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Periodic Motion of a System of Two or Three Charged Particles[1]

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We provide necessary and sufficient conditions for the existence of T-periodic solutions of a system of second-order ordinary differential equations that models the motion of two or three collinear charged particles of the same sign. © 2000 Academic Press

1. INTRODUCTION

This paper is motivated by a previous work of Fonda *et al*. [6] in which the authors studied the behavior of a periodically forced charged particle moving on a line in a field generated by one or two fixed charged particles of the same sign. This model corresponds to a scalar ordinary differential equation with one or two singularities, and by means of critical point theory or the Poincare–Birkhoff theorem, results about existence and multiplicity of periodic and subharmonic solutions were obtained.

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Here our aim is to study a system of two or three particles in which all of them are forced but not fixed, so we have a system of two or three second-order differential equations. We provide necessary and sufficient conditions for the existence of a periodic solution of this system.

The method of proof has two main steps: first, a reduction change of variables, very usual in the n-body problem, that enables us to reduce the dimension of our system by one. After that, the case of two particles is trivial, while for the case of three particles we have a system of dimension two with a singularity on the origin that can be solved by a priori bounds and topological degree.

Forced scalar equations with singularities have been extensively studied since the 1960s (see [13] for a survey). More recently, interest in these equations has been renewed by the papers of Lazer and Solimini [11] for the conservative case and Haberts and Sanchez [9, 10] for the Liénard equation. Other results in the scalar case dealing with singularities of attractive or repulsive type, can be found in [5–7, 12–14, 16] and the references therein.

Systems with singularities have been also considered in several papers, (see, e.g., [1] and the references therein for general results in the variational setting), either with a potential structure in the singular nonlinearity [4, 10, 15] or with a hamiltonian structure [3]. We emphasize that the three particles model presented here cannot be included in any of these possibilities (note in particular, according to a remark at the end of Section 3, that our result applies equally to the damped and to the frictionless cases).

Throughout the article, we denote by $\mathbb{R}^+ = (0, +\infty)$ the set of positive real numbers and set $\overline{w} := \frac{1}{T}\int_0^T w(t)\,dt$.

2. SYSTEMS OF TWO CHARGED PARTICLES

Let us consider the system of second-order differential equations

$$\begin{cases} x'' = -Ag(y-x) + h(t) \\ y'' = Ag(y-x) + p(t), \end{cases} \tag{1}$$

where h, p are continuous T-periodic functions, $A \in \mathbb{R}$, and $g\colon (0, +\infty) \to (0, +\infty)$ is a continuous function such that

$$\lim_{x \to 0^+} g(x) = +\infty, \qquad \lim_{x \to +\infty} g(x) = 0 \tag{2}$$

and

$$\int_0^1 g(s)\, ds = +\infty. \tag{3}$$

We look for T-periodic solutions in the configuration space $\Sigma = \{(x, y): x < y\}$.

Assumption (3) over the potential of g is standard in the related literature [10, 11], and it corresponds to a "strong force" type condition [1, 8]. A possible "model case" of this type of nonlinearity is $g(x) = 1/x^2$. In this case, system (1) governs the motion of two electric charges q_1, q_2 placed on a line in x and y and subject to T-periodic (collinear) forces $h(t)$, $p(t)$ acting on each particle along the line. Then, $A = Kq_1q_2$ (where K is the Coulomb constant). Depending on the sign of the constant A, we have two different situations that are studied in the following two theorems.

THEOREM 1. *Let us suppose that $A > 0$ and let g satisfy assumptions* (2) *and* (3). *Then, a necessary and sufficient condition for the existence of a T-periodic solution of system* (1) *is that*

$$\bar{h} > 0, \qquad \bar{h} + \bar{p} = 0.$$

Proof. The necessity comes from a direct integration on the first equation and then by the addition of both equations averaged on $[0, T]$, respectively.

For the sufficiency, by the change of variables

$$\begin{aligned} s(t) &= x(t) + y(t), \\ d(t) &= y(t) - x(t), \end{aligned}$$

we obtain the equivalent decoupled system

$$s''(t) = h(t) + p(t) \tag{4}$$

$$d''(t) = 2Ag(d(t)) + p(t) - h(t). \tag{5}$$

Clearly, Eq. (4) has a T-periodic solution if and only if $\bar{h} + \bar{p} = 0$, while (5) is a well-studied equation in the related literature. For instance, in [11] it was proved that Eq. (5) has a positive T-periodic solution if and only if $\bar{p} - \bar{h} < 0$, but as $\bar{h} + \bar{p} = 0$, then $\bar{p} = -\bar{h} < 0$ and $\bar{p} - \bar{h} = -2\bar{h} < 0$. ∎

Remark. From [6, 7], we know that there exists a positive integer m_0 such that for each $m \geq m_0$, Eq. (5) has subharmonic solutions with minimal period mT. Then, this property holds for system (1) as well.

