(\xi, \dots, \xi)$ with $\xi = \tilde{g}^{-1}(K)$. If $\tilde{F}'$ is the jacobian matrix of the vector field $\tilde{F}$, by the definition of Brouwer degree and some easy computations we get

$$\deg_B(\tilde{F}, \Omega, 0) = \text{ sign } \det \tilde{F}'(\xi, \dots, \xi) = \text{ sign } \det \tilde{g}'(\xi) A \neq 0.$$

Therefore, the existence of a T-periodic solution of sub-system (7) is proved. Finally, the existence of a T-periodic solution of equation (6) is trivial because using (9) the right-hand member has mean value zero. □

2.3. Proof of Theorem 1.

Taking K a positive fixed number, the idea is to pass to the limit in the finite system (4). By previous Subsection, we have a priori bounds on the T-periodic solutions of system (6) – (7), depending on K, i but not on the total number n of particles of the finite system (see definitions of M_i, N_i). Let $y_0(t)$ be chosen as the T-periodic solution of (6) such that $\int_0^T y_0(t)dy = 0$. Specifically, $\|d_i'\|_\infty \leq M_i$ and $\|d_i\|_\infty \leq N_i$ for all t. Moreover,

$$y_0'' + cy_0' = g_{-1}(d_{-1}) - g_0(d_0) + h_0(t)$$

and by using similar reasonings to those used to find M_i, N_i, it is not hard to deduce that $\|y_0'\|_\infty \leq M_0$ and $\|y_0\|_\infty \leq N_0$.

Now, we are going to invert the change of variables (5) to get estimates on the T-periodic solution of system (4) given by Proposition 1, denoted from now on by $\{x_i^{(n)}\}_{i=-n,\dots,n}$.

Let start defining

$$P_i := \max\{g_i(x) : x \geq -N_i\}$$

for all $i \in \mathbb{Z}$. Then, it is easy to verify from (7) that

$$\|d_i''\|_\infty \leq 2P_{i-1} + P_i + P_{i+1} + \|\widehat{h}_i\|_\infty + cM_i =: D_i.$$

Directly, $\|x_0^{(n)'}\|_\infty \leq M_0$ and $\|x_0^{(n)}\|_\infty \leq N_0$, and moreover, from the equation we deduce that

$$\|x_0^{(n)''}(t)\|_\infty \leq cM_0 + P_{-1} + P_0 + \|h_0\|_\infty =: Q_0.$$

On the other hand, $x_1^{(n)} = d_1 + x_0^{(n)}$, so

$$\|x_1^{(n)}\|_\infty \leq N_1 + N_0 =: \tilde{N}_1$$

$$\|x_1^{(n)'}\|_\infty \leq M_1 + M_0 =: \tilde{M}_1$$

$$\|x_1^{(n)''}\|_\infty \leq D_1 + Q_0 =: Q_1$$

In general, $x_i^{(n)} = d_i + x_{i-1}^{(n)}$ and a recursive argument leads to

$$\|x_i^{(n)}\|_\infty \leq \sum_{j=0}^{i} N_j =: \tilde{N}_i$$

$$\|x_i^{(n)'}\|_\infty \leq \sum_{j=0}^{i} M_j =: \tilde{M}_i$$

$$\|x_i^{(n)''}\|_\infty \leq Q_0 + \sum_{j=1}^{i} D_j =: Q_i$$

for any $i = 1, \dots, n$. A symmetric argument provides bounds if i is negative.

Thus, fixing a position i, the sequences $\{x_i^{(n)}\}_{n\geq i}$ and $\{x_i^{(n)'}\}_{n\geq i}$ are uniformly bounded and equicontinuous. Besides, it is easy to prove that $\{x_i^{(n)''}\}_{n\geq i}$ is also an equicontinuous sequence by using the relation given by (4). Now, Ascoli-Arzela theorem implies the existence of some $x_i \in C^2(\mathbb{R})$ such that $x_i^n \to x_i$ (or at least a subsequence of this) uniformly in C^2, and evidently $\{x_i\}_{i\in\mathbb{Z}}$ is a T-periodic solution of system (1). Besides, passing to the limit in (9), condition (3) is proved. □

2.4. Ordered T-periodic solutions

From the physical point of view, it is interesting to look for T-periodic solutions on the configuration space

$$\begin{aligned}\mathcal{H}^+ = \{x = \{x_i\} \in C^2(\mathbb{R})^{\mathbb{Z}} \ : \ &\int_0^T x_0(t)dt = 0, \\ &x_i(t) < x_{i+1}(t), \ \ \forall i \in \mathbb{Z}, t \in [0,T]\},\end{aligned} \tag{13}$$

which implies an order in the lattice that avoid collision between particles. It is possible to prove the following result. Remember that $\widehat{h}_i = h_{i+1} - h_i$.

