Z. Angew. Math. Phys.

DOI 10.1007/s00033-010-0084-1

Zeitschrift für angewandte Mathematik und Physik ZAMP

A combined variational-topological approach for dispersion-managed solitons in optical fibers

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Abstract. We derive some sufficient conditions for the existence of dispersion-managed solitons in a nonlinear Schrödinger equation with periodically varying coefficients. The proof relies on a combination of the variational method with the development of a novel upper and lower function method.

Mathematics Subject Classification (2000). 35Q55 · 34C25 · 78A60.

Keywords. Optical soliton · Schrödinger equation · Singular equation · Periodic solution · Upper and lower function.

1. Introduction

An optical soliton can be described as an electromagnetic wave which is localized in time and can propagate along an optical medium without significant distortion of its shape [3,6,14]. In a nonlinear optical medium, this physical effect is achieved by means of a suitable balance between the chromatic dispersion and the nonlinear refractive response. From a practical point of view, the concept of soliton is crucial to implement efficient optical fiber communication systems.

From a mathematical point of view, the propagation of an optical pulse in a fiber cable with varying dispersion is governed by the equation

$$i\Psi_z - \frac{1}{2}\beta_2(z)\Psi_{tt} + \sigma(z)\left|\Psi\right|^2\Psi = iG(z)\Psi,$$

where Ψ is the complex-valued envelope function of the electric field, z is the longitudinal coordinate of the fiber line and t is time. The functions β_2, σ, G model respectively the dispersion, nonlinear refractive response and effective gain or loss along the fiber line. In this paper, it is assumed that the optical fiber has a periodic structure so that the coefficients are periodic with a common minimal period which is normalized to $L = 1$. As can be seen in the cited references, this seems to be the most interesting case for practical purposes.

It is customary to remove the right-hand side term of the latter equation by the transformation

$$\Psi(z,t) = A(z,t)\exp\left(\int^{z} G(s)\mathrm{d}s\right).$$

Then, the equation under study is the cubic Schrödinger equation with periodic coefficients

$$iA_z + \mathrm{d}(z)A_{tt} + c(z)\left|A\right|^2 A = 0. \tag{1.1}$$

Now the gain–loss power term is included in the coefficient c. To find soliton-like solutions of Eq. (1.1) is a central problem not only in Nonlinear Optics but also for a variety of physical and biological applications.

Published online: 20 July 2010

A well-known method for the analytical study of Eq. (1.1) is the variational approach described in full detail in [13] (see also the complete list of references therein). Equation (1.1) is rewritten in the Lagrangian form, with the action functional

$$S = \int L\mathrm{d}t\mathrm{d}z = \int \mathrm{d}t\mathrm{d}z \left[\frac{i}{2}(AA_z^* - A^* A_z)\mathrm{d}(z)\left|A_t\right|^2 - \frac{c(z)}{2}\left|A\right|^4\right].$$

The following trial function is chosen

$$A(z,t) = \frac{Q(t/T(z))}{\sqrt{T(z)}} \exp\left(i\frac{M(z)}{T(z)}t^2\right) \tag{1.2}$$

where the shape of the input pulse Q is in principle arbitrary, being the most typical choice a gaussian $Q(x) = C_0 \exp(-x^2/2)$. Inserting this ansatz into the action functional, one obtains the system of ordinary differential equations

$$\begin{aligned} T' &= 4\mathrm{d}(z)M \\ M' &= \frac{\mathrm{d}(z)C_1}{T^3} - \frac{c(z)C_2}{T^2}, \end{aligned} \tag{1.3}$$

with fixed constants

$$C_1 = \frac{\int \left|Q'(x)\right|^2 \mathrm{d}x}{\int x^2 \left|Q(x)\right|^2 \mathrm{d}x}, \quad C_2 = \frac{\int \left|Q(x)\right|^4 \mathrm{d}x}{4\int x^2 \left|Q(x)\right|^2 \mathrm{d}x}. \tag{1.4}$$

The functions $T(z)$ and $M(z)$ describe the optical pulse width and the chirp (time-dependent phase) of the breathing central part of the optical soliton. The dynamics of system (1.3), often known as TM-equations in the related literature, is of key importance on this field. Then the problem is reduced to find conditions for the existence of 1-periodic solutions of system (1.3), that is, T, M verifying $T(0) = T(1), M(0) = M(1)$. Although the variational approach is approximate, it is recognized as an effective theoretical method to gain insight on the dynamics of the system.

At this point, all the theoretical results presented in the literature assume that both coefficients c, d are piecewise constants. This assumption makes possible to apply a matching technique for the respective phase planes in order to find explicit existence conditions. To have piecewise continuous coefficients is a coherent assumption in the framework of Nonlinear Optics, but the main problem is that for a large number of pieces computations become too hard to handle with. The paper [7] solves explicitly the case of c constant (the so-called lossless case) and d composed by two pieces, but for more than two pieces only numerical results are known [11]. Our approach is of a different nature. We propose the use of a classical approach like the upper and lower functions method [1]. This technique is very known in the qualitative analysis of second order ODEs and has been applied to equations with singularities in the recent paper [2]. We take advantage of the techniques developed there to open a new path in the study of DM-solitons in optical fibers. The main technical difficulty is that in the general case with arbitrary coefficients, the system (1.3) can not be written as a second-order ODE (as in fact it is done in the particular case considered in [7]). This has forced us to develop a specific upper and lower function method for this framework. It will be shown that such method is the natural extension to the first-order system of known results for the second order scalar ODE (see, e.g., [10,12] and references therein), so in this sense from a mathematical point of view it is interesting by itself.

The structure of the paper will be as follows. In Sect. 2, we develop a new method of upper and lower functions in consonance with the problem under consideration. In Sect. 3, such a method is properly applied to a family of systems which include the TM-equations.

2. The method of upper and lower functions

2.1. Basic notation and definitions

The following notation is used throughout:

$\mathbb{R}$ is a set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$;
$C([0,\omega];\mathbb{R})$ is a Banach space of all continuous functions $u : [0,\omega] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [0,\omega]\}$;
$C([0,\omega];\mathbb{R}^2)$ is a Banach space of all continuous vector-valued functions $(u,v) : [0,\omega] \to \mathbb{R}^2$ with the norm $\|(u,v)\|_C = \|u\|_C + \|v\|_C$;
$AC([0,\omega];\mathbb{R}^2)$ is a set of all vector-valued functions $(u,v) : [0,\omega] \to \mathbb{R}^2$ with absolutely continuous components;
$L([0,\omega];\mathbb{R})$ is a Banach space of all Lebesgue integrable functions $p : [0,\omega] \to \mathbb{R}$ with the norm $\|p\|_L = \int_0^\omega |p(s)|\,\mathrm{d}s$;
$L([0,\omega];\mathbb{R}_+) = \{p \in L([0,\omega];\mathbb{R}) : p(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega]\}$;
$L([0,\omega];\mathbb{R}^2)$ is a Banach space of all vector-valued Lebesgue integrable functions $(p,q) : [0,\omega] \to \mathbb{R}^2$ with the norm $\|(p,q)\|_L = \|p\|_L + \|q\|_L$;
$K([0,\omega] \times D;\mathbb{R})$, where $D \subseteq \mathbb{R}$, is the Carathéodory class, i.e., the set of functions $f : [0,\omega] \times D \to \mathbb{R}$ such that $f(\cdot,x) : [0,\omega] \to \mathbb{R}$ is measurable for all $x \in D$, $f(t,\cdot) : D \to \mathbb{R}$ is continuous for almost all $t \in [0,\omega]$, and

$$\sup\{|f(\cdot,x)| : x \in D_0\} \in L([0,\omega];\mathbb{R}_+)$$

for any compact set $D_0 \subseteq D$.
if $x \in \mathbb{R}$ then $[x]_+ = \max\{0,x\}$, $[x]_- = \max\{0,-x\}$;
for every $a \in [0,\omega[$ and $b \in]0,\omega]$ such that $a \neq b$ define

$$I(a,b) = \begin{cases}]\,a,b\,[& \text{if } a < b \\ [0,b\,[\cup]\,a,\omega] & \text{if } b < a \end{cases}, \quad I\,[a,b) = I(a,b) \cup \{a\}.$$

In the development of the general method of upper and lower functions we will adopt the more classical use of t as the independent variable. Consider a system of two differential equations

$$u' = p(t)v, \tag{2.1}$$
$$v' = f(t,u) \tag{2.2}$$

with periodic boundary conditions

$$u(0) = u(\omega), \quad v(0) = v(\omega), \tag{2.3}$$

where $p \in L([0,\omega];\mathbb{R})$ and $f \in K([0,\omega] \times D;\mathbb{R})$. By a solution to (2.1, 2.2) is understood a vector-valued function $(u,v) \in AC([0,\omega];\mathbb{R}^2)$ with $u(t) \in D$ for $t \in [0,\omega]$ satisfying (2.1, 2.2) almost everywhere on $[0,\omega]$. By a solution to the problem (2.1)–(2.3) is understood a solution to (2.1, 2.2) satisfying (2.3).