By the same argument, the following theorem is proved.

THEOREM 2. *Let us suppose that* $A < 0$ *and let* g *satisfy assumption* (2) (*but not necessarily* (3)). *Then, a necessary and sufficient condition for the existence of a* T*-periodic solution of system* (1) *is that*

$$\bar{h} < 0, \qquad \bar{h} + \bar{p} = 0.$$

Note that Theorem 1 concerns the model with charges of the same sign, while Theorem 2 deals with the case of charges of different sign. Both theorems remain valid if we assume that $h, p \in L^1_{\text{loc}}$ (and $p - h$ upper bounded in Theorem 2), the solutions being considered in the Carathéodory sense.

The sign of the coefficient A plays a crucial role in the dynamics of the solutions. Indeed, one can see that in case of constant external forcings the structure for $A > 0$ and $A < 0$ is that of a center or a saddle, respectively. The physical implication of condition (3) is that it prevents the possibility of collisions. Actually, using a result from [11] about the nonexistence of positive T-periodic solutions for (5), it is possible to prove in the case of Theorem 1 that if (3) fails there are examples with no T-periodic solutions. This problem does not occur in Theorem 2 where (3) is not needed, since the different sign of the nonlinearity $2Ag$ makes it possible to find a lower solution.

3. A SYSTEM OF THREE CHARGED PARTICLES OF THE SAME SIGN

Let us consider the system of second-order differential equations

$$\begin{cases} x_1'' = -Ag(x_2 - x_1) - Bg(x_3 - x_1) + h(t) \\ x_2'' = Ag(x_2 - x_1) - Cg(x_3 - x_2) + k(t) \\ x_3'' = Bg(x_3 - x_1) + Cg(x_3 - x_2) + p(t), \end{cases} \tag{6}$$

where A, B, C are positive constants, h, k, p are continuous T-periodic functions, and g: $(0, +\infty) \to (0, +\infty)$ is a continuous function that satisfies assumptions (2) and (3). We look for T-periodic solutions in the configuration space $\Sigma = \{(x_1, x_2, x_3) : x_1 < x_2 < x_3\}$.

In the "model case" $g(x) = (1/x^2)$, this system governs the motion of three particles x_1, x_2, x_3 placed on the real line, with charges q_1, q_2, q_3 of the same sign and with T-periodic forces $h(t), k(t), p(t)$ acting on each particle. Then, $A = Kq_1q_2$, $B = Kq_1q_3$, and $C = Kq_2q_3$.

It is easy to verify that a necessary condition for the existence of T-periodic solutions is that

$$\bar{h} > 0 > \bar{p}, \qquad \bar{h} + \bar{k} + \bar{p} = 0. \tag{7}$$

We are going to prove that it is also sufficient.

By the change of variables

$$\begin{aligned} s(t) &= x_1(t) + x_2(t) + x_3(t) \\ u(t) &= x_2(t) - x_1(t) \\ v(t) &= x_3(t) - x_2(t), \end{aligned}$$

we arrive at the equivalent system

$$s''(t) = h(t) + k(t) + p(t)$$

$$\begin{cases} u''(t) = 2Ag(u) + Bg(u+v) - Cg(v) + k(t) - h(t) \\ v''(t) = 2Cg(v) + Bg(u+v) - Ag(u) + p(t) - k(t). \end{cases} \tag{8}$$

Note that the first equation is independent of the rest of the system and has a T-periodic solution if and only if $\bar{h} + \bar{k} + \bar{p} = 0$, so we only have to prove the existence of a T-periodic solution (u, v) of subsystem (8).

