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NONDEGENERACY OF THE PERIODICALLY FORCED LIÉNARD DIFFERENTIAL EQUATION WITH ϕ-LAPLACIAN

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New results on the existence of periodic solutions of a forced Liénard differential equation with ϕ-Laplacian are provided. The method of proof relies on the Schauder fixed point theorem, so some information on the location of the solutions is also obtained, leading to multiplicity results. The flexibility of this approach is tested by comparing our results with some examples taken from the related literature, including the classical pendulum equation.

Keywords: Periodic solution; ϕ-Laplacian; Liénard equation; Schauder fixed point theorem.

Mathematics Subject Classification 2010: 34C25

1. Introduction

The purpose of this paper is to investigate the existence and multiplicity of T-periodic solutions for the equation

$$(\phi(x'))' + h(x)x' + g(x) = e(t) + s \tag{1.1}$$

where h, g are continuous functions, $e \in L^1[0,T], s \in \mathbb{R}$ is a parameter and $\phi :]-a, a[\rightarrow]-b, b[$ is an increasing homeomorphism with $\phi(0) = 0$ and $0 < a, b \leq +\infty$. Following the related literature, a ϕ-Laplacian operator is said *singular* when the domain of ϕ is finite (that is, $a < +\infty$), on the contrary the operator is said *regular*. On the other hand we say that ϕ is *bounded* if its range is finite (that is, $b < +\infty$) and *unbounded* in other case. There are three paradigmatic models in this context:

- $a = b = +\infty$ (Regular unbounded): The p-Laplacian operator

$$\phi_p(x) = |x|^{p-2}x, \quad \text{with } p > 1.$$

- $a < +\infty, b = +\infty$ (Singular unbounded): The relativistic operator

$$\phi_r(x) = \frac{x}{\sqrt{1 - x^2}}.$$

- $a = +\infty, b < +\infty$ (Regular bounded): The one-dimensional mean curvature operator

$$\phi_c(x) = \frac{x}{\sqrt{1 + x^2}}.$$

The number of references concerning the p-Laplacian is huge (only to mention some of them, see [9, 14–16], their references and the papers citing these ones). On the other hand, the interest on periodic equations with a general ϕ-Laplacian has increased with the recent publication of some interesting papers [1–5].

Let us describe the main topic of this paper. Assume that the mean value of e is zero, that is, $\overline{e} = \frac{1}{T}\int_0^T e(t)dt = 0$. Then, the general problem is the description of the set I_e of mean values s for which Eq. (1.1) has at least a T-periodic solution, which we call the *solvability set*. This is an innocent but ambitious question. An important problem is to determine sufficient conditions for such set to be nondegenerate, that is, not reduced to a single point. In the framework of the classical forced pendulum equation, it is known as the *degeneracy problem* and it is still open (for the state of the art, see the review [8] and its references).

Our purpose is to obtain sufficient conditions such that there exist periodic solutions of (1.1) for a nondegenerate interval of s, giving explicit estimates of such interval. The method of proof is inspired in a simple idea exposed in [13] and it is composed of two steps: a nonstandard change of variables and an application of the Schauder fixed point theorem. The flexibility of the method is tested with several examples taken from the literature. We pay special attention to the pendulum equation, for which new results are obtained even in the classical case $\phi = Id$. Other generalizations like the study of systems of equations or the application to equations with singularities are technically possible but should be developed elsewhere.

Some notation is needed. Let C_T be the Banach space of the T-periodic and continuous functions. The space C_T can be decomposed as $C_T = \mathbb{R} \oplus \tilde{C}_T$ where $\tilde{C}_T$ is the space of the T-periodic and continuous functions with zero mean value. In the same way, $L^1[0,T] = \mathbb{R} \oplus \tilde{L}^1[0,T]$. From now on, it is assumed that $e \in \tilde{L}^1[0,T]$.

