Advanced Nonlinear Studies **10** (2010), 927–938

Periodic Solutions of a Singular Equation With Indefinite Weight

José Luis Bravo*

Departamento de Matemáticas,
Universidad de Extremadura, 06071 Badajoz, Spain
e-mail: trinidad@unex.es

Pedro J. Torres†

Departamento de Matemática Aplicada,
Universidad de Granada, 18071 Granada, Spain.
e-mail: ptorres@ugr.es

Received in revised form 18 December 2009
Communicated by Rafael Ortega

Abstract

Motivated by some relevant physical applications, we study the existence and uniqueness of T-periodic solutions for a second order differential equation with a piecewise constant singularity which changes sign. Other questions like the stability and robustness of the periodic solution are considered.

1991 Mathematics Subject Classification. 34C25.

Key words. Periodic solution, singular equation, linear stability.

1 Introduction

The purpose of this paper is to investigate the existence of T-periodic solutions for the equation

$$x'' = \frac{a(t)}{x^3} \tag{1.1}$$

*The author acknowledges the support of a MCYT/FEDER grant number MTM2008-05460

†The author acknowledges the support of a MCYT/FEDER grant number MTM2008-02502 and project FQM2216 (Junta de Andalucía).

where a is a T-periodic function given by

$$a(t) = \begin{cases} a_+ & \text{if } 0 \leq t < t_+ \\ -a_- & \text{if } t_+ \leq t < T := t_+ + t_-, \end{cases} \tag{1.2}$$

with $a_+, a_- > 0$. This problem plays an important role in at least three different physical contexts:

Stabilization of matter-wave breathers in Bose-Einstein condensates: A Bose-Einstein condensate (BEC) is a special state achieved by a boson gas in the zero-temperature limit. The recent review [2] gives a detailed description of this physical model with a complete bibliography. The mean field dynamics of a BEC is described by a nonlinear Schrödinger equation with a cubic term, also known in this context as the Gross-Pitaevskii equation

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\triangle\psi + g|\psi|^2\psi, \tag{1.3}$$

where g is proportional to the s-wave scattering length and measures the type of interactions between bosons (attractive if $g > 0$, repulsive if $g < 0$). The possibility to generate breathers (that is, spatially-located solutions which are periodic in time) by modulating the nonlinear term with alternating periods of attractive and repulsive interaction was first considered in [21]. Mathematically, this modulation is equivalent to considering the parameter g as a sign-changing function $g(t)$. Experimentally, this effect can be obtained by the so-called Feshbach-resonance management [5, 6]. Recently, a square-wave control like (1.2) has been explicitly considered in [19].

A fruitful approach to study eq. (1.3) is the method of moments [14]. The idea is to define a set of integral quantities which are physically relevant and characterize the solution of the partial differential equation. Then, such quantities are related between them by a closed set of ordinary differential equations. For instance, the width of the atom cloud is given by

$$I_2(t) = \int |\psi|^2 \, r^2 d^n x,$$

where n is the spatial dimensionality. When $n = 2$, $x(t) = \sqrt{I_2(t)}$ is a solution of (1.1). See [13, 14] for more details. It is interesting to remark that as an approximation of the full problem, (1.1) can be obtained by variational techniques as well. The paper [8] reviews and compares different approximations presented in the related literature.

Propagation of guided waves in optical fibers [23, 18, 3]: We consider a light beam propagating along an optical medium composed by alternating layers of self-focusing and self-defocusing Kerr material composing a periodic structure. If the axis z is the direction of propagation, the model for the wave evolution is again the nonlinear Schrödinger equation with cubic term

$$i\frac{\partial \psi}{\partial z} = -\frac{1}{2}\triangle\psi + \gamma(z)|\psi|^2\psi, \tag{1.4}$$

where γ is a piecewise constant function whose sign determine the self-focusing or self-defocusing character of the layer. By using either the method of moments or either a variational approach, the existence of guided waves propagating along the fiber leads is equivalent (up to some approximations, see again [14, 8] for a critical review) to the existence of a periodic solution of (1.1), where we have identified $z \leftrightarrow t$.

