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ESTIMATES ON THE NUMBER OF LIMIT CYCLES OF A GENERALIZED ABEL EQUATION

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Abstract. We prove new results about the number of isolated periodic solutions of a first order differential equation with a polynomial nonlinearity. Such results are applied to bound the number of limit cycles of a family of planar polynomial vector fields which generalize the so-called rigid systems.

1. **Introduction.**

Let us consider a first order differential equation

$$x' = \sum_{k=0}^{n} a_k(t)x^k, \tag{1}$$

where a_k are continuous and T-periodic functions for some $T > 0$. A T-periodic solution of (1) which is isolated in the set of all the periodic solutions is called a limit cycle. We are interested in the problem of the existence and multiplicity of limit cycles of (1). Classically, the interest on this problem comes from the study of the number of limit cycles of a planar polynomial vector field.

The first non-trivial situation is the Abel equation $n = 3$. If $a_3(t) > 0$, Pliss [16] proved that (1) has at most three limit cycles, but in the general case Lins Neto [12] gave examples with an arbitrary number of limit cycles.

For $n \geq 4$, a constant sign in the leading coefficient a_n is not sufficient in general to bound uniformly the number of limit cycles, see [12, 6]. At this point, to get a more accurate information on the number of limit cycles, most of the papers in the literature require that only some the polynomial coefficients $a_k(t)$ do not vanish, in such a way that the polynomial nonlinearity has only three or four terms (see for instance [15, 2, 1, 10]).

Comparatively, the number of papers providing an explicit bound on the number of limit cycles in the general case where all the coefficients of the polynomial nonlinearity are present is small. The most known result is due to Ilyashenko [9], who proved that if $a_n(t) \equiv 1$, the number of real periodic solutions does not exceed the bound $8\exp\left\{(3P+2)\exp\left[\frac{3}{2}(2P+3)^n\right]\right\}$, where $P > 1$ is a uniform upper bound for $|a_k(t)|$. The importance of this result relies in the fact that it states the finiteness of the number of limit cycles more that in the explicit estimate, which

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is rather conservative. Later, Calanchi and Ruf [5] proved that there are at most n limit cycles if if n is odd, the leading term is fixed and the remaining terms are small enough. Finally, in a very recent work [3], Alwash have proved related results giving more precise information about the number of limit cycles.

Our purpose is to contribute to the literature with some results which complement the previously mentioned results. To this aim, let us write (1) in the more general form

$$x' = x^m \sum_{k=0}^{n} a_k(t)x^k, \tag{2}$$

where $m \in \mathbb{Z}$. Thus, we consider a polynomial nonlinearity with possible negative powers, a situation which has been scarcely explored in the related literature. We will justify the interest of this case in the last section with the study of a new family of planar vector fields. Along the whole paper, we assume the standing hypothesis

(H1) $a_n(t)a_0(t) \neq 0$ for all t.

Our main result is the following one.

Theorem 1. *Let us assume that for all t,*

$$\sum_{k=1}^{n} |a_k(t)| < |a_0(t)| \tag{3}$$

and

$$\sum_{k=1}^{n-1} |(m+k)a_k(t)| \leq |(m+n)a_n(t)|\,. \tag{4}$$

Then

i) *Equation (2) has at most one positive limit cycle and at most one negative limit cycle.*

ii) *If n is odd, equation (2) has exactly one limit cycle. If n is even and $a_n(t)a_0(t) < 0$ for all t, then equation (2) has exactly one positive limit cycle and one negative limit cycle. If n is even and $a_n(t)a_0(t) > 0$ for all t, then equation (2) has no limit cycles.*

In [3, Theorem 1.3 (i)], it is proved that if $a_n(t) \equiv 1$, $|a_0(t)| > \frac{2n-1}{n-1}$ and $|a_k(t)| < \frac{n}{(n-1)^2}$ for $1 \leq k \leq n-1$ then equation (1) has at most one positive limit cycle and at most one negative limit cycle. This result is generalized by the previous one as it is shown below.

Corollary 1. *For all t, let us assume that $a_n(t) \equiv 1$, $|a_k(t)| \leq P \leq \frac{2}{n-1}$ for $1 \leq k \leq n-1$ and $|a_0(t)| > (n-1)P + 1$. Then equation (1) has at most one positive limit cycle and at most one negative limit cycle.*

We will show that this is a direct corollary of Theorem 1. Then [3, Theorem 1.3 (i)] corresponds to the particular case $P = \frac{n}{(n-1)^2}$ (which obviously is less than $\frac{2}{n-1}$ for the nontrivial case $n \geq 2$.).