2. The Fixed Point Formulation

The aim of this section is to write the problem of finding a T-periodic solution of (1.1) as a fixed point problem for a suitable operator. To this purpose, we will assume as a key hypothesis that there exist real numbers $R_1 < R_2$ such that

$$\text{(H1}^+\text{)} \quad g \in C^1([R_1, R_2]) \quad \text{and} \quad g'(x) > 0 \quad \text{for all } x \in [R_1, R_2].$$

The first step is to perform the formal change of variables

$$y = g(x) - s, \tag{2.1}$$

which is well-defined from the interval $[R_1, R_2]$ to $[g(R_1) - s, g(R_2) - s]$. Then, the original equation (1.1) is transformed into

$$\phi\left(\frac{y'}{g'(g^{-1}(y+s))}\right)' + h(g^{-1}(y+s))\frac{y'}{g'(g^{-1}(y+s))} + y(t) = e(t). \tag{2.2}$$

An integration gives

$$\phi\left(\frac{y'}{g'(g^{-1}(y+s))}\right) = \int_0^t (e(\tau) - y(\tau))d\tau - H(g^{-1}(y+s)) + C,$$

where H is a primitive of the function h and C is a constant to be fixed later. For convenience, let us define

$$E(t) = \int_0^t e(\tau)d\tau$$

and

$$F[y](t) = \int_0^t y(\tau)d\tau - E(t) + H(g^{-1}(y+s)).$$

Then, we have

$$y' = g'(g^{-1}(y+s))\phi^{-1}(-F[y](t) + C).$$

Finally, a new integration gives

$$y(t) = \int_0^t g'(g^{-1}(y+s))\phi^{-1}(-F[y](\tau) + C)d\tau + D.$$

The following step is to choose adequately the constants C, D in the latter formula. Let us define the closed and convex set

$$K = \{y \in \tilde{C}_T : y(t) \in [g(R_1) - s, g(R_2) - s]\}.$$

Lemma 1. *Let us assume that the* ϕ*-Laplacian operator is unbounded* (*that is*, $\operatorname{Range}\phi = \mathbb{R}$). *For any* $y \in K$, *there exists a unique choice of* C_y, D_y (*depending continuously on* y) *such that*

$$\mathcal{T}[y](t) \equiv \int_0^t g'(g^{-1}(y+\overline{p}))\phi^{-1}(-F[y](\tau) + C_y)d\tau + D_y \in \tilde{C}_T. \tag{2.3}$$

Moreover, $C_y \in [A_y, B_y]$, *where*

$$A_y = \min_{t\in[0,T]}\{F[y](t)\}, \quad B_y = \max_{t\in[0,T]}\{F[y](t)\}.$$

Proof. The proof resembles [3, Lemma 1]. The periodicity is equivalent to

$$\int_0^T g'(g^{-1}(y+s))\phi^{-1}(-F[y](\tau) + C_y)d\tau = 0.$$

As a function of C_y, the left-hand side of this equation is continuous and increasing (remember we have fixed the positive sign for g' in $[R_1, R_2]$). Since $\phi^{-1}(-F[y](\tau) + A_y) \le 0 \le \phi^{-1}(-F[y](\tau) + B_y)$ for all τ, the existence of a unique solution $C_y \in [A_y, B_y]$ for such equation follows from a basic application of Bolzano's Theorem.

Once C_y is fixed, D_y is given by

$$D_y = -\frac{1}{T}\int_0^T\int_0^t g'(g^{-1}(y+s))\phi^{-1}(-F[y](\tau)+C_y)d\tau dt. \qquad \square$$

Next lemma gives some uniform bounds for C_y.

Lemma 2. *Let us define the constants*

$$A \equiv T(g(R_1)-s) - \max_t E(t) + \min_{x\in[R_1,R_2]} H(x),$$

$$B \equiv T(g(R_2)-s) - \min_t E(t) + \max_{x\in[R_1,R_2]} H(x).$$

Then, for any $y \in K$,

$$A \le A_y \le B_y \le B.$$

Proof. The proof is direct from the definition of A_y, B_y and the monotonicity of g. $\square$

Lemma 3. *Let us assume that the* ϕ*-Laplacian operator is bounded* (*that is*, $\operatorname{Range}\phi =]-b, b[$). *Suppose that* R_1, R_2 *are such that* $B - A < b$. *Then, for any* $y \in K$, *there exists a unique choice of* C_y, D_y *such that*

$$\mathcal{T}[y](t) \equiv \int_0^t g'(g^{-1}(y+\overline{p}))\phi^{-1}(-F[y](t)+C_y)ds + D_y \in \tilde{C}_T. \tag{2.4}$$

Moreover, $C_y \in [A, B]$.