Electromagnetic trapping of a neutral atom near a charged wire: The reference [7] describes a trapping mechanism for a neutral atom in the vicinity of a wire charged by a time-varying sinusoidal voltage. Later, such a model has been studied in [9, 12]. If the voltage is piecewise constant, the dynamics of the model is described by eq. (1.1).

Evidently, the parameters a_+, a_- depend closely on the nature of each particular model under consideration (for instance, the specific type of gas used in the BEC or the particular optical material which conform the layered structure). Once these parameters are fixed, a question of particular interest from the practical point of view is how to control the switching times t_-, t_+ in order to get periodic states with a particular amplitude. Our aim is to shed some new light on this question. From a mathematical point of view, the dynamics of equations with a indefinite (sign-changing) potential has been considered before for non-singular equations [16, 17] with a different approach. For future developments, the techniques used in our proofs are in principle suitable for the study of more complex scenarios such as those arising with higher order nonlinearities [1], while the analysis of systems of coupled equations [20, 15] will probably require different techniques.

The paper is organized in two main sections. After this introduction, the first section is devoted to the analysis of the conditions for existence and uniqueness of periodic solution. In the second section a criterion for linear stability is provided, as well as some comments and open problems.

2 Existence of periodic solutions

In this section we study the existence and uniqueness of T-periodic solutions of (1.1). To this aim, for any initial condition (x_0, v_0) with $x_0 > 0$, let us denote by $x(t, x_0, v_0)$ to the unique solution of (1.1) such that $x(0) = x_0$, and $x'(0) = v_0$. With this notation, our main result is the following.

Theorem 2.1 *Fix $T, a_-, a_+, t_-, t_+ \in \mathbb{R}^+$ such that $t_+ + t_- = T$.*
1. If $a_-t_- > a_+t_+$, then there exists a unique initial condition (x_0, v_0) such that $x(t, x_0, v_0)$ is a T-periodic solution of (1.1).
2. If $a_-t_- \leq a_+t_+$, then (1.1) *has no T-periodic solutions.*

Our eq. (1.1) is ruled by two alternating autonomous planar systems

$$\left\{ \begin{array}{rcl} x' & = & y, \\ y' & = & \frac{a_+}{x^3} \end{array} \right. \tag{2.1}$$

and

$$\begin{cases} x' &= y, \\ y' &= -\frac{a_-}{x^3}. \end{cases} \tag{2.2}$$

The respective phase diagrams are drawn in Figure 1. It is evident that a T-periodic

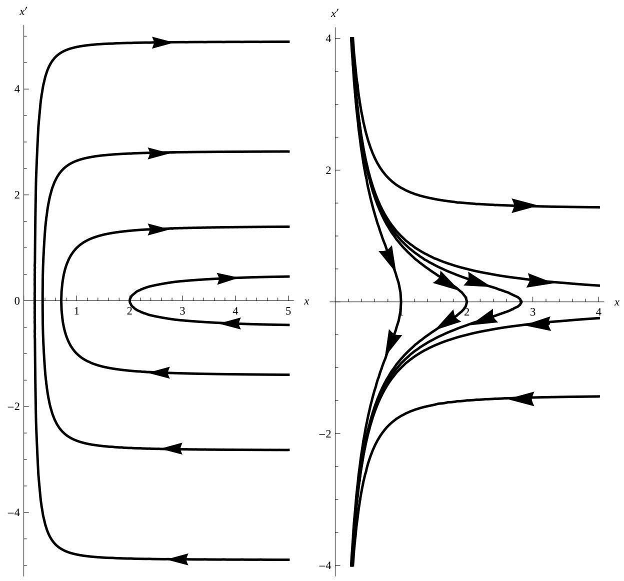

Figure 1: Phase planes of (2.1) (left) and (2.2) (right).