Let us consider the homotopy

$$\begin{cases} u''(t) = 2Ag(u) + \lambda Bg(u+v) - Cg(v) + k_\lambda(t) - h_\lambda(t) \\ v''(t) = 2Cg(v) + \lambda Bg(u+v) - Ag(u) + p_\lambda(t) - k_\lambda(t) \end{cases} \tag{9}$$

with $h_\lambda(t) = \bar{h} + \lambda \tilde{h}(t)$, $k_\lambda(t) = \bar{k} + \lambda \tilde{k}(t)$, $p_\lambda(t) = \bar{p} + \lambda \tilde{p}(t)$, and $\lambda \in [0,1]$, where, as usual, we denote by $\tilde{w}(t) = w(t) - \bar{w}$ the component with mean value zero of a T-periodic function $w(t)$.

LEMMA 3. *Assume that there exists a compact set $K \subset (\mathbb{R}^+)^2$ such that $(u(t), v(t)) \in K$, $\forall t \in [0, T]$ and for every T-periodic solution (u, v) of* (9), *with $0 \leq \lambda \leq 1$. Let us define $F\colon (\mathbb{R}^+)^2 \to \mathbb{R}^2$ as*

$$F(u,v) = \big(2Ag(u) - Cg(v) + \bar{k} - \bar{h}, 2Cg(v) - Ag(u) + \bar{p} - \bar{k}\big)$$

and suppose that

$$\deg_B(F, \Omega, 0) \neq 0$$

for some open and bounded set Ω containing K. Then, there exists a T-periodic solution of (8).

The proof of this result is a consequence of the continuation theorems in [2] (see [2, Corollaries 4 and 6]). Now, in order to prove our existence result, first we find a priori bounds.

PROPOSITION 4. *Suppose that* (2), (3), *and* (7) *hold. Then, there exists some* $\epsilon > 0$ *such that*

$$\epsilon < u(t), v(t) < \frac{1}{\epsilon}, \qquad \forall t \in [0, T]$$

for any T-periodic solution (u, v) *of the homotopic system* (9), *with* $\lambda \in [0, 1]$.

Proof. We start by integrating both equations over a period and find

$$2A\int_0^T g(u)\,dt + \lambda B\int_0^T g(u+v)\,dt - C\int_0^T g(v)\,dt + \bar{k}T - \bar{h}T = 0, \quad (10)$$

$$2C\int_0^T g(v)\,dt + \lambda B\int_0^T g(u+v)\,dt - A\int_0^T g(u)\,dt + \bar{p}T - \bar{k}T = 0. \quad (11)$$

If (10) is multiplied by 2 and added to (11), one obtains

$$3A\int_0^T g(u)\,dt + 3\lambda B\int_0^T g(u+v)\,dt + \left(\bar{p} + \bar{k} - 2\bar{h}\right)T = 0,$$

but $\bar{p} + \bar{k} = -\bar{h}$ from (7), so we get

$$A\int_0^T g(u)\,dt + \lambda B\int_0^T g(u+v)\,dt = \bar{h}T > 0. \quad (12)$$

An analogous argument leads to

$$C\int_0^T g(v)\,dt + \lambda B\int_0^T g(u+v)\,dt = -\bar{p}T > 0. \quad (13)$$

Now, we use hypothesis (2) in order to take $\psi_1 > 0$ such that

$$g(x) < \frac{\bar{h}}{A+B}, \qquad \forall x \geq \psi_1.$$

Note that this is possible because $\bar{h}, A, B > 0$. If $u(t) \geq \psi_1$ for all $t \in [0, T]$, then

$$Ag(u(t)) + \lambda Bg(u(t) + v(t)) < \frac{A\bar{h}}{A+B} + \frac{\lambda B\bar{h}}{A+B} \leq \bar{h}, \qquad t \in [0, T],$$

and integrating we obtain a contradiction with (12). Hence, there exist some t_1 such that $u(t_1) < \psi_1$.

On the other hand, let us fix $\psi_2 < \psi_1$ such that

$$g(x) > \frac{\bar{h}}{A}, \qquad \forall x \in (0, \psi_2].$$

If $u(t) \leq \psi_2$ for all $t \in [0, T]$, then

$$Ag(u(t)) + \lambda Bg(u(t) + v(t)) \geq Ag(u(t)) > \bar{h} \qquad t \in [0, T],$$

and integrating we obtain another contradiction with (12), so that there exists some t_2 with $u(t_2) > \psi_2$. Finally, by using the continuity of u, we can conclude that there exists $\bar{t} \in [0, T]$ such that

$$\psi_2 < u(\bar{t}) < \psi_1.$$

A completely symmetric argument provides some ψ_3, ψ_4 not depending on (u, v) and λ, such that

$$\psi_4 < v(\hat{t}) < \psi_3$$

for some $\hat{t} \in [0, T]$.