Proof. It is similar to that of Lemma 1. Again, the periodicity is equivalent to

$$\int_0^T g'(g^{-1}(y+s))\phi^{-1}(-F[y](\tau)+C_y)d\tau = 0.$$

By Lemma 2,

$$-B + A \le -F[y](\tau) + A \le 0 \le -F[y](\tau) + B \le B - A,$$

for all τ. Moreover, by the hypothesis $B - A < b$, $\phi^{-1}(-F[y](\tau)+C)$ is well defined for all $C \in [A, B]$. The proof is finished as in Lemma 1. $\square$

Therefore, we have a well-defined functional $\mathcal{T} : K \to \tilde{C}_T$. It is easy to prove that it is continuous and compact as a basic application of Ascoli–Arzela Theorem. By ($H1^+$), the change is reversible, so to find a fixed point $y \in K$ of $\mathcal{T}$ is equivalent to find a T-periodic solution of the original equation (1.1).

Finally, let us observe that under the reciprocal assumption

$$(\mathrm{H1}^-) \quad g \in C^1([R_1, R_2]) \quad \text{and} \quad g'(x) < 0 \quad \text{for all } x \in [R_1, R_2].$$

an analogous fixed point problem can be formulated, with the evident changes. In this case, the set is

$$K = \{y \in \tilde{C}_T : y(t) \in [g(R_2) - s, g(R_1) - s]\},$$

and the involved constants read

$$\tilde{A} \equiv T(g(R_2) - s) - \max_t E(t) + \min_{x\in[R_1,R_2]} H(x),$$

$$\tilde{B} \equiv T(g(R_1) - s) - \min_t E(t) + \max_{x\in[R_1,R_2]} H(x).$$

3. The Unbounded Case

In this section we assume $\operatorname{Range}\phi = \mathbb{R}$. From now on, we will denote

$$\delta E = \max_t E(t) - \min_t E(t).$$

The main result of this section is the following one.

Theorem 1. *Let us assume that there exist real numbers* $R_1 < R_2$ *such that* (H1$^+$) *holds and*

(H2) $\frac{T}{\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(B - A) \leq |g(R_2) - g(R_1)|$.

Then, for every

$$s \in \left[g(R_1) + \frac{T}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(B - A), g(R_2) - \frac{T}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(B - A)\right],$$

there exists at least one T*-periodic solution of Eq.* (1.1) *belonging to* $[R_1, R_2]$.

The constants A, B were defined in Lemma 2. Note that $B - A$ does not depend on s, more concretely,

$$B - A = T(g(R_2) - g(R_1)) + \delta E + \max_{x\in[R_1,R_2]} H(x) - \min_{x\in[R_1,R_2]} H(x).$$

Proof. By Lemma 1, we have a well-defined continuous and compact functional $\mathcal{T} : K \to \tilde{C}_T$ and we look for a Fixed Point. By the Schauder Fixed Point Theorem, it is sufficient to prove that $\mathcal{T}(K) \subset K$.

Let us recall the well known Sobolev's inequality,

$$\|u\|_\infty^2 \leq \frac{T}{12}\|u'\|_2^2,$$

for all $u \in \tilde{C}_T$ with continuous u' (see for instance [7, p. 25]). Given $y \in K$,

$$\|\mathcal{T}[y]\|_\infty \leq \frac{\sqrt{T}}{2\sqrt{3}}\|\mathcal{T}[y]'\|_2 \leq \frac{T}{2\sqrt{3}}\|\mathcal{T}[y]'\|_\infty \leq \frac{T}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(B - A).$$

By using the condition over s,

$$g(R_1) - s \leq \mathcal{T}[y](t) \leq g(R_2) - s,$$

for all t. Therefore, $\mathcal{T}[y] \in K$ and the proof is done. ◻

Analogously, it is proved the following theorem.

Theorem 2. *Let us assume that there exist real numbers* $R_1 < R_2$ *such that* (H1$^-$) *holds and*

(H2) $\frac{T}{\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(\tilde{B}-\tilde{A}) \leq |g(R_2)-g(R_1)|$.

Then, for every

$$s \in \left[g(R_2) + \frac{T}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(\tilde{B}-\tilde{A}), g(R_1) \right.$$
$$\left. - \frac{T}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(\tilde{B}-\tilde{A}) \right],$$

there exists at least one T*-periodic solution of Eq.* (1.1) *belonging to* $[R_1, R_2]$.