solution will be a closed loop composed by two arcs or piece of orbits of such systems. For a given initial condition (x_0, v_0), let us call $(x_+(t), y_+(t))$ (resp. $(x_-(t), y_-(t))$) the unique solution of (2.1) (resp. (2.2)) such that $x_+(0) = x_0, y_+(0) = v_0$ (resp. $x_-(0) = x_0, y_-(0) = v_0$). If $\gamma^\pm(x_0, v_0)$ are the respective orbits, from the geometry of the phase planes one can see that γ^+ and γ^- intersect in at most two points, one of them being (x_0, v_0). Now, for a given (x_0, v_0) with $x_0 > 0$, we define the functions $\tau_-(x_0, v_0)$, $\tau_+(x_0, v_0) > 0$ as the unique positive times such that

$$\Big(x_+(\tau_+(x_0, v_0)), y_+(\tau_+(x_0, v_0))\Big) = \Big(x_-(-\tau_-(x_0, v_0)), y_-(-\tau_-(x_0, v_0))\Big),$$

whenever such times exist. The next result provides the explicit domain of definition and expressions of τ_+, τ_-.

Theorem 2.2 *The domain of definition of the functions $\tau_+(x_0, v_0), \tau_-(x_0, v_0)$ is*

$$R = \left\{(x_0, v_0) : x_0 > 0, v_0 < 0, x_0^2 v_0^2 < a_-\right\}.$$

Moreover,

$$\tau_+(x_0, v_0) = \frac{-2v_0 x_0^3}{v_0^2 x_0^2 + a_+}, \quad \tau_-(x_0, v_0) = \frac{2v_0 x_0^3}{v_0^2 x_0^2 - a_-}.$$

Proof. If $v_0 \geq 0$, then for any $t > 0$, $x_+(t)$ is strictly increasing, $x_-(-t)$ is strictly decreasing, and $y_+(t)$ and $y_-(-t)$ are strictly increasing. Therefore, the graphs $(x_+(t), v_+(t))$ and $(x_-(-t), y_-(-t))$ never intersect for $t > 0$; thus, the functions $\tau_-(x_0, v_0)$ and $\tau_+(x_0, v_0)$ are not defined. In consequence, one may assume that $v_0 < 0$.

Now, fix $x_0 > 0$, $v_0 < 0$. We have that for every $t > 0$ $x_+(t)$ is strictly decreasing whenever $y_+(t) < 0$, and $x_-(-t)$ is strictly increasing whenever $y_-(-t) < 0$. In such a case, $(x_+(t), v_+(t))$ and $(x_-(-t), y_-(-t))$ do not intersect for $t > 0$. But if both graphs intersect the axis $v = 0$, then arguing as above one obtain that the graphs of $(x_+(t), v_+(t))$ and $(x_-(-t), y_-(-t))$ never intersect in exactly one point for $t > 0$. Therefore, to complete the proof, we only need to determine the values of $x_0 > 0$, $v_0 < 0$ for which there exist $t_1, t_2 > 0$ such that $y_+(t_1, x_0, v_0) = 0$ and $y_-(-t_2, x_0, v_0) = 0$, and compute the functions $\tau_-(x_0, v_0)$ and $\tau_+(x_0, v_0)$ for such initial conditions.

Fix $x_0 > 0$, $v_0 < 0$. Then there exist $c_1, c_2 \in \mathbb{R}$ such that for $t > 0$,

$$\frac{y_+^2(t)}{2} + \frac{a_+}{2x_+^2(t)} = c_1, \tag{2.3}$$

$$\frac{y_-^2(-t)}{2} - \frac{a_-}{2x_-^2(-t)} = c_2, \tag{2.4}$$

where

$$c_1 = \frac{v_0^2}{2} + \frac{a_+}{2x_0^2}, \quad c_2 = \frac{v_0^2}{2} - \frac{a_-}{2x_0^2}.$$

In particular, if $x_0^2 v_0^2 < a_-$, then $c_1 > 0, c_2 < 0$ and both graphs intersect $v = 0$, so by the previous argument $\tau_-(x_0, v_0)$, $\tau_+(x_0, v_0)$ are defined. In the sequel, we may skip the dependence on (x_0, v_0) for simplicity.