The question of the existence of a periodic solution to the two-dimensional system of the type (2.1, 2.2) was studied by Kiguradze and Mukhigulashvili in [5]. However, the results obtained by them are not applicable to our system because the function p is assumed to be sign-constant in their paper. The results dealing with general nonlinear two-dimensional system one can find, e.g., in [4]. In [8,9] one can find conditions guaranteeing the existence of a periodic solution to the n-dimensional linear system of both ordinary and functional differential equations.

Definition 2.1. A vector-valued function $(\gamma_1, \gamma_2) \in AC\left([0,\omega];\mathbb{R}^2\right)$ is said to be an upper function (resp. a lower function) to the problem (2.1)–(2.3) if $\gamma_1(t) \in D$ for $t \in [0,\omega]$,

$$\gamma_1'(t) = p(t)\gamma_2(t) \quad \text{for a. e. } t \in [0,\omega],$$
$$\gamma_2'(t) \le f(t,\gamma_1(t)) \quad (\text{resp. } \gamma_2'(t) \ge f(t,\gamma_1(t))) \quad \text{for a. e. } t \in [0,\omega],$$

and the boundary conditions

$$\gamma_1(0) = \gamma_1(\omega), \quad \gamma_2(0) \le \gamma_2(\omega) \quad (\text{resp. } \gamma_2(0) \ge \gamma_2(\omega))$$

hold.

Definition 2.2. A vector-valued function $(p_1, p_2) \in L\left([0,\omega];\mathbb{R}^2\right)$ is said to verify a property (P_+) if every vector-valued function $(u,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfying

$$u'(t) = p_1(t)v(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.4}$$
$$v'(t) \ge p_2(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.5}$$
$$u(0) = u(\omega), \quad v(0) \ge v(\omega) \tag{2.6}$$

admits the inequality

$$u(t) \ge 0 \quad \text{for } t \in [0,\omega]. \tag{2.7}$$

Definition 2.3. A vector-valued function $(p_1, p_2) \in L\left([0,\omega];\mathbb{R}^2\right)$ is said to verify a property (P_-) if every vector-valued function $(u,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfying (2.4)–(2.6) admits the inequality

$$u(t) \le 0 \quad \text{for } t \in [0,\omega]. \tag{2.8}$$

Remark 2.1. Note that (p_1, p_2) verifies a property (P_+) iff $(-p_1, -p_2)$ verifies a property (P_-). Indeed, let $(u,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfy (2.4)–(2.6). Put $w(t) = -u(t)$ for $t \in [0,\omega]$. Then $(w,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfies

$$w'(t) = -p_1(t)v(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.9}$$
$$v'(t) \ge -p_2(t)w(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.10}$$
$$w(0) = w(\omega), \quad v(0) \ge v(\omega), \tag{2.11}$$

and vice versa, if $(w,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfies (2.9)–(2.11) then, having defined $u(t) = -w(t)$ for $t \in [0,\omega]$ we obtain that $(u,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfies (2.4)–(2.6).

2.2. On the properties (P_+) and (P_-)

Theorem 2.1. *Let $p \in L([0,\omega];\mathbb{R})$ and let $\varphi \in L([0,\omega];\mathbb{R}_+)$ satisfy*

$$\int_0^\omega \varphi(s)\mathrm{d}s \ne 0. \tag{2.12}$$

If either the inequalities

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s < 1, \tag{2.13}$$

$$\frac{\int_0^\omega [p(s)]_+\mathrm{d}s}{1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s} \le \int_0^\omega [p(s)]_-\mathrm{d}s \tag{2.14}$$

hold or the inequalities

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s < 1, \tag{2.15}$$

$$\frac{\int_0^\omega [p(s)]_-\mathrm{d}s}{1-\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s} \le \int_0^\omega [p(s)]_+\mathrm{d}s \tag{2.16}$$

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s \le 2+2\sqrt{1-\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s}, \tag{2.17}$$

are fulfilled then a vector-valued function $(p,-\varphi)$ verifies a property (P_+).

Theorem 2.2. *Let $p \in L([0,\omega];\mathbb{R})$ and let $\varphi \in L([0,\omega];\mathbb{R}_+)$ satisfy (2.12). If either the inequalities*

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s < 1,$$

$$\frac{\int_0^\omega [p(s)]_-\mathrm{d}s}{1-\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s} \le \int_0^\omega [p(s)]_+\mathrm{d}s$$

hold or the inequalities

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s < 1,$$

$$\frac{\int_0^\omega [p(s)]_+\mathrm{d}s}{1-\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s} \le \int_0^\omega [p(s)]_-\mathrm{d}s$$

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s \le 2+2\sqrt{1-\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s},$$

are fulfilled then a vector-valued function (p,φ) verifies a property (P_-).

2.3. Existence theorems

Theorem 2.3. *Let $(\beta_1,\beta_2) \in AC\big([0,\omega];\mathbb{R}^2\big)$ and $(\alpha_1,\alpha_2) \in AC\big([0,\omega];\mathbb{R}^2\big)$ be an upper and a lower function to (2.1)–(2.3), respectively, with*

$$\beta_1(t) \le \alpha_1(t) \quad \text{for } t \in [0,\omega]. \tag{2.18}$$

Let, moreover, there exist $\varphi \in L([0,\omega];\mathbb{R}_+)$ such that

$$f(t,\beta_1(t)) + \varphi(t)\beta_1(t) \le f(t,x) + \varphi(t)x \le f(t,\alpha_1(t)) + \varphi(t)\alpha_1(t)$$
$$\text{for a. e. } t \in [0,\omega], \quad \beta_1(t) \le x \le \alpha_1(t) \tag{2.19}$$

and a vector-valued function $(p,-\varphi)$ verifies a property (P_+). If, in addition,

$$\int_0^\omega p(s)\mathrm{d}s \ne 0, \quad \int_0^\omega \varphi(s)\mathrm{d}s \ne 0 \tag{2.20}$$

then the problem (2.1)–(2.3) *has at least one solution* (u, v) *such that*

$$\beta_1(t) \le u(t) \le \alpha_1(t) \quad \text{for } t \in [0, \omega].$$

Theorem 2.4. *Let* $(\beta_1, \beta_2) \in AC\left([0, \omega]; \mathbb{R}^2\right)$ *and* $(\alpha_1, \alpha_2) \in AC\left([0, \omega]; \mathbb{R}^2\right)$ *be an upper and a lower function to* (2.1)–(2.3)*, respectively, with*

$$\alpha_1(t) \le \beta_1(t) \quad \text{for } t \in [0, \omega]. \tag{2.21}$$

Let, moreover, there exist $\varphi \in L([0, \omega]; \mathbb{R}_+)$ *such that*

$$f(t, \beta_1(t)) - \varphi(t)\beta_1(t) \le f(t, x) - \varphi(t)x \le f(t, \alpha_1(t)) - \varphi(t)\alpha_1(t)$$
$$\text{for a. e. } t \in [0, \omega], \quad \alpha_1(t) \le x \le \beta_1(t) \tag{2.22}$$

and a vector-valued function (p, φ) *verifies a property* (P_-)*. If, in addition,* (2.20) *holds then the problem* (2.1)–(2.3) *has at least one solution* (u, v) *such that*

$$\alpha_1(t) \le u(t) \le \beta_1(t) \quad \text{for } t \in [0, \omega].$$

Remark 2.2. Note that there exists φ satisfying (2.19), resp. (2.22), e.g. if there exists a partial derivative $\frac{\partial f}{\partial x}$ which belongs to $K([0, \omega] \times D; \mathbb{R})$. Then we can put

$$\varphi(t) = \sup\left\{\left|\frac{\partial f}{\partial x}(t, x)\right| : \gamma_1(t) \le x \le \gamma_2(t)\right\} \quad \text{for a. e. } t \in [0, \omega],$$

where

$$\gamma_1 = \min\{\alpha_1, \beta_1\}, \quad \gamma_2 = \max\{\alpha_1, \beta_1\}.$$

2.4. Auxiliary propositions

Lemma 2.1. *Let* $(u, v) \in AC\left([0, \omega]; \mathbb{R}^2\right)$ *satisfy*

$$u'(t) = p(t)v(t) \quad \text{for a. e. } t \in [0, \omega], \tag{2.23}$$
$$v'(t) \ge -\varphi(t)u(t) \quad \text{for a. e. } t \in [0, \omega], \tag{2.24}$$
$$u(0) = u(\omega), \quad v(0) \ge v(\omega) \tag{2.25}$$

with $p \in L([0, \omega]; \mathbb{R}), \varphi \in L([0, \omega]; \mathbb{R}_+)$*. Let, moreover,* u *assume both positive and negative values. If* (2.13) *holds then* v *does not vanish.*

Proof. First we will show that there exists $t_0 \in [0, \omega]$ such that

$$u(t_0) = 0, \quad v(t_0) \ne 0. \tag{2.26}$$

Assume on the contrary that if u has a zero at some point, then v has a zero at the same point. Obviously, according to the assumptions of the lemma, there exist $t_1 \in [0, \omega[\,, t_2 \in \,]0, \omega], t_1 \ne t_2$ such that

$$u(t_1) = 0, \quad u(t_2) = 0, \quad v(t_1) = 0, \quad v(t_2) = 0, \tag{2.27}$$
$$u(t) < 0 \quad \text{for } t \in I(t_1, t_2), \tag{2.28}$$

Then, (2.24) in view of (2.28) yields $v'(t) \ge 0$ for a. e. $t \in I(t_1, t_2)$, which together with (2.25) and (2.27) results in

$$v(t) = 0 \quad \text{for } t \in I(t_1, t_2). \tag{2.29}$$

However, (2.23) and (2.29) yield $u'(t) = 0$ for a. e. $t \in I(t_1, t_2)$, which together with (2.27) implies $u(t) = 0$ for $t \in I(t_1, t_2)$. The last equality contradicts (2.28).