Using the same technique of proof that in Theorem 1, the following result holds.

Theorem 2. *Let us assume* (3). *Then,*

i) *If*

$$\sum_{k=1}^{n-1} |(m+k)(m+k-1)a_k(t)| \leq |(m+n)(m+n-1)a_n(t)|, \tag{5}$$

for all t, then equation (2) *has at most two positive limit cycles and at most two negative limit cycles.*

ii) *If*

$$\sum_{k=1}^{n-1} |(m+k)(m+k-1)(m+k-2)a_k(t)| \\ \leq |(n+m)(n+m-1)(n+m-2)a_n(t)|, \tag{6}$$

for all t, then (2) *has at most three positive limit cycles and at most three negative limit cycles.*

Note that condition (6) is weaker than (5) and (5) is weaker than (4). By using Theorem 2 i), the following corollary is direct.

Corollary 2. *For all t, let us assume that $a_n(t) \equiv 1$, $|a_k(t)| \leq P \leq \frac{3}{n-2}$ for $1 \leq k \leq n-1$ and $|a_0(t)| > (n-1)P + 1$. Then equation* (1) *has at most two positive limit cycles and at most two negative limit cycles.*

This result is complementary to the main result of [8], where the required assumptions imply a fixed sign for $n-1$ coefficients.

The paper is organized as follows: Section 2 contains some useful results for the main proofs, which are implemented on Section 3. Finally, Section 4 contains an application to the estimation of the number of limit cycles of a family of polynomial planar systems which generalize the well-known rigid systems by using action-angle variables instead of polar coordinates.

2. **Preliminary results.** This section collects some elementary results employed on the proofs of the main results. Let us consider a general first order equation

$$x' = g(t,x), \tag{7}$$

with g continuous and T-periodic in t and with continuous derivatives in x up to order 3. Let $x(t,c)$ be the unique solution of (7) with initial condition $x(0,c) = c$. Then $x(t,c)$ is a limit cycle of (7) if and only if it is an isolated zero of the displacement function $q(c) := x(T,c) - c$. A given limit cycle $x(t,c)$ is said *hyperbolic* if the characteristic exponent $\delta = \int_0^T g_x(t,x(t,c))dt$ is different from zero. Derivation with respect to the initial condition gives $q'(c) = \exp(\delta) - 1$. As a consequence, a limit cycle is asymptotically stable if $\delta < 0$ and unstable if $\delta > 0$. Similarly, $q''(c)$ and $q'''(c)$ can be explicitly computed in terms of the derivatives of g up to the third order (see for instance [14]). By using this information, the following result was proved in [17]. Here the notation $f \succ 0$ stands for a T-periodic function which is nonnegative and positive in a set of positive measure. Similarly, $f \prec 0$ if and only if $-f \succ 0$.

Proposition 1. *Let J be an open interval,*

i) *If $g_x(t,x) \succ 0$ for all $x \in J$ (resp. $g_x(t,x) \prec 0$ for all $x \in J$), then (7) has at most one limit cycle with range contained in J.*
ii) *If $g_{xx}(t,x) \succ 0$ for all $x \in J$ (resp. $g_{xx}(t,x) \prec 0$ for all $x \in J$), then the (7) has at most two limit cycles with range contained in J.*
iii) *If $g_{xxx}(t,x) \succ 0$ for all $x \in J$ (resp. $g_{xxx}(t,x) \prec 0$ for all $x \in J$), then (7) has at most three limit cycles with range contained in J.*

The next notion is classical (see for instance [13]).

Definition 1. *A T-periodic function α is called a strict lower (resp. upper) solution of equation (7) if*

$$\alpha'(t) < g(t,\alpha(t)) \quad (\textit{resp. } \alpha'(t) > g(t,\alpha(t))), \forall t.$$

Some basic results about upper and lower solutions are collected below.