The systems (2.1)-(2.2) have symmetric orbits with respect to the horizontal axis. Note that $(x_+(t, x_0, 0), y_+(t, x_0, 0))$, $(x_+(-t, x_0, 0), -y_+(-t, x_0, 0))$ are both solutions of (2.1) with the same initial condition at $t = 0$, hence they are equal. Analogously, $x_-(t, x_0, 0) = x_-(-t, x_0, 0)$, $y_-(t, x_0, 0) = -y_-(-t, x_0, 0)$. Therefore, τ_+ is twice the time between 0, and the intersection time of the orbit with $v = 0$ $(t = \tau_+/2)$. The same holds for τ_-.

Now,

$$\frac{dx_+(t)}{dt} = y_+(t) = \sqrt{2c_1 - \frac{a_+}{x_+^2(t)}} \neq 0$$

for any time $t \in (0, \tau_+/2)$. Therefore,

$$\frac{dt}{dx_+} = \frac{1}{\sqrt{2c_1 - \frac{a_+}{x_+^2}}},$$

and the result follows after some computations by direct integration,

$$\tau_+(x_0, v_0) = 2\int_{\sqrt{\frac{a_+}{2c_1}}}^{x_0} \frac{dx}{\sqrt{2c_1 - \frac{a_+}{x^2}}} = \frac{-2v_0 x_0^3}{v_0^2 x_0^2 + a_+}.$$

The computation is analogous for τ_-. ■

The previous result provides a period map $\mathcal{T} : R \to I\!R$ defined as

$$\mathcal{T}(x_0, v_0) := \tau_+(x_0, v_0) + \tau_-(x_0, v_0).$$

Some properties are collected in the next propositions.

Proposition 2.1 *1. If $(x_0, v_0) \in \partial R$ and $x_0 v_0 = 0$, then*

$$\lim_{(x,v)\to(x_0,v_0)} \mathcal{T}(x, v) = 0.$$

2. If $(x_0, v_0) \in \partial R$ and $v_0^2 x_0^2 = a_-$, then

$$\lim_{(x,v)\to(x_0,v_0)} \mathcal{T}(x, v) = +\infty.$$

3. For any $v_0 < 0$, the function $x_0 \to \mathcal{T}(x_0, v_0)$ is strictly increasing. In particular for any fixed $T > 0$, the Implicit Function Theorem implies the existence of a regular function $x(v_0)$ defined by $\mathcal{T}(x(v_0), v_0) = T$.

Proof. The first two properties follow directly by taking limits in the explicit expressions of of $\tau_-(x_0, v_0)$ and $\tau_+(x_0, v_0)$.

Fix $x_0 \geq 0, v_0 \leq 0$ such that $v_0^2 x_0^2 < a_-$. Then

$$\frac{\partial \tau_+(x_0, v_0)}{\partial x_0} = -\frac{2v_0 x_0^2(3a_+ + v_0^2 x_0^2)}{(a_+ + v_0^2 x_0^2)^2} > 0,$$

and

$$\frac{\partial \tau_-(x_0, v_0)}{\partial x_0} = \frac{2v_0 x_0^2(-3a_- + v_0^2 x_0^2)}{(-a_- + v_0^2 x_0^2)^2} > 0.$$

In consequence, $\frac{\partial \mathcal{T}}{\partial x_0}(x_0, v_0) > 0$, and the result follows. ■

Proposition 2.2 *Fixed $T > 0$, let $x(v_0)$ be the function defined implicitly by $\mathcal{T}(x(v_0), v_0) = T$. Then*

$$\lim_{v\to-\infty} \frac{\tau_-(x(v), v)}{\tau_+(x(v), v)} = +\infty,$$

$$\lim_{v\to 0^-} \frac{\tau_-(x(v), v)}{\tau_+(x(v), v)} = \frac{a_+}{a_-}.$$

Moreover, for every

$$\frac{a_+}{a_-} < \lambda < +\infty,$$

there exists a unique v_0 such that

$$\frac{\tau_-(x(v_0), v_0)}{\tau_+(x(v_0), v_0)} = \lambda.$$

Proof. Firstly, for any $v<0$,

$$T=\tau_+(x(v),v)+\tau_-(x(v),v)=-\frac{2(a_++a_-)vx^3(v)}{(a_--v^2x^2(v))(a_++v^2x^2(v))}. \tag{2.5}$$