Let, therefore, $t_0 \in [0, \omega]$ be such that (2.26) holds. Obviously, either

$$v(t_0) > 0 \tag{2.30}$$

or

$$v(t_0) < 0. \tag{2.31}$$

We will show that v has no zero. Assume on the contrary that v has a zero in the interval $[0,\omega]$. If (2.30) is satisfied then in view of (2.25) we can assume without loss of generality that $t_0 \neq \omega$ and, furthermore, there exists $t_1 \in]0,\omega], t_1 \neq t_0$ such that

$$v(t_1) = 0, \quad v(t) > 0 \quad \text{for } t \in I[t_0, t_1). \tag{2.32}$$

The integration of (2.23) over $I(t_0, t)$ in view of (2.25, 2.26), and (2.32) yields

$$u(t) = \int\limits_{I(t_0,t)} p(s)v(s)\mathrm{d}s \le \int\limits_{I(t_0,t)} [p(s)]_+ v(s)\mathrm{d}s \quad \text{for } t \in I(t_0, t_1). \tag{2.33}$$

The integration of (2.24) over $I(t, t_1)$ in view of (2.25) and (2.32), results in

$$-v(t) \ge -\int\limits_{I(t,t_1)} \varphi(s)u(s)\mathrm{d}s \quad \text{for } t \in I[t_0, t_1). \tag{2.34}$$

Using (2.33) in (2.34) we obtain

$$v(t) \le \int\limits_{I(t,t_1)} \varphi(s) \int\limits_{I(t_0,s)} [p(\xi)]_+ v(\xi)\mathrm{d}\xi\mathrm{d}s \quad \text{for } t \in I[t_0, t_1). \tag{2.35}$$

Now let $t_2 \in I[t_0, t_1)$ be such that

$$v(t_2) = \max\{v(t) : t \in I[t_0, t_1)\}. \tag{2.36}$$

Then from (2.35) on account of (2.32) and (2.36) we get

$$0 < v(t_2) \le v(t_2) \int\limits_{I(t_2,t_1)} \varphi(s) \int\limits_{I(t_0,s)} [p(\xi)]_+ \mathrm{d}\xi\mathrm{d}s \le v(t_2) \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+ \mathrm{d}s. \tag{2.37}$$

However, by using (2.13) it follows that $v(t_2) < v(t_2)$, a contradiction.

Now assume that (2.31) holds. Put

$$\overline{u}(t) = u(\omega - t) \quad \text{for } t \in [0,\omega], \quad \overline{v}(t) = -v(\omega - t) \quad \text{for } t \in [0,\omega], \tag{2.38}$$

$$\overline{p}(t) = p(\omega - t) \quad \text{for a. e. } t \in [0,\omega], \quad \overline{\varphi}(t) = \varphi(\omega - t) \quad \text{for a. e. } t \in [0,\omega]. \tag{2.39}$$

Then it can be easily verified that $(\overline{u}, \overline{v}) \in AC([0,\omega]; \mathbb{R}^2)$ satisfy

$$\overline{u}'(t) = \overline{p}(t)\overline{v}(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.40}$$

$$\overline{v}'(t) \ge -\overline{\varphi}(t)\overline{u}(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.41}$$

$$\overline{u}(0) = \overline{u}(\omega), \quad \overline{v}(0) \ge \overline{v}(\omega), \tag{2.42}$$

and

$$\int_0^\omega [\overline{p}(s)]_+ \mathrm{d}s = \int_0^\omega [p(s)]_+ \mathrm{d}s, \quad \int_0^\omega \overline{\varphi}(s)\mathrm{d}s = \int_0^\omega \varphi(s)\mathrm{d}s. \tag{2.43}$$

Moreover, in view of (2.26, 2.31), and (2.38) we have

$$\overline{u}(\omega - t_0) = 0, \quad \overline{v}(\omega - t_0) > 0,$$

and thus the lemma follows from the above-proven. ☐

Lemma 2.2. *Let $(u,v) \in AC([0,\omega]; \mathbb{R}^2)$ satisfy (2.23)–(2.25) with $p \in L([0,\omega]; \mathbb{R}), \varphi \in L([0,\omega]; \mathbb{R}_+)$. Let, moreover, u take both positive and negative values. If (2.15) and (2.17) hold then v does not vanish.*

Proof. Assume on the contrary that v has a zero. Put

$$M_v = \max\{v(t) : t \in [0,\omega]\}, \quad m_v = \max\{-v(t) : t \in [0,\omega]\}, \tag{2.44}$$

$$M_u = \max\{u(t) : t \in [0,\omega]\}, \quad m_u = \max\{-u(t) : t \in [0,\omega]\}. \tag{2.45}$$

According to our assumptions,

$$M_u > 0, \quad m_u > 0. \tag{2.46}$$

Moreover, if $v \equiv 0$ then from (2.23) it follows that u is a constant function, which in view of (2.45) yields $M_u = -m_u$. However, the latter equality contradicts (2.46). Therefore, we have

$$M_v \geq 0, \quad m_v \geq 0, \quad M_v + m_v > 0. \tag{2.47}$$

Choose $t_0 \in [0,\omega[\,, t_1 \in]0,\omega]$ such that

$$v(t_0) = M_v, \quad v(t_1) = -m_v. \tag{2.48}$$

Obviously, $t_0 \neq t_1$, and the integration of (2.24) over $I(t_0,t_1)$ on account of (2.25, 2.45), and (2.48) yields

$$M_v + m_v \leq \int\limits_{I(t_0,t_1)} \varphi(s)u(s)\mathrm{d}s \leq M_u \int\limits_0^\omega \varphi(s)\mathrm{d}s. \tag{2.49}$$

Now choose $t_2 \in [0,\omega[\,, t_3 \in]0,\omega]$ such that

$$u(t_2) = -m_u, \quad u(t_3) = M_u. \tag{2.50}$$

Obviously, $t_2 \neq t_3$, and the integration of (2.23) over $I(t_2,t_3)$ and over $J \overset{def}{=} [0,\omega] \setminus I(t_2,t_3)$, respectively, in view of (2.25, 2.44), and (2.50), result in

$$M_u + m_u = \int\limits_{I(t_2,t_3)} p(s)v(s)\mathrm{d}s \leq M_v \int\limits_{I(t_2,t_3)} [p(s)]_+\mathrm{d}s + m_v \int\limits_{I(t_2,t_3)} [p(s)]_-\mathrm{d}s, \tag{2.51}$$

resp.

$$-m_u - M_u = \int\limits_J p(s)v(s)\mathrm{d}s \geq -m_v \int\limits_J [p(s)]_+\mathrm{d}s - M_v \int\limits_J [p(s)]_-\mathrm{d}s. \tag{2.52}$$

Now from (2.51) and (2.52), with respect to (2.46), we obtain

$$M_u < M_v \int\limits_{I(t_2,t_3)} [p(s)]_+\mathrm{d}s + m_v \int\limits_{I(t_2,t_3)} [p(s)]_-\mathrm{d}s, \tag{2.53}$$

and

$$M_u < m_v \int\limits_J [p(s)]_+\mathrm{d}s + M_v \int\limits_J [p(s)]_-\mathrm{d}s. \tag{2.54}$$

Note that from (2.47) and (2.49) it follows that

$$\int\limits_0^\omega \varphi(s)\mathrm{d}s > 0. \tag{2.55}$$

Thus if we multiply both sides of (2.53), resp. (2.54), by $\int_0^\omega \varphi(s)\mathrm{d}s$, on account of (2.55) we get

$$M_u \int\limits_0^\omega \varphi(s)\mathrm{d}s < M_v \int\limits_0^\omega \varphi(s)\mathrm{d}s \int\limits_{I(t_2,t_3)} [p(s)]_+\mathrm{d}s + m_v \int\limits_0^\omega \varphi(s)\mathrm{d}s \int\limits_{I(t_2,t_3)} [p(s)]_-\mathrm{d}s, \tag{2.56}$$

resp.