Let us return to (12) and (13). We have that

$$A\int_0^T g(u)\, dt \leq \bar{h}T, \qquad \lambda B\int_0^T g(u+v)\, dt < \bar{h}T, \qquad C\int_0^T g(v)\, dt \leq -\bar{p}T. \tag{14}$$

Therefore, taking absolute values in the homotopic system and integrating on $[0, T]$ we get

$$\|u''\|_1 \leq 4\bar{h}T - \bar{p}T + |\bar{k}|T + \|\tilde{k}\|_1 + \|\tilde{h}\|_1 =: M_1$$

and

$$\|v''\|_1 \leq -4\bar{p}T + \bar{h}T + |\bar{k}|T + \|\tilde{p}\|_1 + \|\tilde{k}\|_1 =: M_2.$$

If $u(t_*) = \min\{u(t) \colon t \in [0, T]\}$ and we take t with $t_* \leq t \leq t_* + T$, we obtain

$$|u'(t)| = \left|\int_{t_*}^t u''(s)\, ds\right| \leq \int_{t_*}^t |u''(s)|\, ds \leq \|u''\|_1 \leq M_1,$$

so $\|u'\|_\infty \leq M_1$, and by the same reasoning $\|v'\|_\infty \leq M_2$. Accordingly, for $\tilde{t} \leq t \leq \tilde{t} + T$,

$$u(t) - u(\tilde{t}) = \int_{\tilde{t}}^t u'(s)\, ds \leq \int_{\tilde{t}}^t |u'(s)|\, ds \leq T\|u'\|_\infty \leq TM_1.$$

Similarly, for $\hat{t} \leq t \leq \hat{t} + T$,

$$v(t) - v(\hat{t}) = \int_{\hat{t}}^{t} v'(s)\, ds \leq \int_{\hat{t}}^{t} |v'(s)|\, ds \leq T\|v'\|_\infty \leq TM_2$$

and finally we conclude that

$$u(t) \leq TM_1 + u(\tilde{t}) < TM_1 + \psi_1,$$
$$v(t) \leq TM_2 + u(\hat{t}) < TM_2 + \psi_3$$

for all $t \in [0, T]$.

Now, we find a priori bounds from below. Let us define

$$w_1(t) = \lambda Bg(u(t) + v(t)) - Cg(v(t)) + k_\lambda(t) - h_\lambda(t).$$

This function is bounded in the L^1-norm by the estimates in (14). In fact,

$$\|w_1\|_1 \leq 2\bar{h}T - \bar{p}T + |\bar{k}|T + \|\tilde{k}\|_1 + \|\tilde{h}\|_1 =: W_1.$$

Now, multiplying the first equation of (9) by u' and integrating on $[\tilde{t}, t]$, we have

$$\frac{u'(t)^2}{2} - \frac{u'(\tilde{t})^2}{2} - 2A\int_{\tilde{t}}^{t} g(u(s))u'(s)\, ds = \int_{\tilde{t}}^{t} w_1(s)u'(s)\, ds$$

and then

$$2A\int_{u(t)}^{u(\tilde{t})} g(s)\, ds \leq \int_{\tilde{t}}^{t} |w_1(s)u'(s)|\, ds + \frac{u'(\tilde{t})^2}{2}$$
$$\leq \|w_1\|_1\|u'\|_\infty + \frac{M_1^2}{2} \leq W_1M_1 + \frac{M_1^2}{2}.$$

By assumption (3), it is possible to take $\epsilon_1 > 0$ such that

$$\int_{\epsilon_1}^{\psi_2} g(s)\, ds \geq \frac{1}{2A}\left(W_1M_1 + \frac{M_1^2}{2}\right).$$

Then,

$$\int_{u(t)}^{u(\tilde{t})} g(s)\, ds \leq \int_{\epsilon_1}^{\psi_2} g(s)\, ds$$

and as $u(\tilde{t}) > \psi_2$, we conclude that

$$u(t) > \epsilon_1, \qquad \forall t \in [0, T].$$

An analogous argument can be applied to v, defining

$$w_2(t) = \lambda Bg(u(t) + v(t)) - Ag(u(t)) + p_\lambda(t) - k_\lambda(t),$$

so we find $\epsilon_2 > 0$ such that

$$v(t) > \epsilon_2, \qquad \forall t \in [0, T].$$

In conclusion, the proof is completed by taking

$$\epsilon = \min\left\{ \epsilon_1, \epsilon_2, \frac{1}{TM_1 + \psi_1}, \frac{1}{TM_2 + \psi_3} \right\}.$$

∎

Note that the presence of a priori bounds is a remarkable difference from the situation in [6].