Consequently, $x(v)\to+\infty$ when $v\to 0^-$. On the other hand, since $v^2x^2(v)<a_-$, then $|vx(v)|<\sqrt{a_-}$, and from here $x(v)\to 0$ when $v\to-\infty$. Therefore,

$$\lim_{v\to-\infty}\frac{\tau_-(x(v),v)}{\tau_+(x(v),v)}=\lim_{v\to-\infty}\frac{a_++v^2x^2(v)}{a_--v^2x^2(v)}.$$

Obtaining the value of $a_--v^2x^2(v)$ from (2.5), replacing into equation above, and taking into account that $vx(v)$ is bounded and that $x(v)\to 0$ as $v\to-\infty$, one has

$$\lim_{v\to-\infty}\frac{\tau_-(x(v),v)}{\tau_+(x(v),v)}=\lim_{v\to-\infty}\frac{-T\left(a_++v^2x^2(v)\right)^2}{2(a_++a_-)vx^3(v)}=+\infty.$$

Now, let us assume by contradiction that $v\to 0$ and $vx(v)\not\to 0$. Since $vx(v)$ is bounded, there exists a sequence $v_n\to 0$ such that $v_nx(v_n)\to\nu<0$. Taking limits in (2.5), as the denominator is bounded by $a_-(a_++a_-)$, we get $T\to+\infty$, but this is impossible because T is a fixed number, thus if $v\to 0$ then $vx(v)\to 0$. Therefore,

$$\lim_{v\to 0^-}\frac{\tau_-(x(v),v)}{\tau_+(x(v),v)}=\lim_{v\to 0^-}\frac{a_++v^2x^2(v)}{a_--v^2x^2(v)}=\frac{a_+}{a_-}.$$

Finally, denote

$$\Lambda(v)=\frac{\tau_-(x(v),v)}{\tau_+(x(v),v)},$$

and let

$$\frac{a_+}{a_-}<\lambda<+\infty.$$

By continuity, there exists v_0 such that $\Lambda(v_0)=\lambda$. The objective is to prove that v_0 is unique by showing the $\Lambda(v)$ is a monotone function. Deriving in $T=\mathcal{T}(x(v),v)$, one obtains

$$0=\mathcal{T}_x(x(v),v)x'(v)+\mathcal{T}_v(x(v),v),$$

therefore

$$\begin{aligned}x'(v)&=-\frac{\mathcal{T}_v(x(v),v)}{\mathcal{T}_x(x(v),v)}\\&=-\frac{x\left[a_-a_+-(a_--a_+)v^2x^2+3v^4x^4\right]}{v\left[3a_-a_++(a_--a_+)v^2x^2+v^4x^4\right]}.\end{aligned}\tag{2.6}$$

If we write

$$\Lambda(x,v)=\frac{\tau_-(x,v)}{\tau_+(x,v)}=\frac{a_++x^2v^2}{a_--x^2v^2},$$

then

$$\begin{aligned}\Lambda'(v) &= \Lambda_x(x(v),v)x'(v) + \Lambda_v(x(v),v)\\ &= \frac{2(a_- + a_+)vx(v)}{(a_- - v^2x^2(v))^2}\left[x(v) + vx'(v)\right]\\ &= \frac{2(a_- + a_+)vx^2(v)}{(a_- - v^2x^2(v))^2}\left[1 + \frac{vx'(v)}{x(v)}\right].\end{aligned}$$

Since $a_- > v^2x^2(v)$ and $v < 0$, in order to prove that $\Lambda'(v) < 0$, it remains to show that the function $h(v) := 1 + \frac{vx'(v)}{x(v)}$ is positive for all $v < 0$. By using (2.6), we can write

$$\begin{aligned}h(v) &= \frac{2a_-a_+ + 2(a_- - a_+)v^2x^2 - 2v^4x^4}{3a_-a_+ + (a_- - a_+)v^2x^2 + 3v^4x^4}\\ &= \frac{2(a_+ + v^2x^2)(a_- - v^2x^2)}{2a_-a_+ + 4v^2x^2 + (a_+ + v^2x^2)(a_- - v^2x^2)}\end{aligned}$$

and it is clearly positive because $a_- > v^2x^2(v)$.