Theorem 2. *Let consider a system of Toda type such that* $\overline{h}_i = 0$ *for all* $i \in \mathbb{Z}$. *Then, for any* $n \in \mathbb{N}$ *there exists a* $K_n \in \mathbb{R}^+$ *such that if* $0 < K < K_n$, *system* (1) *has a* T*-periodic solution* $x \in \mathcal{H}$ *satisfying* (3) *and such that*

$$x_i(t) < x_{i+1}(t), \ \ \forall t \in [0,T], \ i = -n, \dots, n.$$

Moreover, if $\|\widehat{h}_i\|_{L^1}$ *is uniformly bounded for every* $i \in \mathbb{Z}$, *then there exists some* $K_\infty \in \mathbb{R}^+$ *such that if* $0 < K < K_\infty$, *system* (1) *has a* T*-periodic solution* $x \in \mathcal{H}^+$ *satisfying* (3).

Proof. The key idea of the proof is a revision of a priori bounds found on Lemma 1. Remember that we have defined

$$M_i := (4KT + \|\widehat{h}_i\|_{L^1})e^{cT}$$

holding that $\|d_i'\|_\infty \leq M_i$, and $\psi_1^i < \psi_2^i$ fixed numbers satisfying

$$g_i(x) > K \qquad \forall x < \psi_1^i,$$

$$g_i(x) < K \qquad \forall x > \psi_2^i.$$

If we look at ψ_1^i as a function of K it is clear from its own definition that $\lim_{K\to 0+} \psi_1^i(K) = +\infty$. Hence, it is possible to fix K_n such that

$$\psi_1^i(K) > TM_i, \qquad i = -n, \dots, n$$

for all $0 < K < K_n$. Now, repeating the arguments of Lemma 1, for any $i = -n, \dots, n$ there exists some $\tilde{t}$ such that $d_i(\tilde{t}) > \psi_1^i$, so in consequence

$$d_i(t) = d_i(t) + d_i(\tilde{t}) - d_i(\tilde{t}) \geq d_i(\tilde{t}) - |d_i(t) - d_i(\tilde{t})| > \psi_1^i - TM_i > 0$$

for all $t \in [0, T]$ and $i = -n, \dots, n$.

Finally, if there exists some $M > 0$ such that $\|\widehat{h}_i\|_{L^1} < M$ for every $i \in \mathbb{Z}$, then it is easy to verify that there exists some K_∞ such that if $0 < K < K_\infty$, then

$$\psi_1^i(K) > TM_i, \qquad \forall i \in \mathbb{Z},$$

which again implies after some easy computations that $d_i(t) > 0$ for all $t \in [0, T]$ and $i \in \mathbb{Z}$. □

Clearly, the assumption imposed over $\hat{h}_i$ is quite restrictive. Next Section will be devoted to a class of lattices in which T-periodic solutions are in $\mathcal{H}^+$ without further assumptions.

3. Lattices of singular type

In this Section, we are going to consider lattices in which forces between particles present a repulsive singularity in the origin.

Definition 2. *System* (1) *is said* of singular type *if for all i, $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that*

$$\lim_{x\to 0^+} g_i(x) = +\infty, \qquad \lim_{x\to +\infty} g_i(x) = 0 \tag{14}$$

and

$$\int_0^1 g_i(s)ds = +\infty. \tag{15}$$

This type of system describes the motion of a 1-dimensional lattice with external time-periodic forces and singular repulsive interaction between neighbor particles. The "model case" is $g_i(x) = x^{-\alpha}$ with $\alpha > 0$. A simple mechanical model for this system is an infinite chain of coupled pistons filled with a perfect gas, with time-periodic forces and a possible viscous friction. It is possible to think on a long train with pistons between wagons acting like shock absorbers. Also, forces between charged particles of the same sign (Coulomb forces) are included, assuming a short-ranged interaction between particles.

We are going to look for T-periodic solutions of singular lattices on the configuration space $\mathcal{H}^+$ defined in (13). As we have already noted, a conservation on the order of the particles is imposed. In this sense, hypothesis (15), the so-called "strong force condition", is standard on the related literature about singular forces (see for instance [6, 7, 11, 12]) in order to avoid collisions between particles. On the model case, hypothesis (15) holds if $\alpha \geq 1$.

The main result of this Section is the following.

Theorem 3. *Let consider a system of singular type such that $\overline{h}_i = 0$ for all $i \in \mathbb{Z}$. Then, for any $K \in \mathbb{R}^+$ there exists a T-periodic solution $x \in \mathcal{H}^+$ of* (3) *such that*

$$\int_0^T g_i(x_{i+1}(t) - x_i(t))dt = KT, \qquad \forall i \in \mathbb{Z}.$$

The idea of proof is the same that in Section 2. We consider the associate finite system (4), and by using the change of variables (5) we can prove the following result about the corresponding homotopic system (8).