$$M_u \int_0^\omega \varphi(s)\mathrm{d}s < m_v \int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_+\mathrm{d}s + M_v \int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_-\mathrm{d}s. \tag{2.57}$$

Now (2.49, 2.56), and (2.57) result in

$$m_v \left(1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_{I(t_2,t_3)} [p(s)]_-\mathrm{d}s\right) < M_v \left(\int_0^\omega \varphi(s)\mathrm{d}s \int_{I(t_2,t_3)} [p(s)]_+\mathrm{d}s - 1\right), \tag{2.58}$$

$$M_v \left(1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_-\mathrm{d}s\right) < m_v \left(\int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_+\mathrm{d}s - 1\right). \tag{2.59}$$

Note that (2.58) and (2.59) in view of (2.15) and (2.47) yields $M_v > 0, m_v > 0$. Therefore, multiplying the corresponding sides of (2.58) and (2.59) we obtain

$$\begin{aligned}&\left(1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_{I(t_2,t_3)} [p(s)]_-\mathrm{d}s\right)\left(1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_-\mathrm{d}s\right)\\ &\quad< \left(\int_0^\omega \varphi(s)\mathrm{d}s \int_{I(t_2,t_3)} [p(s)]_+\mathrm{d}s - 1\right)\left(\int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_+\mathrm{d}s - 1\right).\end{aligned} \tag{2.60}$$

Note that

$$\left(1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_{I(t_2,t_3)} [p(s)]_-\mathrm{d}s\right)\left(1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_-\mathrm{d}s\right) \geq 1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s \tag{2.61}$$

and, in view of the inequality $AB \leq \frac{1}{4}(A+B)^2$,

$$\left(\int_0^\omega \varphi(s)\mathrm{d}s \int_{I(t_2,t_3)} [p(s)]_+\mathrm{d}s - 1\right)\left(\int_0^\omega \varphi(s)\mathrm{d}s \int_J [p(s)]_+\mathrm{d}s - 1\right) \leq \frac{1}{4}\left(\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s - 2\right)^2. \tag{2.62}$$

Therefore, using (2.61) and (2.62) in (2.60) we obtain

$$1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s < \frac{1}{4}\left(\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s - 2\right)^2. \tag{2.63}$$

Note also, that from (2.58) and (2.59) it follows that

$$\int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s > 2. \tag{2.64}$$

Thus from (2.63), in view of (2.15) and (2.64) we get

$$2 + 2\sqrt{1 - \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s} < \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s,$$

which contradicts (2.17). □

Lemma 2.3. *Let* $(u,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ *satisfy* (2.23)–(2.25) *with* $p \in L\left([0,\omega];\mathbb{R}\right), \varphi \in L\left([0,\omega];\mathbb{R}_+\right)$, *and let* (2.12) *hold. Let, moreover, u assume both positive and negative values. If either* (2.13) *and* (2.14) *hold or* (2.15) *and* (2.16) *are fulfilled, then v has a zero.*

Proof. We will prove the lemma in the case when (2.13) and (2.14) are fulfilled. The case when (2.15) and (2.16) are satisfied can be proven analogously.

Let, therefore, (2.13) and (2.14) hold and assume to the contrary that v has no zero. First assume that v is a positive function. Put

$$M_v = \max\left\{v(t) : t \in [0,\omega]\right\}, \quad m_v = \min\left\{v(t) : t \in [0,\omega]\right\}, \tag{2.65}$$

and define numbers M_u and m_u by (2.45). Then, according to our assumptions, we have (2.46) and

$$M_v > 0, \quad m_v > 0. \tag{2.66}$$

Choose $t_0, t_2 \in [0,\omega[\,, t_1, t_3 \in]0,\omega]$ such that (2.50) holds and

$$v(t_0) = M_v, \quad v(t_1) = m_v. \tag{2.67}$$

Now the integration of (2.24) over $I(t_0,t_1)$, on account of (2.25, 2.45), and (2.67), yields

$$M_v - m_v \le \int\limits_{I(t_0,t_1)} \varphi(s)u(s)\mathrm{d}s \le M_u \int\limits_0^\omega \varphi(s)\mathrm{d}s. \tag{2.68}$$

Further, the integration of (2.23) over $I(t_2,t_3)$, in view of (2.25, 2.50, 2.65), and (2.66), results in

$$M_u + m_u = \int\limits_{I(t_2,t_3)} p(s)v(s)\mathrm{d}s \le M_v \int\limits_0^\omega [p(s)]_+\mathrm{d}s. \tag{2.69}$$

From (2.69), with respect to (2.46) and (2.12), we obtain

$$M_u \int\limits_0^\omega \varphi(s)\mathrm{d}s < M_v \int\limits_0^\omega \varphi(s)\mathrm{d}s \int\limits_0^\omega [p(s)]_+\mathrm{d}s. \tag{2.70}$$

Now (2.68) and (2.70) imply

$$M_v < m_v + M_v \int\limits_0^\omega \varphi(s)\mathrm{d}s \int\limits_0^\omega [p(s)]_+\mathrm{d}s. \tag{2.71}$$

On the other hand, the integration of (2.23) from 0 to ω, in view of (2.25) and (2.65), yields

$$0 = \int\limits_0^\omega p(s)v(s)\mathrm{d}s \le M_v \int\limits_0^\omega [p(s)]_+\mathrm{d}s - m_v \int\limits_0^\omega [p(s)]_-\mathrm{d}s,$$

i.e.,

$$m_v \int\limits_0^\omega [p(s)]_-\mathrm{d}s \le M_v \int\limits_0^\omega [p(s)]_+\mathrm{d}s. \tag{2.72}$$

Note that having assumed u not to be a constant function we find $p \not\equiv 0$. Therefore, from (2.14) it follows that

$$\int_0^\omega [p(s)]_-\mathrm{d}s > 0. \tag{2.73}$$

Thus if we multiply both sides of (2.71) by $\int_0^\omega [p(s)]_-\mathrm{d}s$, then, in view of (2.66, 2.72), and (2.73) we obtain

$$\int_0^\omega [p(s)]_-\mathrm{d}s < \int_0^\omega [p(s)]_+\mathrm{d}s + \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s. \tag{2.74}$$

However, (2.74) contradicts (2.14).

Now assume that v is a negative function. Define $\overline{u}, \overline{v}, \overline{p}$, and $\overline{\varphi}$ by (2.38) and (2.39). Then it can be easily verified that $(\overline{u}, \overline{v}) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfies (2.40)–(2.42). Furthermore, (2.43) is fulfilled and, in addition, also

$$\int_0^\omega [\overline{p}(s)]_-\mathrm{d}s = \int_0^\omega [p(s)]_-\mathrm{d}s.$$

Moreover, $\overline{v}$ is a positive function, and thus the lemma follows from the above-proven. □

Lemma 2.4. *Let a vector-valued function $(p, -\varphi)$ verify the property (P_+) with $p \in L([0,\omega];\mathbb{R}), \varphi \in L([0,\omega];\mathbb{R}_+)$. Let, moreover, (2.20) holds. Then the problem*

$$u'(t) = p(t)v(t) + h(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.75}$$

$$v'(t) = -\varphi(t)u(t) + q(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.76}$$

$$u(0) = u(\omega), \quad v(0) = v(\omega) \tag{2.77}$$

has a unique solution (u, v) for each $(h, q) \in L\left([0,\omega];\mathbb{R}^2\right)$. Moreover, there exists $\rho_0 > 0$ independent on h and q such that the estimate

$$\|(u,v)\|_C \le \rho_0 \|(h,q)\|_L \tag{2.78}$$

holds.

Proof. According to the well-known Fredholm alternative principle, it is sufficient to show that the corresponding homogeneous system

$$u'(t) = p(t)v(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.79}$$

$$v'(t) = -\varphi(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.80}$$

with boundary conditions (2.77) has only the trivial solution. Let, therefore, $(u, v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfy (2.77, 2.79), and (2.80). By Definition 2.2, u is non-negative, which together with (2.80) results in $v'(t) \le 0$ for a. e. $t \in [0,\omega]$. Then, the periodicity condition (2.77) implies that v is a constant function. Having in mind this fact, the integration of (2.79) from 0 to ω, in view of (2.77) gives

$$0 = v(0)\int_0^\omega p(s)\mathrm{d}s. \tag{2.81}$$

Now (2.81) on account of (2.20) results in

$$v(t) = 0 \quad \text{for } t \in [0,\omega]. \tag{2.82}$$

Using (2.82) in (2.79) we get that u is a constant function. Therefore, the integration of (2.80) from 0 to ω in view of (2.77) yields

$$0 = u(0)\int_0^\omega \varphi(s)\mathrm{d}s. \tag{2.83}$$

By (2.20), we have

$$u(t) = 0 \quad \text{for} \;\; t \in [0, \omega]. \tag{2.84}$$

Thus (2.82) and (2.84) ensure that the only solution to (2.77, 2.79, 2.80) is the trivial one.