In conclusion, there exists a unique v_0 such that $\Lambda(v_0) = \lambda$ and the proof is finished. ■

Now, using the above results, we prove our main result.

Proof of Theorem 1. First, let us prove that if $a_-t_- \leq a_+t_+$, then there are no T-periodic solutions. By contradiction, let us assume that x is a T-periodic solution of (1.1). Then, $x^3x'' = a(t)$. Integrating by parts over $[0,T]$,

$$-3\int_0^T x^2(x')^2 dt = \int_0^T a(t)dt = a_+t_+ - a_-t_- \geq 0.$$

Therefore, $\int_0^T x^2(x')^2 dt \leq 0$, which lead to a straightforward contradiction. In consequence, (1.1) has no T-periodic solutions.

Now, let us assume that $a_-t_- > a_+t_+$. To prove that there exists a unique initial condition (x_0, v_0) such that $x(t, x_0, v_0)$ is a T-periodic solution of (1.1), is equivalent to prove that there exists a unique (x_0, v_0) such that $\tau_-(x_0, v_0) = t_-$, and $\tau_+(x_0, v_0) = t_+$.

If (x_0, v_0) is such that $\tau_-(x_0, v_0) = t_-$, and $\tau_+(x_0, v_0) = t_+$, then $x_0 = x(v_0)$, where $x(v)$ is the function defined implicitly by $T = \mathcal{T}(x(v), v)$.

Finally, since

$$\frac{t_-}{t_+} > \frac{a_+}{a_-},$$

by Proposition 2.2 and continuity, there exists a unique v_0 such that $\tau_-(x(v_0), v_0) = t_-$, and $\tau_+(x(v_0), v_0) = t_+$. ■

Remark 2.1 Fixed $T > 0$, we may obtain how the initial condition (x_0, v_0) of the unique T-periodic solution of (1.1) depends on t_-, t_+. Concretely, define

$$\lambda = \frac{t_-}{t_+},$$

suppose that $\lambda > a_+/a_-$, and let (x_0, v_0) be the initial condition of the unique T-periodic solution of (1.1). Then

$$(x_0, v_0) \to (0, -\infty) \quad \text{as } \lambda \to a_+/a_-, \quad (x_0, v_0) \to (+\infty, 0) \quad \text{as } \lambda \to +\infty.$$

3 Stability

For a given $(x_0, v_0) \in R$, the aim of this section is to study the stability of $x(t) \equiv x(t, x_0, v_0)$ as a T-periodic function of eq. (1.1). We are going to use a classical stability result by Krein [10] (see also [4]).

Proposition 3.1 ([10]) *The periodic Hill's equation $y'' + p(t)y = 0$ is stable if $\int_0^T p(t)dt \geq 0$ and*

$$T \int_0^T p^+(t)dt \leq 4,$$

where $p^+ = \frac{|p|+p}{2}$ is the positive part of p.

The main result of this section is the following one.

Theorem 3.1 *Let us assume that*

$$3a_+\tau_+(x_0, v_0)\mathcal{T}(x_0, v_0)\left(\frac{v_0^2}{a_+} + \frac{1}{x_0^2}\right)^2 \leq 4 \tag{3.1}$$

Then (1.1) has a T-periodic solution with initial condition (x_0, v_0) for $t_- = \tau_-(x_0, v_0)$, $t_+ = \tau_+(x_0, v_0)$ and $T = t_- + t_+$, which is stable in the linear sense.