Lemma 3. *There exists $\epsilon = (\epsilon_{-n}, \dots, \epsilon_{n-1}) \in (\mathbb{R}^+)^{2n}$ such that*

$$\epsilon_i < d_i(t) < \frac{1}{\epsilon_i}, \quad \forall t \in [0,T], \quad i = -n, \dots, n-1$$

for any T-periodic solution $d = (d_i)$ of (8).

Proof. As in Lemma 1, by using now assumption (14), it is possible to fix $\psi_1^i < \psi_2^i$ for each i satisfying

$$g_i(x) > K \qquad \forall x < \psi_1^i,$$
$$g_i(x) < K \qquad \forall x > \psi_2^i.$$

Repeating the proof of Lemma 1, there are M_i, N_i only depending on K, i such that

$$\|d_i\|_\infty \leq N_i, \qquad \|d_i'\|_\infty \leq M_i$$

for each $i \in \mathbb{Z}$. Now, let find a priori bounds from below.

Multiplying each equation of (8) by d_i' and integrating on $[t, \tilde{t}]$, some easy computations lead to

$$\frac{d_i'(\tilde{t})^2}{2} - \frac{d_i'(t)^2}{2} + c\int_t^{\tilde{t}} d_i'(s)^2 ds + \int_t^{\tilde{t}} g_i(d_i(s))d_i'(s)ds \leq (3KT + \|\widehat{h}_i\|_{L^1})\|d_i'\|_\infty$$

and in consequence

$$\int_{d_i(t)}^{d_i(\tilde{t})} g_i(s)ds \leq (3KT + \|\widehat{h}_i\|_{L^1})\|d_i'\|_\infty + \frac{d_i'(t)^2}{2} \leq (3KT + \|\widehat{h}_i\|_{L^1})M_i + \frac{M_i^2}{2}$$

for all $t < \tilde{t}$. The periodicity of d_i implies that this inequality holds for all t. By condition (15), it is possible to fix $\delta_i > 0$ such that

$$\int_{\delta_i}^{\psi_1^i} g_i(s)ds > (3KT + \|\widehat{h}_i\|_{L^1})M_i + \frac{M_i^2}{2}. \tag{16}$$

Then,

$$\int_{d_i(t)}^{d_i(\tilde{t})} g_i(s)ds < \int_{\delta_i}^{\psi_1^i} g(s)ds$$

and as $d_i(\tilde{t}) > \psi_1^i$, we deduce that $d_i(t) > \delta_i$ for all t.

To end the proof, we only have to take ϵ_i as the maximum of δ_i and $\frac{1}{N_i}$. □

The analogous of Lemma 2 in this context is the following result.

Lemma 4. *Let $F : (\mathbb{R}^+)^{2n} \longrightarrow \mathbb{R}^{2n}$ be a continuous function of components $F = (F_{-n}, \dots, F_{n-1})$ defined by* (2). *Assume that there exists a compact set $D \subset (\mathbb{R}^+)^{2n}$ such that*

$$(d_{-n}(t), \dots, d_{n-1}(t)) \in D, \qquad \forall t \in [0, T]$$

for any T-periodic solution of (8)*, $\lambda \in [0, 1]$. Then, if*

$$deg_B(F, \Omega, 0) \neq 0$$

for some Ω open bounded set containing D, there exists at least a T-periodic solution of (7).

Computation of degree is done by repeating exactly the proof of Proposition 1. In fact, the remaining proof is identical, with the only detail that the existence of an a priori bound from below of the solutions insures that the T-periodic solution obtained belongs to $\mathcal{H}^+$.

4. Further remarks

By means of a revision of proofs, it is clear that other types of forces between particles can be considered without further difficulties. For instance, the addition and subtraction of a fixed number on each equation of (1) enable us to study not only positive nonlinearities but also nonlinearities uniformly bounded from below. Also, limits (2) can be interchanged.

With respect to the singular lattice, it is interesting to note that condition (15) is used only in (16), so it is possible to weaken this hypothesis assuming that

$$\int_0^{\psi_1^i} g_i(s)ds > (3KT + \|\widehat{h}_i\|_{L^1})M_i + \frac{M_i^2}{2}$$

for every $i \in \mathbb{Z}$.

As a final remark, an open problem that naturally arises in this context is to find necessary and sufficient conditions over the mean values $\overline{h}_i$ for existence and multiplicity of T-periodic solutions of system (1). Another line of research would be to study systems with a more complex interaction between particles. Finally, it would be interesting to find conditions for the existence of periodic solutions with finite energy.

Acknowledgment

I would like to thank the referee for many useful comments and for bring the reference [13] to my attention.

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(Received: April 9, 1998; revised: October 9, 1998)