In this case,

$$\tilde{B} - \tilde{A} = T(g(R_1) - g(R_2)) + \delta E + \max_{x\in[R_1,R_2]} H(x) - \min_{x\in[R_1,R_2]} H(x).$$

Note that in both theorems above, (H2) is just the required condition for the interval to be non-empty.

3.1. *Some examples for the regular case*

In this subsection the main results given before are illustrated with some examples with a regular unbounded operator ϕ, that is, $a = b = +\infty$, which include the p-Laplacian ϕ_p.

First, we consider the pendulum equation

$$(\phi(x'))' + cx' + k\sin x = e(t) + s \tag{3.1}$$

with $c \geq 0, k > 0$.

Theorem 3. *Let us assume*

$$\frac{T}{\sqrt{3}}\phi^{-1}(2T + c\pi + \delta E) < 2.$$

Then, for every

$$s \in \left] -k + \frac{kT}{2\sqrt{3}}\phi^{-1}(2T + c\pi + \delta E), k - \frac{kT}{2\sqrt{3}}\phi^{-1}(2T + c\pi + \delta E) \right[,$$

Eq. (3.1) *possesses two different solutions* x_1, x_2 *which verify* $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$.

Proof. Take $R_1 = -\frac{\pi}{2} + \epsilon$ and $R_2 = \frac{\pi}{2} - \epsilon$, with $\epsilon > 0$ small enough so that

$$\frac{T}{\sqrt{3}}\phi^{-1}(2T + c\pi + \delta E) < \sin\left(\frac{\pi}{2} - \epsilon\right) - \sin\left(-\frac{\pi}{2} + \epsilon\right) < 2$$

and

$$B - A = T\Big(\sin\Big(\frac{\pi}{2} - \epsilon\Big) - \sin\Big(-\frac{\pi}{2} + \epsilon\Big)\Big) + c(\pi - 2\epsilon) + \delta E < 2T + c\pi + \delta E.$$

It is easy to verify the conditions of Theorem 1, so we get x_1. Analogously, the second solution x_2 is obtained by applying Theorem 2 with $R_1 = \frac{\pi}{2} + \epsilon$ and $R_2 = \frac{3\pi}{2} - \epsilon$. □

Up to my knowledge, this result is new even for the classical pendulum equation

$$x'' + cx' + k \sin x = e(t) + s. \tag{3.2}$$

Observe that $s = 0$ is included in our interval of solvability I_e. In [10], it proved that (3.2) is non-degenerate and $s = 0 \in I_e$ under the condition

$$T\|e\|_2 < c\pi\sqrt{3}. \tag{3.3}$$

In our case, if

$$T(2T + c\pi + \delta E) < 2\sqrt{3}, \tag{3.4}$$

we arrive to the same conclusion, giving in addition an explicit interval for solvability. Both assumptions are independent, but (3.4) has the advantage that it is valid for the undamped case $c = 0$.

For the damped case $c > 0$, it was proved in [12] that for any positive constants k and T, there exists $e \in \tilde{C}_T$ such that Eq. (3.1) has no T-periodic solutions for $s = 0$. Hence our result can be seen as a partial counterpart of the previous one.

Apart from the pendulum equation, many other corollaries can be derived from our main theorem. We have selected two examples.

Corollary 1. *Let us assume the following conditions*:

(i) *There exists* $M > 0$ *such that* $0 < g'(x) < M$ *for all* $x \in \mathbb{R}$.
(ii) $\lim_{|x|\to+\infty} |g(x)| = +\infty$.
(iii) $\lim_{|x|\to+\infty} \frac{\phi(x)}{x} = +\infty$.

Then, *the equation*

$$(\phi(x'))' + cx' + g(x) = e(t) + s$$

has at least one T*-periodic solution for any choice of* $c, s \in \mathbb{R}, e \in \tilde{L}^1[0, T]$.

Corollary 2. *If* $2 < q < p$, *the equation*

$$(\phi_p(x'))' + cx' + \phi_q(x) = e(t) + s$$

has at least one T*-periodic solution for any choice of* $c, s \in \mathbb{R}, e \in \tilde{L}^1[0, T]$.

It is interesting to compare these results with those available in the literature. In both cases, the proof is just an asymptotic analysis by taking $R_2 = -R_1 \to +\infty$, then the solvability interval becomes unbounded.