Remark 3.1 Let us recall that we have derived explicit expressions for $\tau_+(x_0, v_0)$, $\tau_-(x_0, v_0)$, $\mathcal{T}(x_0, v_0)$. then condition (3.1) can be written explicitly in terms of the initial condition (x_0, v_0). It provides a region of initial conditions where the linear stability is guaranteed. Such region is depicted in Fig. 2 for the values $a_+ = a_- = 1$.

Proof. The existence follows from the previous section. Let us call the solution $x(t)$. The linearized equation is

$$y'' + \frac{3a(t)}{x^4}y = 0.$$

Let us call $p(t) = \frac{3a(t)}{x^4}$. By using that x is a positive solution of (1.1),

$$p(t) = 3\frac{x''}{x}.$$

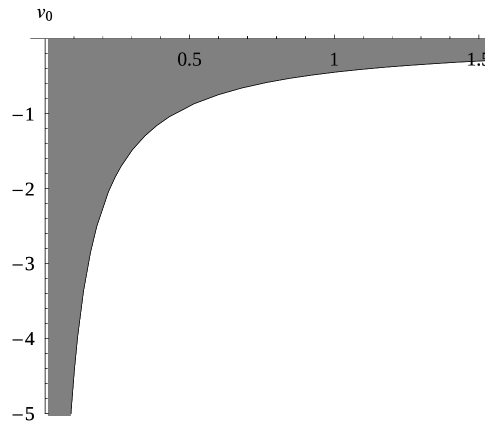

Figure 2: The stability region for the values $a_+ = a_- = 1$.

Then, an integration by parts gives

$$\int_0^T p(t)dt = 3\int_0^T \frac{x'^2}{x^2} > 0.$$

On the other hand, it is easy to see that

$$x_* \equiv \min_{t\in[0,T]} x(t) = \sqrt{\frac{a_+}{2c_1}},$$

where $c_1 = \frac{v_0^2}{2} + \frac{a_+}{2x_0^2}$. Then,

$$T\int_0^T p^+(t)dt = T\int_0^{t_+} \frac{3a_+}{x^4}dt \le \frac{3a_+Tt_+}{x_*^4} = 3a_+Tt_+\left(\frac{v_0^2}{a_+} + \frac{1}{x_0^2}\right)^2 \le 4.$$

Hence, we are in the conditions of the previous proposition. ■

The equation under study is conservative, so the stability in the sense of Lyapunov can not be directly derived from the first approximation because of the possible synchronized influence of higher terms leading to resonance. After the works of Moser [22], it is well known that the stability in the nonlinear sense depends generically on the third approximation around of the periodic solution. By using the results in [11] it would be possible to get a more restrictive region in the semiplane (x_0, v_0) where nonlinear stability is assured. We will not follow this way because a heavy theoretical machinery is required.

A different type of stability is the structural stability or robustness. A physical system is called *robust* if its dynamical response is preserved under small perturbations of the involved parameters. Generally speaking, hyperbolicity implies robustness, but in a hamiltonian ambient it is not always easy to prove that a given solution is hyperbolic. Hence, in our context, an interesting problem is whether the unique T-periodic solution of (1.1) remains after small perturbations of the function $a(t)$. More generally, for an arbitrary T-periodic function $a(t)$ we think that it should be possible to find a necessary and sufficient condition for existence of periodic solutions by using degree arguments, although it does not seem an easy task.

Acknowledgements

J.L.B. is grateful to the Departament of Applied Mathematicas of the University of Granada, for the hospitality during the visit in which this work was started.

References

[1] F. Kh. Abdullaev, J. Garnier, *Dynamical stabilization of solitons in cubic-quintic nonlinear Schrödinger model*, Phys. Rev. E **72** (2005), 035603 .

[2] R. Carretero-González, D.J. Frantzeskakis and P.G. Kevrekidis, *Nonlinear waves in Bose-Einstein condensates: physical relevance and mathematical techniques*, Nonlinearity **21** (2008), R139–R202.

[3] M. Centurion, M. A. Porter, Ye Pu, P. G. Kevrekidis, D. J. Frantzeskakis, D. Psaltis, *Modulational Instability in a Layered Kerr Medium: Theory and Experiment*, Phys. Rev. Lett. **97** (2006), 234101 .