Lemma 1. *Let $\alpha(t)$ be a strict lower (resp. upper) solution of (7). If $q(\alpha(0))$ exists, then $q(\alpha(0)) > 0$ (resp. $q(\alpha(0)) < 0$).*

Proof. Let us assume that $\alpha(t)$ is a strict lower solution and define $d(t) = x(t,\alpha(0)) - \alpha(t)$. By hypothesis, $d(t)$ is well-defined in the interval $[0,T]$, $d(0) = 0$ and $d'(0) = g(0,\alpha(0)) - \alpha'(0) > 0$. Let us prove that $d(t) > 0$ for all $0 < t \le T$. By contradiction, let us suppose that there exists $t_1 > 0$ such that $d(t_1) = 0$ and $d(t) > 0$ for all $0 < t < t_1$. Then $x(t_1,\alpha(0)) = \alpha(t_1)$ and $d'(t_1) = g(t_1,\alpha(t_1)) - \alpha'(t_1) > 0$, which is a contradiction with the fact that t_1 is the first positive zero. Therefore, $d(T) = x(T,\alpha(0)) - \alpha(T) > 0$, but because $\alpha(t)$ is T-periodic, this means that $q(\alpha(0)) = x(T,\alpha(0)) - \alpha(0) > 0$. The proof for the upper solution is analogous. □

Lemma 2.

i) *Let $\alpha(t)$ be a strict lower solution. Then the set $\{(t,x) : x \ge \alpha(t)\}$ is positively invariant.*
ii) *Let $\beta(t)$ be a strict upper solution. Then the set $\{(t,x) : x \le \beta(t)\}$ is positively invariant.*

Proof. Assume that $\alpha(t)$ is a strict lower solution of equation (7). Take (t_0,x_0) such that $x_0 \ge \alpha(t_0)$ and let $x(t)$ the unique solution such that $x(t_0) = x_0$, with maximal interval of definition $]\omega_-,\omega_+[$. We are going to prove that $x(t) > \alpha(t)$ for every $t_0 < t < \omega_+$. Since the solutions of a first-order differential equation are ordered, we can assume without loss of generality that $x_0 = \alpha(t_0)$. Define $d(t) = x(t) - \alpha(t)$, then $d'(t_0) = g(t_0,\alpha(t_0)) - \alpha'(t_0) > 0$. By contradiction, assume that there exists t_1 such that $x(t_1) = \alpha(t_1)$ and $x(t) \ge \alpha(t)$ for every $t_0 < t < t_1$. Define $d(t) = x(t) - \alpha(t)$, then $d(t) > 0$ for $t_0 < t < t_1$, $d(t_1) = 0$ and

$$d'(t_1) = x'(t_1) - \alpha'(t_1) > g(t_1,x(t_1)) - g(t_1,\alpha(t_1)) = 0,$$

a clear contradiction. The proof for the upper solution in analogous. □

Lemma 3. *A T-periodic solution does not intersect any eventual upper or lower solution.*

Proof. It is a direct corollary of the previous lemma. □

Proposition 2. *Let us assume that $\alpha(t),\beta(t)$ are strict lower and upper solutions such that $\alpha(t) < \beta(t)$ (resp $\alpha(t) > \beta(t)$). Then, there exists a T-periodic solution $x(t)$ such that $\alpha(t) < x(t) < \beta(t)$ (resp. $\beta(t) < x(t) < \alpha(t)$).*

Proof. Suppose first that $\alpha(t) < \beta(t)$. By Lemma 2, the set $\{(t,x) : \alpha(t) \leq x \leq \beta(t)\}$ is positively invariant. Then the displacement function is well-defined and continuous in the interval $[\alpha(0), \beta(0)]$, and the result is a direct consequence of Bolzano's theorem and Lemmas 1. The result when $\beta(t) < x(t) < \alpha(t)$ follows directly after a change of time $\tau = -t$. □

The next lemma provides a classical bound for the roots of a polynomial. It can be found in a lot of classical textbooks, but a short proof is given for completeness.

Lemma 4. *Any solution of the polynomial equation*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{8}$$

with real or complex coefficients and $a_n \neq 0$, verifies the bound

$$|x| \leq \max\left\{1, \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|\right\}.$$

Proof. Assume that $|x| \geq 1$. For $k \leq n-1$ we have $1 \leq |x|^k \leq |x|^{n-1}$.
From equation (8),

$$x^n = \frac{-1}{a_n}(a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0).$$

Then,

$$|x|^n \leq \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right||x|^k \leq (\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|)|x|^{n-1}.$$

Hence $|x| \leq \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|$. □

Let us consider the polynomial function

$$f(t,x) = \sum_{k=0}^{n} a_k(t)x^k, \tag{9}$$

where the coefficients a_i are continuous and T-periodic functions.

Two simple consequences of the previous result are stated below.