Now we will show the estimate (2.78). Let $\Omega : L\left([0,\omega];\mathbb{R}^2\right) \to C\left([0,\omega];\mathbb{R}^2\right)$ be the operator assigning to every $(h,q) \in L\left([0,\omega];\mathbb{R}^2\right)$ the unique solution (u,v) of (2.75)–(2.77). Thus, Ω is a linear continuous operator. Then, (2.78) obviously holds with

$$\rho_0 = \sup\left\{\|\Omega(x,y)\|_C : \|(x,y)\|_L = 1\right\}.$$

□

2.5. Proofs of main results

Proof of Theorem 2.1. Let $(u,v) \in AC\left([0,\omega];\mathbb{R}^2\right)$ satisfy (2.23)–(2.25). According to Definition 2.2 it is sufficient to show that (2.7) holds. Assume on the contrary that there exists $t_0 \in [0,\omega]$ such that

$$u(t_0) < 0. \tag{2.85}$$

First note that if $p \equiv 0$ then from (2.23) it follows that u is a constant function. Consequently, the integration of (2.24) from 0 to ω in view of (2.25) yields

$$0 \geq -u(t_0) \int_0^{\omega} \varphi(s)\mathrm{d}s \tag{2.86}$$

which together with (2.12) contradicts (2.85). Therefore, in what follows we can assume that

$$p \not\equiv 0. \tag{2.87}$$

According to Lemmas 2.1, 2.2, and 2.3 it follows that u does not assume positive values. Therefore,

$$u(t) \leq 0 \quad \text{for} \;\; t \in [0,\omega]. \tag{2.88}$$

However, from (2.24, 2.25), and (2.88) it follows that v is a constant function. Thus the integration of (2.23) from 0 to ω, on account of (2.25) results in

$$0 = v(0) \int_0^{\omega} p(s)\mathrm{d}s. \tag{2.89}$$

On the other hand, from (2.12, 2.87) and (2.14), resp. (2.16), it follows that $\int_0^{\omega} p(s)\mathrm{d}s \neq 0$. Therefore, from (2.89) we obtain $v \equiv 0$. Thus from (2.23) we get that u is a constant function which is, according to (2.85), negative. Now the integration of (2.24) from 0 to ω, in view of (2.25), results in (2.86). However, (2.86) in view of (2.12) contradicts (2.85). □

Proof of Theorem 2.2. It immediately follows from Theorem 2.1 and Remark 2.1. □

Proof of Theorem 2.3. Let ρ_0 be a number appearing in Lemma 2.4. Put

$$\rho(t) = \sup\left\{|f(t,x)| : \beta_1(t) \leq x \leq \alpha_1(t)\right\} \quad \text{for a. e.} \;\; t \in [0,\omega] \tag{2.90}$$

and

$$\rho_1 = \rho_0\left(\|\varphi\|_L \|(\alpha_1,\beta_1)\|_C + \|\rho\|_L\right). \tag{2.91}$$

Let, moreover,

$$U_1 = \left\{y \in C\left([0,\omega];\mathbb{R}\right) : \beta_1(t) \leq y(t) \leq \alpha_1(t) \text{ for } \; t \in [0,\omega]\right\}, \tag{2.92}$$

$$U_2 = \left\{z \in C\left([0,\omega];\mathbb{R}\right) : \|z\|_C \leq \rho_1\right\}, \tag{2.93}$$

and let $\Omega : U_1 \times U_2 \to C\left([0,\omega];\mathbb{R}^2\right)$ be an operator which to every (y,z) assigns the unique solution of

$$u' = p(t)v \tag{2.94}$$
$$v' = -\varphi(t)u + \varphi(t)y + f(t,y), \tag{2.95}$$
$$u(0) = u(\omega), \quad v(0) = v(\omega). \tag{2.96}$$

According to Lemma 2.4, the operator Ω is defined correctly. Also note that $\Omega(y,z) = \Omega(y,0)$ for any $(y,z) \in U_1 \times U_2$. Furthermore, the operator Ω is continuous. We will show that Ω transforms $U_1 \times U_2$ into itself. According to Lemma 2.4 we have

$$\|\Omega(y,z)\|_C = \|(u,v)\|_C \le \rho_0 \left\|(0, \varphi y + f(\cdot,y))\right\|_L . \tag{2.97}$$

From (2.97), in view of (2.90)–(2.92) and the inclusion $y \in U_1$, we obtain

$$\|v\|_C \le \|(u,v)\|_C \le \rho_1. \tag{2.98}$$

Consequently, $v \in U_2$. On the other hand, in view of (2.19, 2.94)–(2.96),

$$u'(t) - \beta_1'(t) = p(t)\left(v(t) - \beta_2(t)\right) \quad \text{for a. e. } t \in [0,\omega], \tag{2.99}$$
$$v'(t) - \beta_2'(t) \ge -\varphi(t)\left(u(t) - \beta_1(t)\right) \quad \text{for a. e. } t \in [0,\omega], \tag{2.100}$$
$$u(0) - \beta_1(0) = u(\omega) - \beta_1(\omega), \quad v(0) - \beta_2(0) \ge v(\omega) - \beta_2(\omega). \tag{2.101}$$

However, a pair $(p, -\varphi)$ verifies a property (P_+), and therefore from (2.99)–(2.101) we get

$$u(t) \ge \beta_1(t) \quad \text{for } t \in [0,\omega]. \tag{2.102}$$

Analogously, we find

$$u(t) \le \alpha_1(t) \quad \text{for } t \in [0,\omega]. \tag{2.103}$$

Now (2.102) and (2.103) imply $u \in U_1$. Thus we have shown that Ω transforms a set $U_1 \times U_2$ into itself. Furthermore, on account of (2.90, 2.98, 2.102, 2.103), and the inclusion $y \in U_1$, from (2.94) and (2.95) it follows

$$|u'(t)| \le \rho_1 \, |p(t)| \quad \text{for a. e. } t \in [0,\omega],$$
$$|v'(t)| \le 2\,|\varphi(t)|\,\|(\alpha_1,\beta_1)\|_C + \rho(t) \quad \text{for a. e. } t \in [0,\omega].$$

Consequently, Ω transforms $U_1 \times U_2$ into its relatively compact subset. According to Schauder fixed point theorem, there exists $(u_0, v_0) \in U_1 \times U_2$ such that

$$(u_0, v_0) = \Omega(u_0, v_0). \tag{2.104}$$

However, in view of (2.94)–(2.96), from (2.104) it follows that (u_0, v_0) is a solution to (2.1)–(2.3). ☐

Proof of Theorem 2.4. It immediately follows from Theorem 2.3 and Remark 2.1. ☐

3. Applications to the optical problem

In this section we will consider the system

$$u' = p(t)v, \tag{3.1}$$
$$v' = \frac{h(t)}{u^3} + \frac{g(t)}{u^2}, \tag{3.2}$$

together with the boundary conditions

$$u(0) = u(\omega), \quad v(0) = v(\omega). \tag{3.3}$$

Here $p, h, g \in L\left([0,\omega];\mathbb{R}\right)$. We establish conditions for the existence of a solution (u,v) to (3.1)–(3.3) with $u(t) > 0$ for $t \in [0,\omega]$.

For the sake of brevity we will use the following notation:

$$H_+ = \int_0^\omega [h(s)]_+ \mathrm{d}s, \quad H_- = \int_0^\omega [h(s)]_- \mathrm{d}s,$$

$$G_+ = \int_0^\omega [g(s)]_+ \mathrm{d}s, \quad G_- = \int_0^\omega [g(s)]_- \mathrm{d}s.$$

The main results of this section are the following ones.

Theorem 3.1. *Let*

$$\int_0^\omega p(s)\mathrm{d}s \neq 0, \tag{3.4}$$

$H_+ > \left(\frac{9}{8}\right)^3 H_-, G_- > G_+$, *and let*

$$\frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}\left(G_-H_+ - \frac{9}{8}G_+H_-\right) \leq \frac{8^8\left(H_+ - \left(\frac{9}{8}\right)^3 H_-\right)^4}{9^9\left(G_- - \left(\frac{8}{9}\right)^2 G_+\right)^3}. \tag{3.5}$$

Then the problem (3.1)–(3.3) *has at least one solution* (u, v) *with* $u(t) > 0$ *for* $t \in [0, \omega]$.