[4] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations (third edition), Springer-Verlag, 1971.

[5] S.L. Cornish, N.R. Claussen, J.L. Roberts, E.A. Cornell, C.E. Wieman, *Stable ^{85}Rb Bose-Einstein Condensates with widely tunable interactions*, Phys. Rev. Lett. **85** (2000), 1795–1798.

[6] S.L. Cornish, N.R. Claussen, J.L. Roberts, E.A. Cornell, C.E. Wieman, *Dynamics of collapsing and exploding Bose-Einstein condensates*, Nature **412** (2001), 295–299.

[7] L.V. Hau, M.M. Burns, J.A. Golovchenko, *Bound states of guided matter waves: An atom and a charged wire*, Physical Rev. A **45**, no. 9 (1996), 6468–6478.

[8] A. Itin, T. Morishita, S. Watanabe, *Reexamination of dynamical stabilization of matter-wave solitons*, Phys. Rev. A **74** (2006), 033613.

[9] C. King, A. Lésniewski, *Periodic motion of atoms near a charged wire*, Letters in Mathematical Physics **39** (1997), 367–378.

[10] M.G. Krein, *On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones on stability*, Prikl. Mat. Meh. SSSR **15** (1951), 323–348.

[11] J. Lei, P.J. Torres, *L1 criteria for stability of periodic solutions of a newtonian equation*, Math. Proc. Cambridge Phil. Soc. **140**, no. 2 (2006), 359–368.

[12] J. Lei, M. Zhang, *Twist Property of Periodic Motion of an Atom Near a Charged Wire*, Letters in Mathematical Physics **60** (2002), 9–17.

[13] G.D. Montesinos, V.M. Pérez-García, P.J. Torres, *Stabilization of solitons of the multidimensional nonlinear Schrödinger equation: matter-wave breathers*, Phys. D **191** (2004), 193–210.

[14] G.D. Montesinos, P.J. Torres, V.M. Pérez-García, *The method of moments for nonlinear Schrödinger equations: theory and applications*, SIAM J. Appl. Math. **67** n.4 (2007), 990–1015.

[15] G. D. Montesinos, M. I. Rodas-Verde, V. M. Pérez-García, *n-body dynamics of stabilized vector solitons*, Chaos **15**, 033501 (2005).

[16] D. Papini, F. Zanolin, *A topological approach to superlinear indefinite boundary value problems*, Topol. Methods Nonlinear Anal. **15** (2000), no. 2, 203–233.

[17] D. Papini, F. Zanolin, *On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill's equations*, Adv. Nonlinear Stud. **4** (2004), no. 1, 71–91.

[18] V.M. Pérez-García, P.J. Torres, J.J. García-Ripoll, H. Michinel, *Moment analysis of paraxial propagation in a nonlinear graded index fibre*, J. Opt. B: Quantum Semiclass. Opt. **2** (2000), 353–358.

[19] A.S. Rodrigues, P.G. Kevrekidis, M.A. Porter, D.J. Frantzeskakis, P. Schmelcher, A.R. Bishop, *Matter-Wave Solitons with a Periodic, Piecewise-Constant Scattering Length*, Physical Review A **78** (2008), No. 013611.

[20] H. Saito, R. G. Hulet, M. Ueda, *Stabilization of a Bose-Einstein droplet by hyperfine Rabi oscillations*, Phys. Rev. A **76** (2007), 053619.

[21] H. Saito, M. Ueda, *Dynamically Stabilized Bright Solitons in a Two-Dimensional Bose-Einstein Condensate*, Phys. Rev. Lett. **90**-4 (2003), 040403.

[22] C. L. Siegel, J. K. Moser, Lectures on Celestial Mechanics, Springer-Verlag, New York. Berlin, 1971.

[23] I. Towers, B.A. Malomed, *Stable (2+1)-dimensional solitons in a layered medium with sign-alternating Kerr nonlinearity*, J. Opt. Soc. Am. B **19**, no. 3 (2002), 537–543.