Finally, we prove our existence result computing the degree.

THEOREM 5. *The condition*

$$\bar{h} > 0 > \bar{p} \qquad \text{and} \qquad \bar{h} + \bar{k} + \bar{p} = 0$$

is necessary and sufficient for the existence of a T-periodic solution of system (6).

Proof. We only have to prove that

$$\deg_B(F, \Omega, 0) \neq 0$$

for $F: (\mathbb{R}^+)^2 \to \mathbb{R}^2$ defined by

$$F(u, v) = \big(2Ag(u) - Cg(v) + \bar{k} - \bar{h}, 2Cg(v) - Ag(u) + \bar{p} - \bar{k}\big).$$

To this purpose, we do a convex homotopy between F and some F_0: $(\mathbb{R}^+)^2 \to \mathbb{R}^2$ defined by

$$F_0(u, v) = \big(2Ag_0(u) - Cg_0(v) + \bar{k} - \bar{h}, 2Cg_0(v) - Ag_0(u) + \bar{p} - \bar{k}\big),$$

where g_0: $(0, +\infty) \to (0, +\infty)$ is a continuous strictly decreasing function satisfying (2) and such that

$$g_0(x) < g(x) \qquad \forall x \in (0, +\infty). \tag{15}$$

Then, the respective degrees are the same (maybe with a greater Ω) if we find a priori bounds for the solutions of

$$\lambda F(u, v) + (1 - \lambda) F_0(u, v) = 0, \qquad \lambda \in [0, 1];$$

that is,

$$\begin{cases} 2A(\lambda g(u) + (1-\lambda) g_0(u)) - C(\lambda g(v) + (1-\lambda) g_0(v)) + \bar{k} - \bar{h} = 0 \\ 2C(\lambda g(v) + (1-\lambda) g_0(v)) - A(\lambda g(u) + (1-\lambda) g_0(u)) + \bar{p} - \bar{k} = 0. \end{cases} \tag{16}$$

If these equations are added, we find

$$A(\lambda g(u) + (1-\lambda) g_0(u)) + C(\lambda g(v) + (1-\lambda) g_0(v)) + \bar{p} - \bar{h} = 0,$$

so that

$$\lambda g(u) + (1-\lambda) g_0(u) < \frac{\bar{h} - \bar{p}}{A}$$

and by using (15),

$$g_0(u) < \frac{\bar{h} - \bar{p}}{A}.$$

But g_0 is strictly decreasing and satisfies (2); hence, there exists the inverse g_0^{-1} and

$$u > g_0^{-1}\left(\frac{\bar{h} - \bar{p}}{A}\right).$$

On the other hand, if we multiply the first equation of (16) by 2 and then we add it to the second one, we find

$$3A(\lambda g(u) + (1-\lambda) g_0(u)) = 3\bar{h};$$

that is,

$$g(u) - (1-\lambda)(g(u) - g_0(u)) = \frac{\bar{h}}{A}.$$

Hence, $g(u) \geq \frac{\bar{h}}{A} > 0$, and, using (2), we conclude that there exists some fixed $M > 0$ such that $u < M$.

By symmetric arguments, we get a priori bounds for v. Therefore, we have proved that

$$\deg_B(F, \Omega, 0) = \deg_B(F_0, \Omega, 0),$$

but it is easy to see that this last degree is not zero, making the change of variables

$$r = g_0(u), \qquad s = g_0(v)$$

and taking into account that

$$\det\begin{pmatrix} 2A & -C \\ -A & 2C \end{pmatrix} = AC\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \neq 0.$$

∎

Clearly Theorem 5 remains valid if we assume that h, k, $p \in L^1_{\mathrm{loc}}$, the solutions being considered in the Carathéodory sense.

We also note that all the results of this paper (except the remark after Theorem 1) are true if we add to the ith equation, a "friction term" of the form cx_i', with c a constant. In this case, the left-hand side of system (8) contains the terms $u'' + cu'$ and $v'' + cv'$ and Proposition 4 is proved, putting λc in the homotopy.

It could be interesting to ask if there is an analog of Theorem 2 for the three particle case. Results in this direction will be discussed elsewhere.

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