3.2. *The singular case*

In this subsection, we analyze the singular case $\phi :]-a, a[\to \mathbb{R}$ with $a < +\infty$ as in the relativistic operator ϕ_r. In the case of a singular unbounded ϕ-Laplacian, by using the results in [3] it is easy to prove that I_e is a non-empty interval: [3, Lemma 4] gives at least one point belonging to I_e, whereas the upper and lower solutions theory given in that paper shows that it is indeed an interval. A direct application of Theorem 1 provides conditions for I_e to be non-degenerate *not depending on e*.

Theorem 4. *Let us assume that there exist real numbers $R_1 < R_2$ such that* (H1$^+$) *holds and*

$$\frac{aT}{\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)| \le |g(R_2) - g(R_1)|.$$

Then, for every $e \in \tilde{L}^1[0,T]$ and

$$s \in \left[g(R_1) + \frac{aT}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|, g(R_2) - \frac{aT}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)| \right],$$

Eq. (1.1) *has a T-periodic solution belonging to $[R_1, R_2]$.*

The proof is immediate from Theorem 1 since $\phi^{-1}(B - A) < a$. Besides, by Theorem 2 the result holds true if (H1$^+$) is replaced by (H1$^-$) and R_1, R_2 are interchanged in the interval. It is interesting to note that no assumption over the friction coefficient $h(x)$ is required.

A direct consequence for the relativistic pendulum equation is the following corollary, which improve [13].

Corollary 3. *Let us assume that $aT < 2\sqrt{3}$. Then, for every $e \in \tilde{L}^1[0,T]$ and*

$$|s| < k\left(1 - \frac{aT}{2\sqrt{3}}\right),$$

the relativistic pendulum equation

$$\phi(x')' + h(x)x' + k \sin x = e(t) + s$$

possesses two different solutions x_1, x_2 which verify $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$.

4. The Bounded Case

The analysis of a bounded ϕ-Laplacion operator $\phi : \mathbb{R} \to]-b, b[$ with $b < +\infty$ is in general more difficult and an additional condition is required.

Theorem 5. *Let us assume that there exist real numbers $R_1 < R_2$ such that* (H1$^+$) (*respectively* (H1$^-$)), (H2) *are verified and*

$$B - A < b \quad (\textit{respectively } \tilde{B} - \tilde{A} < b).$$

Then, the conclusion of Theorem 1 (*respectively Theorem* 2) *holds true.*

Note that the additional condition assures that $\phi^{-1}(B - A)$ is well-defined, therefore the proof remains unchanged.

To illustrate this result, a corollary for the pendulum equation with the curvature operator is easily deduced.

Corollary 4. *Let us consider the equation*

$$\phi_c(x')' + cx' + k \sin x = e(t) + s,$$

where ϕ_c is the curvature operator defined in the introduction. Then, if

$$2T + c\pi + \delta E < \phi_c\left(\frac{2\sqrt{3}}{T}\right) := \frac{2\sqrt{3}}{\sqrt{T^2 + 12}},$$

the previous equation has at least two T periodic solutions for any

$$|s| < k\left(1 - \frac{T}{2\sqrt{3}}\phi_c^{-1}(2T + c\pi + \delta E)\right).$$

Here it is pertinent to observe that $\phi_c^{-1} \equiv \phi_r$.

We conclude the paper by considering an example taken from [4]. In Example 3 of that paper, the authors consider the equation

$$(\phi_c(x'))' + k \exp(-x^2) = e(t) + s \tag{4.1}$$

and prove that if $\|e\|_\infty \leq k < \frac{1}{2T}$ then there is s^* such that the equation has zero, at least one or at least two T-periodic solutions according to $s \notin (0, s^*]$, $s = s^*$ or $s \in (0, s^*)$. By applying our method, we get the following independent result.

Corollary 5. *If*

$$kT + \delta E < \phi_c\left(\frac{k \exp(1/4)\sqrt{3}}{T}\right),$$

and

$$s \in \left]\frac{T}{2\sqrt{3}}\phi_c^{-1}(kT + \delta E), k - \frac{T}{2\sqrt{3}}\phi_c^{-1}(kT + \delta E)\right[,$$

then Eq. (4.1) *has at least two solutions* (*one positive and one negative*).

For the proof, just take $R_1 = -1/\epsilon, R_2 = -\epsilon$ for ϵ small enough to find the negative solution, and the positive one is found by reversing the signs in R_1, R_2.

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