Theorem 3.2. *Let* (3.4) *hold,* $H_- > \left(\frac{9}{8}\right)^3 H_+, G_+ > G_-$, *and let*

$$\frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}\left(G_+H_- - \frac{9}{8}G_-H_+\right) \leq \frac{8^8\left(H_- - \left(\frac{9}{8}\right)^3 H_+\right)^4}{9^9\left(G_+ - \left(\frac{8}{9}\right)^2 G_-\right)^3}. \tag{3.6}$$

Then the problem (3.1)–(3.3) *has at least one solution* (u, v) *with* $u(t) > 0$ *for* $t \in [0, \omega]$.

Such results are directly applicable to the TM–equations (1.3), giving some interesting corollaries.

Corollary 3.1. *Let us consider the ansatz* (1.2) *with a general input pulse profile* $Q(x)$. *Assume that* $c, d \in L([0, 1]; \mathbb{R})$ *verify*

$$\int_0^1 [d(s)]_+ \mathrm{d}s > \left(\frac{9}{8}\right)^3 \int_0^1 [d(s)]_- \mathrm{d}s, \quad \int_0^1 [c(s)]_+ \mathrm{d}s > \int_0^1 [c(s)]_- \mathrm{d}s \tag{3.7}$$

Then, there exists a constant $K \equiv K(d, C_1, C_2)$ *(where* C_1, C_2 *are defined by* (1.4)*), such that the TM-equations* (1.3) *have a* 1*-periodic solution provided that*

$$\int_0^1 [c(s)]_+ \mathrm{d}s < K(d, C_1, C_2).$$

Corollary 3.2. *Let us consider the ansatz* (1.2) *with a gaussian input pulse profile* $Q(x) = C_0 \exp(-x^2/2)$. *Assume that* $c, d \in L([0, 1]; \mathbb{R})$ *verify* (3.7). *Then, there exists a constant* $H \equiv H(c, d)$ *such that the TM-equations* (1.3) *have a* 1*-periodic solution provided that*

$$C_0 < H(c, d).$$

Both corollaries are direct applications of Theorem 3.1, taking into account in the second corollary that for a gaussian profile $Q(x) = C_0 \exp(-x^2/2)$, we can compute $C_1 = 1, C_2 = \frac{C_0^2}{2\sqrt{2}}$. Of course, explicit expressions of constant K, H are easily derived. We omit further details.

The following two corollaries can be directly derived from Theorem 3.2.

Corollary 3.3. *Let us consider the ansatz (1.2) with a general input pulse profile $Q(x)$. Assume that $c, d \in L([0,1];\mathbb{R})$ verify*

$$\int_0^1 [d(s)]_- \mathrm{d}s > \left(\frac{9}{8}\right)^3 \int_0^1 [d(s)]_+ \mathrm{d}s, \quad \int_0^1 [c(s)]_- \mathrm{d}s > \int_0^1 [c(s)]_+ \mathrm{d}s \tag{3.8}$$

Then, there exists a constant $K \equiv K(d, C_1, C_2)$ (where C_1, C_2 are defined by (1.4)), such that the TM-equations (1.3) have a 1-periodic solution provided that

$$\int_0^1 [c(s)]_- \mathrm{d}s < K(d, C_1, C_2).$$

Corollary 3.4. *Let us consider the ansatz (1.2) with a gaussian input pulse profile $Q(x) = C_0 \exp(-x^2/2)$. Assume that $c, d \in L([0,1];\mathbb{R})$ verify (3.8). Then, there exists a constant $H \equiv H(c,d)$ such that the TM-equations (1.3) have a 1-periodic solution provided that*

$$C_0 < H(c,d).$$

The proofs of Theorem 3.1 and 3.2 require the construction of adequate upper and lower functions. For brevity, we will only give the complete proof of Theorem 3.1, since Theorem 3.2 can be proven analogously. Some lemmas are needed.

Lemma 3.1. *Let $p, q \in L([0,\omega];\mathbb{R})$ be such that*

$$\int_0^\omega p(s)\mathrm{d}s \neq 0, \quad \int_0^\omega q(s)\mathrm{d}s = 0. \tag{3.9}$$

Then every solution to the problem

$$u' = p(t)v, \tag{3.10}$$

$$v' = q(t), \tag{3.11}$$

$$u(0) = u(\omega), \quad v(0) = v(\omega) \tag{3.12}$$

is given by

$$\begin{aligned} u(t) = c - \frac{1}{\int_0^\omega p(s)\mathrm{d}s} \Bigg(&\int_t^\omega p(s)\mathrm{d}s \int_0^t q(s) \int_0^s p(\xi)\mathrm{d}\xi\mathrm{d}s \\ &+ \int_0^t p(s)\mathrm{d}s \int_t^\omega q(s) \int_s^\omega p(\xi)\mathrm{d}\xi\mathrm{d}s \Bigg) \quad \text{for } t \in [0,\omega], \end{aligned} \tag{3.13}$$

$$v(t) = \frac{1}{\int_0^\omega p(s)\mathrm{d}s} \left(\int_0^t q(s) \int_0^s p(\xi)\mathrm{d}\xi\mathrm{d}s - \int_t^\omega q(s) \int_s^\omega p(\xi)\mathrm{d}\xi\mathrm{d}s \right) \quad \text{for } t \in [0,\omega], \tag{3.14}$$

where $c \in \mathbb{R}$.

Proof. If u and v are given by (3.13) and (3.14) then, obviously, they satisfy (3.10) and (3.11), and in view of (3.9), also (3.12) is fulfilled.

Let $(u,v) \in AC([0,\omega];\mathbb{R}^2)$ satisfy (3.10)–(3.12). Then the integration of (3.10) from 0 to t and from t to ω, respectively, gives

$$u(t) = u(0) + \int_0^t p(s)v(s)\mathrm{d}s \tag{3.15}$$

and

$$u(t) = u(\omega) - \int_t^\omega p(s)v(s)\mathrm{d}s. \tag{3.16}$$

The integration by parts of (3.15) and (3.16), in view of (3.11) and the periodicity of u, results in

$$u(t) = u(0) + v(t)\int_0^t p(s)\mathrm{d}s - \int_0^t q(s)\int_0^s p(\xi)\mathrm{d}\xi\mathrm{d}s, \tag{3.17}$$

$$u(t) = u(0) - v(t)\int_t^\omega p(s)\mathrm{d}s - \int_t^\omega q(s)\int_s^\omega p(\xi)\mathrm{d}\xi\mathrm{d}s. \tag{3.18}$$

If we multiply both sides of (3.17) by $\int_t^\omega p(s)\mathrm{d}s$ and both sides of (3.18) by $\int_0^t p(s)\mathrm{d}s$, we arrive at

$$u(t)\int_t^\omega p(s)\mathrm{d}s = u(0)\int_t^\omega p(s)\mathrm{d}s + v(t)\int_t^\omega p(s)\mathrm{d}s\int_0^t p(s)\mathrm{d}s - \int_t^\omega p(s)\mathrm{d}s\int_0^t q(s)\int_0^s p(\xi)\mathrm{d}\xi\mathrm{d}s, \tag{3.19}$$

$$u(t)\int_0^t p(s)\mathrm{d}s = u(0)\int_0^t p(s)\mathrm{d}s - v(t)\int_t^\omega p(s)\mathrm{d}s\int_0^t p(s)\mathrm{d}s - \int_0^t p(s)\mathrm{d}s\int_t^\omega q(s)\int_s^\omega p(\xi)\mathrm{d}\xi\mathrm{d}s. \tag{3.20}$$

Now if we sum the corresponding sides of (3.19) and (3.20) we find that (3.13) holds true. The equality (3.14) follows from (3.17) and (3.18). □

Lemma 3.2. *Let* $p, q \in L([0,\omega];\mathbb{R})$ *satisfy* (3.9) *and let* $u, v \in AC\left([0,\omega];\mathbb{R}^2\right)$ *satisfy* (3.10)–(3.12). *Then*

$$M_u - m_u \le \frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}\int_0^\omega [q(s)]_-\mathrm{d}s, \tag{3.21}$$

where

$$M_u = \max\{u(t) : t \in [0,\omega]\}, \quad m_u = \min\{u(t) : t \in [0,\omega]\}. \tag{3.22}$$

Proof. According to (3.12) we can extend functions u, v, p, and q to the interval $[0, 3\omega]$ periodically. Then, obviously, (3.10) and (3.11) hold for almost every $t \in [0, 3\omega]$, and

$$u(t) = u(t+\omega), \quad v(t) = v(t+\omega) \quad \text{for } t \in [0, 2\omega].$$