Lemma 5. *Let us assume that $a_0(t) \neq 0$ for all t. If $x \neq 0$ is a solution of $f(t,x) = 0$ for some t, then*

$$|x| \geq \frac{1}{\max\left\{1, \max_t \sum_{k=1}^{n}\left|\frac{a_k}{a_0}\right|\right\}}.$$

Proof. By using the change of variable $y = \frac{1}{x}$, equation $f(t,x) = 0$ becomes

$$a_n + a_{n-1}y + \cdots + a_1 y^{n-1} + a_0 y^n = 0,$$

Using Lemma 4,

$$|y| = \frac{1}{|x|} \leq \max\left\{1, \max_t \sum_{k=1}^{n}\left|\frac{a_k}{a_0}\right|\right\}.$$

□

Analogously, one can prove the next lemma.

Lemma 6. *Let us assume that $a_n(t) \neq 0$ for all t. If $x \neq 0$ is a solution of $f_x(t,x) = 0$ for some t, then*

$$|x| \leq \max\left\{1, \max_t \sum_{k=1}^{n-1} \left|\frac{ka_k}{na_n}\right|\right\}.$$

3. **Proofs of main results.** During this section, we will call

$$g(t,x) = x^m \sum_{k=0}^{n} a_k(t) x^k.$$

Let us define the constants

$$\lambda = \min\{|x| \neq 0 \,:\, g(t,x) = 0 \text{ for some } t\}$$

$$\mu = \max\{|x| \neq 0 \,:\, g_x(t,x) = 0 \text{ for some } t\}.$$

Note that such quantities are well-defined by hypothesis $(H1)$.

Proposition 3. *If $\mu < \lambda$, then (2) has at most one positive limit cycle and at most one negative limit cycle.*

Proof. First, we prove that a given limit cycle does not change its sign. If $m > 0$, it is trivial since $x \equiv 0$ is a solution, so other solutions never cross it. In the case $m = 0$, note that $(H1)$ implies that $\alpha \equiv 0$ is an strict upper or lower solution (depending on the sign of a_0). Finally, if $m < 0$ the equation is not defined in 0 and changes of sign are not allowed. Therefore, eventual limit cycles have constant sign. Let us prove that there exists at most one positive limit cycle, being analogous the proof for the existence of at most one negative limit cycle. This is done in three steps:

- **Step 1**: *Any eventual positive limit cycle $x(t)$ verifies*
  $$x(t) \geq \lambda, \qquad \forall t.$$
  It $x(t_m) > 0$ is the global minimum of $x(t)$, then $x'(t_m) = 0$, hence $g(t_m, x(t_m)) = 0$ and the inequality holds by the own definition of λ.
- **Step 2**: $g_x(t,x) \neq 0$ *for all* $x > \mu$. This is a direct consequence of the definition of μ.
- **Step 3**: *Conclusion:* Let us call $J = (\mu, +\infty)$. By continuity, $g_x(t,x) > 0$ (if $a_n > 0$) or $g_x(t,x) < 0$ (if $a_n < 0$) for all $x \in J$. Then Proposition 1 says that there exists at most one limit cycle in J, but by Step 1 and the condition $\mu < \lambda$, all the positive limit cycles belong to J, therefore the proof is done.

□

Proof of Theorem 1.
(i) Condition (3) means that $\max\left\{1, \sum_{k=1}^n \left|\frac{a_k}{a_0}\right|\right\} = 1$ for every t, hence we conclude that $\lambda \geq 1$ by Lemma 5. Moreover, By Lemma 6 and condition (4), we get $\mu \leq 1$. Therefore $\mu \leq \lambda$. In order to apply Proposition 3, it remains to prove that $\lambda > 1$. By contradiction, if $\lambda = 1$ then $g(t,1) = 0$ for some t, that is,

$$\sum_{k=0}^{n} a_k(t) = 0,$$

but then

$$|a_0(t)| = \left|-\sum_{k=1}^{n} a_k(t)\right| \leq \sum_{k=1}^{n} |a_k(t)|,$$

contradicting (3).
(ii) To fix ideas, along the whole proof we can consider that $a_0(t) > 0$ for all t. In fact the remaining case can be reduced to this one by inverting the time variable $\tau = -t$.

Assume that n is odd and $a_n(t) < 0$ for all t. Using the sign of a_0, it is easy to verify that $\alpha(t) \equiv \epsilon > 0$ is a strict lower solution for a small enough ϵ. On the other hand, $\beta(t) \equiv M > 0$ is a strict upper solution for a big enough M. This gives a positive limit cycle. Having in mind the sign of a_n, such limit cycle has negative characteristic exponent and therefore it is asymptotically stable. Let us prove that it is unique. By the first part of the theorem, if there is a second limit cycle $\varphi(t)$, it should be negative and the function $g_x(t, \varphi(t))$ does not vanish. By using that n is odd and $a_n < 0$, the sign of this function is negative by continuity, so the characteristic exponent of $\varphi(t)$ is again negative. Therefore, we have two zeros of the displacement function $q(c)$ with negative derivative, this means that it should be a third solution in between, but this is a contradiction with part (i). The case $a_n(t) > 0$ is solved in the same way, resulting in this case a unique limit cycle which will be negative.

Assume now that n is even and $a_n(t)a_0(t) < 0$. Then, taking $\epsilon, M > 0$ small and big enough respectively, $-M$ is a strict upper solution and $-\epsilon$ is a strict lower solution, so there exists a negative limit cycle. Similarly, ϵ is a strict lower solution and M is a strict upper solution, so there exists a positive limit cycle, and there are no more by part (i).

Finally, let us assume that n is even and $a_n(t)a_0(t) > 0$. Assume by contradiction that there exists a limit cycle $\varphi(t)$, say a positive one (the remaining case is analogous). Then, we can fix $\epsilon, M > 0$ small and big enough respectively, in such a way that ϵ, M are a strict lower solutions and $\epsilon < \varphi(t) < M$ for all t. By using Lemma 2, the set $\{(t, x) : \epsilon \le x \le \varphi(t)\}$ is positively invariant, therefore the displacement function q is well-defined in *epsilon* and $q(\epsilon)$ is positive by Lemma 1. On the other hand, by using the arguments of the proof of Lemma 2, the set $\{(t, x) : \varphi(t) \le x \le M\}$ is negatively invariant. Thus, if $\tilde{x}(t)$ is the unique solution such that $\tilde{x}(T) = M$, then $\varphi(0) < \tilde{x}(0) < M$ and $q(\tilde{x}(0)) = M - \tilde{x}(0) > 0$.

By part (i), $\varphi(t)$ should be the unique limit cicle, so from $q(\epsilon), q(\tilde{x}(0)) > 0$ we have that $\varphi(0)$ as a zero of the displacement function must be degenerate, that is, the characteristic exponent $\delta = \int_0^T g_x(t, \varphi(t))dt = 0$. But using the arguments contained in part (i), under our conditions $\varphi(t) > 1$ and $g_x(t, \varphi(t)) \neq 0$ for all t, which is a clear contradiction with $\delta = 0$. □

Let us remark that Theorem 1 remains true if $<$ and $\le$ are interchanged in conditions (3) – (4). The proof of Theorem 2 is analogous to the first part of Theorem 1 by using λ as defined above and modifying the definition of μ with the use of the second or third derivative of g.

Proof of Corollaries 1 and 2. Take $m = 0$. By the hypothesis on $a_0(t)$, condition (3) is trivially verified. On the other hand, $\sum_{k=1}^{n-1} |ka_k(t)| \le P \sum_{k=1}^{n-1} k = P\frac{n(n-1)}{2} \le P$, hence (4) holds. The proof of Corollary 2 is analogous, by using in this case $\sum_{k=1}^{n-1} k(k-1) = \frac{1}{3}n(n-1)(n-2$. □

4. **Applications to polynomial planar systems.** The main motivation for the study of the number of limit cycles in first-order equations with a polynomial nonlinearity is the search for information about the maximum number of limit cycles of a given autonomous polynomial planar system, the so-called Hilbert number.

The most simple family of polynomial planar systems which can be reduced to a generalized Abel equation are the so-called *rigid systems*

$$\begin{cases} x' = \lambda x - y + xP(x,y) \\ y' = x + \lambda y + yP(x,y) \end{cases}$$

If $P(x,y) = R_1(x,y)+R_2(x,y)+\ldots+R_{n-1}(x,y))$, where each R_k is a homogeneous polynomial of degree k , the system in polar coordinates becomes

$$\begin{cases} r' = r^n R_{n-1}(cos\theta, sin\theta) + r^{n-1}R_{n-2}(cos\theta, sin\theta) + \ldots + r^2 R_1(cos\theta, sin\theta) + \lambda r \\ \theta' = 1. \end{cases}$$

Limit cycles of the system correspond to positive limit cycles of

$$\frac{dr}{d\theta} = r^n R_{n-1}(cos\theta, sin\theta) + r^{n-1}R_{n-2}(cos\theta, sin\theta) + \ldots + r^2 R_1(cos\theta, sin\theta) + \lambda r.$$

Many examples contained in the literature belong to this class (see [2, 3, 4, 6, 7, 8] and their references only to cite some of them). Of course, our results can be applied directly to this kind of systems, but we are going to focus our attention in a more general family which seem not to have been studied in previous works.