Choose $t_m \in [0,\omega[$ and $t_M \in]t_m, t_m+\omega[$ such that

$$u(t_m) = m_u, \quad u(t_M) = M_u. \tag{3.23}$$

According to Lemma 3.1, the function u has a representation

$$u(t) = u(t_m) - \frac{1}{\int_{t_m}^{t_m+\omega} p(s)\mathrm{d}s}\left(\int_t^{t_m+\omega} p(s)\mathrm{d}s\int_{t_m}^t q(s)\int_{t_m}^s p(\xi)\mathrm{d}\xi\mathrm{d}s + \int_{t_m}^t p(s)\mathrm{d}s\int_t^{t_m+\omega} q(s)\int_s^{t_m+\omega} p(\xi)\mathrm{d}\xi\mathrm{d}s\right)$$
$$\text{for } t \in [t_m, t_m+\omega]. \tag{3.24}$$

Now from (3.24), for $t = t_M$, in view of (3.23) and the periodicity of p it follows that

$$M_u - m_u = -\frac{1}{\int_0^\omega p(s)\mathrm{d}s}\left(\int_{t_M}^{t_m+\omega} p(s)\mathrm{d}s\int_{t_m}^{t_M} q(s)\int_{t_m}^s p(\xi)\mathrm{d}\xi\mathrm{d}s + \int_{t_m}^{t_M} p(s)\mathrm{d}s\int_{t_M}^{t_m+\omega} q(s)\int_s^{t_m+\omega} p(\xi)\mathrm{d}\xi\mathrm{d}s\right). \tag{3.25}$$

On the other hand, according to Lemma 3.1 again, the function u has also a representation

$$u(t) = u(t_M) - \frac{1}{\int_{t_M}^{t_M+\omega} p(s)\mathrm{d}s} \left(\int_t^{t_M+\omega} p(s)\mathrm{d}s \int_{t_M}^{t} q(s) \int_{t_M}^{s} p(\xi)\mathrm{d}\xi\mathrm{d}s \right.$$
$$\left. + \int_{t_M}^{t} p(s)\mathrm{d}s \int_t^{t_M+\omega} q(s) \int_s^{t_M+\omega} p(\xi)\mathrm{d}\xi\mathrm{d}s \right) \quad \text{for } t \in [t_M, t_M+\omega]. \tag{3.26}$$

Note that $t_m+\omega \in]t_M, t_M+\omega[$. Therefore, from (3.26) for $t = t_m+\omega$ in view of (3.23) and the periodicity of p and q it follows that

$$M_u - m_u = \frac{1}{\int_0^\omega p(s)\mathrm{d}s} \left(\int_{t_m}^{t_M} p(s)\mathrm{d}s \int_{t_M}^{t_m+\omega} q(s) \int_{t_M}^{s} p(\xi)\mathrm{d}\xi\mathrm{d}s + \int_{t_M}^{t_m+\omega} p(s)\mathrm{d}s \int_{t_m}^{t_M} q(s) \int_s^{t_M} p(\xi)\mathrm{d}\xi\mathrm{d}s \right). \tag{3.27}$$

Now if we sum the corresponding sides of (3.25) and (3.27) we find

$$2\,(M_u - m_u) = \frac{1}{\int_0^\omega p(s)\mathrm{d}s} \left(\int_{t_m}^{t_M} p(s)\mathrm{d}s \int_{t_M}^{t_m+\omega} q(s) \left(\int_{t_M}^{s} p(\xi)\mathrm{d}\xi - \int_s^{t_m+\omega} p(\xi)\mathrm{d}\xi \right) \mathrm{d}s \right.$$
$$\left. + \int_{t_M}^{t_m+\omega} p(s)\mathrm{d}s \int_{t_m}^{t_M} q(s) \left(\int_s^{t_M} p(\xi)\mathrm{d}\xi - \int_{t_m}^{s} p(\xi)\mathrm{d}\xi \right) \mathrm{d}s \right). \tag{3.28}$$

Note that

$$\left| \int_{t_M}^{s} p(\xi)\mathrm{d}\xi - \int_s^{t_m+\omega} p(\xi)\mathrm{d}\xi \right| \le \int_{t_M}^{t_m+\omega} |p(\xi)|\,\mathrm{d}\xi \quad \text{for } s \in [t_M, t_m+\omega], \tag{3.29}$$

and

$$\left| \int_s^{t_M} p(\xi)\mathrm{d}\xi - \int_{t_m}^{s} p(\xi)\mathrm{d}\xi \right| \le \int_{t_m}^{t_M} |p(\xi)|\,\mathrm{d}\xi \quad \text{for } s \in [t_m, t_M]. \tag{3.30}$$

Therefore, from (3.28), in view of (3.29), (3.30) and the periodicity of q we obtain

$$2\,(M_u - m_u) \le \frac{1}{\left|\int_0^\omega p(s)\mathrm{d}s\right|} \int_{t_m}^{t_M} |p(s)|\,\mathrm{d}s \int_{t_M}^{t_m+\omega} |p(s)|\,\mathrm{d}s \int_0^\omega |q(s)|\,\mathrm{d}s. \tag{3.31}$$

Now using the inequality $AB \le \frac{1}{4}(A+B)^2$ in (3.31) and the periodicity of p, we arrive at

$$M_u - m_u \le \frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{8\left|\int_0^\omega p(s)\mathrm{d}s\right|} \int_0^\omega |q(s)|\,\mathrm{d}s. \tag{3.32}$$

Note that (3.9) implies

$$\int_0^\omega |q(s)|\,\mathrm{d}s = 2\int_0^\omega [q(s)]_-\mathrm{d}s, \tag{3.33}$$

and consequently, (3.32) and (3.33) result in (3.21). □

Lemma 3.3. *Let $p \in L([0,\omega];\mathbb{R}), \varphi \in L([0,\omega];\mathbb{R}_+)$, (2.20) holds and*

$$\int_0^\omega \varphi(s)\mathrm{d}s \le \frac{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}. \tag{3.34}$$

Then $(p,-\varphi)$ verifies the property (P_+).

Proof. In the case where $\int_0^\omega p(s)\mathrm{d}s > 0$ we put

$$x = \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s, \quad y = \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s, \tag{3.35}$$

while if $\int_0^\omega p(s)\mathrm{d}s < 0$ then we put

$$x = \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_+\mathrm{d}s, \quad y = \int_0^\omega \varphi(s)\mathrm{d}s \int_0^\omega [p(s)]_-\mathrm{d}s. \tag{3.36}$$

In both cases (3.35) and (3.36), from (3.34) it follows that

$$(y+x)^2 \le 4(y-x). \tag{3.37}$$

Now from (3.37) it follows that

$$x \le \frac{1}{2}, \tag{3.38}$$

$$y \le 2 - x + 2\sqrt{1-2x} \le 2 + 2\sqrt{1-x}, \tag{3.39}$$

and, using the inequality $4AB \le (A+B)^2$, also

$$yx \le y - x. \tag{3.40}$$

Therefore, if $\int_0^\omega p(s)\mathrm{d}s > 0$ then in view of (3.35) from (3.38)–(3.40) we obtain the inequalities (2.15)–(2.17). If $\int_0^\omega p(s)\mathrm{d}s < 0$ then in view of (3.36) from (3.38) and (3.40) we get (2.13) and (2.14). Thus the conclusion of the lemma follows from Theorem 2.1. □

Proof of Theorem 3.1. At first we construct an upper function. Put

$$A = G_- - \left(\frac{8}{9}\right)^2 G_+, \quad D = H_+ - \left(\frac{9}{8}\right)^3 H_-, \tag{3.41}$$

$$B = \left(\frac{9}{8}\right)^3 A, \quad C = \left(\frac{8}{9}\right)^2 D, \quad x = \left(\frac{9}{8}\right)^6 \frac{A^2}{D^3}. \tag{3.42}$$

Then, obviously,

$$AH_+ - BH_- + CG_+ - DG_- = 0,$$

and, therefore, according to Lemma 3.1, the problem

$$w_1' = p(t)w_2 \tag{3.43}$$

$$w_2' = A[h(t)]_+ - B[h(t)]_- + C[g(t)]_+ - D[g(t)]_- \tag{3.44}$$

$$w_1(0) = w_1(\omega), \quad w_2(0) = w_2(\omega) \tag{3.45}$$

has a solution $(w_1,w_2) \in AC\left([0,\omega];\mathbb{R}^2\right)$. Put

$$m_w = \min\{w_1(t) : t \in [0,\omega]\}, \tag{3.46}$$

$$M_w = \max\{w_1(t) : t \in [0,\omega]\}. \tag{3.47}$$

Then, according to Lemma 3.2, in view of (3.41) and (3.42) we have

$$M_w - m_w \le \frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}\left(G_-H_+ - \frac{9}{8}G_+H_-\right). \tag{3.48}$$

Put

$$\beta_1(t) = \left(\frac{1}{xD}\right)^{1/2} + x\left(w_1(t) - m_w\right) \quad \text{for } t \in [0,\omega], \tag{3.49}$$

$$\beta_2(t) = xw_2(t) \quad \text{for } t \in [0,\omega]. \tag{3.50}$$

Then, obviously,

$$\beta_1(t) > 0 \quad \text{for } t \in [0,\omega] \tag{3.51}$$

and, on account of (3.46),

$$\left(\frac{1}{xD}\right)^{1/2} \le \beta_1(t) \quad \text{for } t \in [0,\omega]. \tag{3.52}$$