Let us consider the system

$$x' = -y^{2p-1} + \frac{x}{q}P(x,y) \quad , \quad y' = x^{2q-1} + \frac{y}{p}P(x,y) \tag{10}$$

where $P(x,y)$ is a polynomial and p, q are natural numbers. When $p = q = 1$, it is a rigid system. Let us introduce action-angle variables r, θ such that

$$x = r^p C(\theta), \qquad y = r^q S(\theta), \tag{11}$$

where $C(\theta)$ and $S(\theta)$ are functions of θ defined implicitly by

$$\frac{C^{2q}(\theta)}{2q} + \frac{S^{2p}(\theta)}{2p} = 1. \tag{12}$$

Note that $(C(\theta), S(\theta))$ is a solution of the autonomous hamiltonian system

$$x' = -y^{2p-1}, \quad y' = x^{2q-1},$$

and it is uniquely determined by fixing the initial condition $(C(0), S(0)) = ((2q)^{\frac{1}{2q}}, 0)$. The notion of action-angle variables is a classical tool in Celestial Mechanics and Stability Theory (a similar change was proposed by Liapunov, see [11, Pag. 43]).

In the new coordinates, the system (10) is rewritten as

$$r' = \frac{1}{pq} r P(r^p C(\theta), r^q S(\theta)), \qquad \theta' = r^{2pq-p-q}. \tag{13}$$

By taking r as a function of θ we get the single differential equation

$$\frac{dr}{d\theta} = Ar^m P(r^p C(\theta), r^q S(\theta)), \tag{14}$$

where $A = \frac{1}{pq}$ and $m = -2pq + p + q + 1$. The coefficients will be polynomials in the generalized trigonometrical functions $C(\theta), S(\theta)$. Now we can use (12) in order to get $|C(\theta)| \leq (2q)^{\frac{1}{2q}}, |S(\theta)| \leq (2p)^{\frac{1}{2p}}$, and such bounds can be used for practical application our conditions. Instead of concrete examples, we will prove a somewhat general result.

Corollary 3. *Let $p, q \in \mathbb{N}$ and $Q(x,y)$ be a given polynomial of degree h. Consider the system (10) where $P(x,y) = \left(\frac{x^{2q}}{2q} + \frac{y^{2p}}{2p}\right)^n + Q(x,y) + \lambda$. Then, there exist $n_0, \lambda_0 > 0$ (only depending on Q, p, q), such that (10) has*

i) *exactly one limit cycle if* $n > n_0$ *and* $\lambda < -\lambda_0$.
ii) *no limit cycles if* $n > n_0$ *and* $\lambda > \lambda_0$.

Proof. After the change (11), the resulting first order equation is

$$\frac{dr}{d\theta} = \frac{1}{pq} r^m P(r^p C(\theta), r^q S(\theta))) = \frac{1}{pq} r^m \left[r^{2pqn} + Q(r^p C(\theta), r^q S(\theta)) + \lambda \right].$$

For this equation, the leading coefficient is $a_{2pqn} \equiv \frac{1}{pq}$, whereas $a_0 \equiv \frac{\lambda}{pq}$. Finally, we can write

$$Q(r^p C(\theta), r^q S(\theta)) = \sum_{k=1}^{h \max\{p,q\}} a_k(\theta) r^k.$$

Then, condition (3) can be expressed as

$$|\lambda| > \lambda_0 := 1 + \max_\theta \sum_{k=1}^{h \max\{p,q\}} |a_k(\theta)| \, .$$

On the other hand, condition (4) is

$$|m + n| > \max_\theta \sum_{k=1}^{h \max\{p,q\}} |(m + k) a_k(\theta)| \, .$$

so it is sufficient to take

$$n > n_0 := |m| + \max_\theta \sum_{k=1}^{h \max\{p,q\}} |(m + k) a_k(\theta)| \, .$$

Now the result is a direct consequence of Theorem 1.

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