Furthermore, from (3.48) in view of (3.5, 3.41), and (3.42) we obtain

$$x\left(M_w - m_w\right) \le \left(\frac{1}{xA}\right)^{1/3} - \left(\frac{1}{xD}\right)^{1/2}. \tag{3.53}$$

Now from (3.49) with respect to (3.53) it follows that

$$\beta_1(t) \le \left(\frac{1}{xA}\right)^{1/3} \quad \text{for } t \in [0,\omega]. \tag{3.54}$$

On the other hand, in view of (3.41) and (3.42) we have

$$\left(\frac{1}{xB}\right)^{1/3} = \left(\frac{1}{xD}\right)^{1/2}, \quad \left(\frac{1}{xC}\right)^{1/2} = \left(\frac{1}{xA}\right)^{1/3}. \tag{3.55}$$

Therefore, (3.52, 3.54), and (3.55) result in

$$xA \le \frac{1}{\beta_1^3(t)} \le xB \quad \text{for } t \in [0,\omega], \tag{3.56}$$

$$xC \le \frac{1}{\beta_1^2(t)} \le xD \quad \text{for } t \in [0,\omega]. \tag{3.57}$$

Thus, from (3.49) and (3.50), in view of (3.43)–(3.45, 3.56), and (3.57) we obtain

$$\beta_1'(t) = p(t)\beta_2(t) \quad \text{for a. e. } t \in [0,\omega],$$

$$\beta_2'(t) \le \frac{h(t)}{\beta_1^3(t)} + \frac{g(t)}{\beta_1^2(t)} \quad \text{for a. e. } t \in [0,\omega],$$

$$\beta_1(0)=\beta_1(\omega), \quad \beta_2(0)=\beta_2(\omega).$$

Therefore, (β_1,β_2) is an upper function to (3.1)–(3.3).

Now we construct a lower function. Let $y_0\in\,]1,G_-/G_+[$ be such that

$$\frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}G_- \le \frac{y_0^4H_+^3}{(G_- - y_0G_+)^3}\left(y_0^{1/2}-1\right) \tag{3.58}$$

and put

$$y_1=\frac{G_- - y_0G_+}{H_+}. \tag{3.59}$$

Obviously,

$$y_1H_+ + y_0G_+ - G_- = 0$$

and, therefore, according to Lemma 3.1, the problem

$$z_1' = p(t)z_2 \tag{3.60}$$
$$z_2' = y_1[h(t)]_+ + y_0[g(t)]_+ - [g(t)]_- \tag{3.61}$$
$$z_1(0)=z_1(\omega), \quad z_2(0)=z_2(\omega) \tag{3.62}$$

has a solution $(z_1,z_2)\in AC\left([0,\omega];\mathbb{R}^2\right)$. Put

$$m_z=\min\{z_1(t): t\in[0,\omega]\}, \tag{3.63}$$
$$M_z=\max\{z_1(t): t\in[0,\omega]\}. \tag{3.64}$$

Then, according to Lemma 3.2, we have

$$M_z-m_z\le \frac{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}G_-. \tag{3.65}$$

Put

$$\alpha_1(t)=\frac{y_0}{y_1}+\frac{y_1^2}{y_0^3}\left(z_1(t)-m_z\right) \quad \text{for } t\in[0,\omega], \tag{3.66}$$
$$\alpha_2(t)=\frac{y_1^2}{y_0^3}z_2(t) \quad \text{for } t\in[0,\omega]. \tag{3.67}$$

Then, obviously,

$$\alpha_1(t)>0 \quad \text{for } t\in[0,\omega] \tag{3.68}$$

and, on account of (3.63),

$$\frac{y_0}{y_1}\le\alpha_1(t) \quad \text{for } t\in[0,\omega]. \tag{3.69}$$

Furthermore, from (3.65) in view of (3.58) and (3.59) we obtain

$$\frac{y_1^2}{y_0^3}\left(M_z-m_z\right)\le\frac{y_0}{y_1}\left(y_0^{1/2}-1\right). \tag{3.70}$$

Now from (3.66) with respect to (3.64) and (3.70) it follows that

$$\alpha_1(t)\le\frac{y_0^{3/2}}{y_1} \quad \text{for } t\in[0,\omega]. \tag{3.71}$$

Therefore, (3.69) and (3.71) result in

$$\frac{y_1^2}{y_0^3} \le \frac{1}{\alpha_1^2(t)} \le \left(\frac{y_1}{y_0}\right)^2 \quad \text{for } t \in [0,\omega], \tag{3.72}$$

$$\frac{1}{\alpha_1^3(t)} \le \left(\frac{y_1}{y_0}\right)^3 \quad \text{for } t \in [0,\omega]. \tag{3.73}$$

Thus, from (3.66) and (3.67), in view of (3.60)–(3.62, 3.72), and (3.73) we obtain

$$\begin{aligned} &\alpha_1'(t) = p(t)\alpha_2(t) \quad \text{for a. e. } t \in [0,\omega],\\ &\alpha_2'(t) \ge \frac{h(t)}{\alpha_1^3(t)} + \frac{g(t)}{\alpha_1^2(t)} \quad \text{for a. e. } t \in [0,\omega],\\ &\alpha_1(0) = \alpha_1(\omega), \quad \alpha_2(0) = \alpha_2(\omega). \end{aligned}$$

Therefore, (α_1, α_2) is a lower function to (3.1)–(3.3). Moreover, according to (3.41, 3.42, 3.54, 3.59, 3.69), and $y_0 > 1$, we have

$$\beta_1(t) \le \alpha_1(t) \quad \text{for } t \in [0,\omega].$$

Note that functions

$$\psi_1(t,y) = \frac{3}{\beta_1^4(t)} y + \frac{1}{y^3}, \quad \psi_2(t,y) = \frac{2}{\beta_1^3(t)} y + \frac{1}{y^2}$$

are non-decreasing for every $t \in [0,\omega]$ in the second argument whenever $y \ge \beta_1(t)$. Therefore, if we put

$$\varphi(t) = \frac{3[h(t)]_+}{\beta_1^4(t)} + \frac{2[g(t)]_+}{\beta_1^3(t)} \quad \text{for a. e. } t \in [0,\omega], \tag{3.74}$$

$$f(t,x) = \frac{h(t)}{x^3} + \frac{g(t)}{x^2} \quad \text{for a. e. } t \in [0,\omega], \quad x > 0,$$

then one can easily verified that (2.19) is fulfilled. Moreover, in view of (3.41, 3.42, 3.52, 3.55), and (3.74), we have

$$\int_0^\omega \varphi(s)\mathrm{d}s \le 3H_+(xD)^2 + 2G_+xB = \left(\frac{9}{8}\right)^9 \frac{\left(G_- - \left(\frac{8}{9}\right)^2 G_+\right)^3}{\left(H_+ - \left(\frac{9}{8}\right)^3 H_-\right)^4} \left(3\left(\frac{9}{8}\right)^3 G_-H_+ - 2\left(\frac{9}{8}\right)^3 G_+H_- - \frac{11}{8} G_+H_+\right). \tag{3.75}$$

Now, in view of the inequality $H_+ > \left(\frac{9}{8}\right)^3 H_-$ we have

$$-\frac{11}{8} G_+H_+ < -\frac{11}{8}\left(\frac{9}{8}\right)^3 G_+H_-. \tag{3.76}$$

Using (3.76) in (3.75) we get

$$\int_0^\omega \varphi(s)\mathrm{d}s \le 3\left(\frac{9}{8}\right)^{12} \frac{\left(G_- - \left(\frac{8}{9}\right)^2 G_+\right)^3}{\left(H_+ - \left(\frac{9}{8}\right)^3 H_-\right)^4} \left(G_-H_+ - \frac{9}{8} G_+H_-\right). \tag{3.77}$$

Now from (3.77), in view of (3.5), we obtain

$$\int_0^\omega \varphi(s)\mathrm{d}s \le \frac{3^7}{2^{12}} \cdot \frac{4\left|\int_0^\omega p(s)\mathrm{d}s\right|}{\left(\int_0^\omega |p(s)|\,\mathrm{d}s\right)^2}.$$

Thus, according to Lemma 3.3 it follows that $(p, -\varphi)$ verifies the property (P_+).

Now the conclusion follows from Theorem 2.3. □

Acknowledgments

Robert Hakl supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503 and by Junta de Andalucía, Spain, Project FQM2216; Pedro J. Torres supported by Ministerio de Educación y Ciencia, Spain, project MTM2008-02502, and by Junta de Andalucía, Spain, Project FQM2216

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(Received: January 14, 2010; Revised: April 24, 2010)