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On periodic solutions of second-order differential equations with attractive–repulsive singularities

Robert Hakl[a,*,1], Pedro J. Torres[b,2]

[a] Institute of Mathematics AS CR, Žižkova 22, 616 62 Brno, Czech Republic
[b] Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva s/n, 18071 Granada, Spain

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ABSTRACT

Sufficient conditions for the existence of a solution to the problem

$$u''(t)=\frac{g(t)}{u^{\mu}(t)}-\frac{h(t)}{u^{\lambda}(t)}+f(t)\quad \text{for a.e. } t\in[0,\omega],$$
$$u(0)=u(\omega),\qquad u'(0)=u'(\omega)$$

are established.

1. Introduction

In this paper, we are concerned with the periodic problem

$$u''(t)=\frac{g(t)}{u^{\mu}(t)}-\frac{h(t)}{u^{\lambda}(t)}+f(t)\quad \text{for a.e. } t\in[0,\omega], \tag{1.1}$$

$$u(0)=u(\omega),\qquad u'(0)=u'(\omega), \tag{1.2}$$

* Corresponding author.
E-mail addresses: hakl@ipm.cz (R. Hakl), ptorres@ugr.es (P.J. Torres).

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where $g, h \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R}_+)$, $f \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$, and $\lambda, \mu > 0$. By a solution to (1.1), (1.2) we understand a function $u \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ satisfying (1.1). Special cases of Eq. (1.1) are

$$u''(t) = \frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} \quad \text{for a.e. } t \in [0, \omega], \tag{1.3}$$

$$u''(t) = -\frac{h(t)}{u^\lambda(t)} + f(t) \quad \text{for a.e. } t \in [0, \omega], \tag{1.4}$$

$$u''(t) = \frac{g(t)}{u^\mu(t)} + f(t) \quad \text{for a.e. } t \in [0, \omega]. \tag{1.5}$$

In the related literature, it is said that (1.4) has an attractive singularity, whereas (1.5) has a repulsive singularity. The interest on this type of equations began with the paper of Lazer and Solimini [7], where the authors provide necessary and sufficient conditions for existence of periodic solutions of Eqs. (1.4) and (1.5) with constant positive functions h, g and a continuous forcing term f. Their proofs can be easily extended to the case when the function h, respectively g is bounded from below by some positive constant (see the generalized results presented in the paper of Habets and Sanchez [3]), but in their arguments this hypothesis is essential and cannot be omitted. In the repulsive case, a strong force assumption ($\mu \geqslant 1$) is also essential.

Eq. (1.3) is interesting due to a mixed type of singularity on the right-hand side. Since the functions g and h are possibly zero on some sets of positive measure, the singularity may combine attractive and repulsive effects. If h, g are positive constants, the singular term can be regarded as a generalized Lennard–Jones force or van der Waals attraction/repulsion force and it is widely use in Molecular Dynamics to model the interaction between atomic particles (see for instance [4,9,12,15] and the references therein). In a different physical context, a periodic solution of Eq. (1.3) is equivalent to a matter-wave breather in a Bose–Einstein condensate with a periodic control of the scattering length (the mathematical model is a nonlinear Schrödinger equation with a cubic term, then the method of moments leads to the study of a particular case of (1.3), see [8] for more details). Finally, a third different range of applicability is the evolution of optical pulses in dispersion-managed fiber communication devices [6].

In spite of the variety of physical applications, the analysis of differential equations with mixed singularities is at this moment very incomplete, and few references can be cited (see [1,5,13]) if compared with the large number of references devoted to singular equations, either of attractive or repulsive type (see the review [10] and the references therein). Our main purpose in this paper is to contribute to the literature trying to fill partially this gap in the study of singularities of mixed type with an approach that should be useful as a starting point for further studies. Incidentally, our main results can be applied to the original Lazer–Solimini equations both in the attractive and in the repulsive case, giving new sufficient conditions for existence of periodic solutions when the functions h and g are possibly zero on the sets of positive measure.

The structure of the paper is as follows: Section 2 contains the tools needed in the proofs. In Section 3 we state and prove the main results and develop some corollaries for the equation with a singularity of mixed type. To illustrate the results, an application to the dynamics of a trapless Bose–Einstein condensate is given. This model and related ones deserve a different treatment more oriented to a physical audience, that will be performed elsewhere. Finally, due its relevance in the related literature we have decided to devote Sections 4 and 5 to perform a comparative study of the equation with attractive (respectively repulsive) singularity. Along the paper, some open problems are posed. We feel that their consideration will bring light to this subject in the future.

The following notation is used throughout the paper:

$\mathbb{R}$ is a set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$, $[x]_+ = \max\{x, 0\}$, $[x]_- = \max\{-x, 0\}$.

$L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ is the Banach space of ω-periodic Lebesgue integrable functions $p : \mathbb{R}/\omega\mathbb{Z} \to \mathbb{R}$.

$AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ is a set of all ω-periodic functions $u : \mathbb{R}/\omega\mathbb{Z} \to \mathbb{R}$ such that u and u' are absolutely continuous.

$L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R}_+) = \{p \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R}):\ p(t) \geqslant 0 \text{ for a.e. } t \in [0, \omega]\}$.

Notation 1.1. For the sake of brevity we will use the following notation throughout the paper:

$$G=\int_0^\omega g(s)\,ds,\qquad H=\int_0^\omega h(s)\,ds,$$

$$F=\int_0^\omega f(s)\,ds,\qquad F_+=\int_0^\omega \big[f(s)\big]_+\,ds,\qquad F_-=\int_0^\omega \big[f(s)\big]_-\,ds.$$

Note that $F=F_+-F_-$.

2. Auxiliary results

The proofs of our results rely on the method of upper and lower functions. The following two lemmas are classical and can be found, e.g., in [2,14]. We introduce them in a form suitable for us.

Lemma 2.1. *Let there exist positive functions $\alpha,\beta\in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ such that*

$$\alpha''(t)\geqslant\frac{g(t)}{\alpha^\mu(t)}-\frac{h(t)}{\alpha^\lambda(t)}+f(t)\quad\textit{for a.e. } t\in[0,\omega],\tag{2.1}$$

$$\beta''(t)\leqslant\frac{g(t)}{\beta^\mu(t)}-\frac{h(t)}{\beta^\lambda(t)}+f(t)\quad\textit{for a.e. } t\in[0,\omega],\tag{2.2}$$

$$\alpha(t)\leqslant\beta(t)\quad\textit{for } t\in[0,\omega].$$

Then there exists at least one positive solution to (1.1), (1.2).

A function $\alpha\in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ (respectively $\beta\in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$) verifying (2.1) (respectively (2.2)) is called lower (respectively upper) function. When the order between the lower and the upper functions is the inverse, an additional hypothesis is needed.

Definition 2.1. A function $\varphi\in L(\mathbb{R}/\omega\mathbb{Z};\mathbb{R}_+)$ is said to verify the property (P) if the implication

$$\left.\begin{array}{l} u\in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})\\ u''(t)+\varphi(t)u(t)\geqslant 0\quad\text{for a.e. } t\in[0,\omega]\end{array}\right\}\quad\Longrightarrow\quad u(t)\geqslant 0\quad\text{for } t\in[0,\omega]$$

holds.

Lemma 2.2. *Let there exist positive functions $\alpha,\beta\in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ satisfying* (2.1), (2.2) *and*

$$\beta(t)\leqslant\alpha(t)\quad\textit{for } t\in[0,\omega].$$

Let, moreover, there exist $\varphi\in L(\mathbb{R}/\omega\mathbb{Z};\mathbb{R}_+)$ with the property (P) *and such that*

$$\frac{g(t)}{u^\mu(t)}-\frac{h(t)}{u^\lambda(t)}-\left(\frac{g(t)}{v^\mu(t)}-\frac{h(t)}{v^\lambda(t)}\right)\leqslant\varphi(t)\big(v(t)-u(t)\big)\quad\textit{for a.e. } t\in[0,\omega],\tag{2.3}$$

whenever $\beta(t)\leqslant u(t)\leqslant v(t)\leqslant\alpha(t)$ for $t\in[0,\omega]$. Then there exists at least one positive solution to (1.1), (1.2).

Property (P) is just a maximum principle for the linear operator $Lu := u'' + \varphi(t)u$ with periodic boundary conditions, and it is equivalent to have a nonnegative Green function. Ref. [14] provides sufficient conditions in the L^p-norm for $\varphi(t)$ to verify property (P). In particular, we have the following lemma.

Lemma 2.3. *Let us assume that $\varphi \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R}_+)$, $\varphi \not\equiv 0$, and at least one of the following conditions holds*:

(i) $\varphi(t) \leqslant (\frac{\pi}{\omega})^2$ *for a.e.* $t \in [0, \omega]$,
(ii) $\int_0^\omega \varphi(t)\,dt \leqslant \frac{4}{\omega}$.

Then, φ verifies the property (P).

To finish this section, we show a technical bound on the amplitude of oscillation of a periodic function.

Lemma 2.4. *Given $v \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$, then*

$$M_v - m_v \leqslant \frac{\omega}{4} \int_0^\omega \big[v''(s)\big]_+ \,ds, \tag{2.4}$$

where

$$M_v = \max\big\{v(t)\colon t \in [0, \omega]\big\}, \qquad m_v = \min\big\{v(t)\colon t \in [0, \omega]\big\}. \tag{2.5}$$

Moreover, (2.4) *is fulfilled as an equality if and only if v is a constant function.*

Proof. If v is a constant function, then (2.4) follows trivially.

Let, therefore, v be a non-constant function and choose $t_0, t_1 \in [0, \omega]$ such that

$$v(t_0) = M_v, \qquad v(t_1) = m_v.$$

Without loss of generality we can assume that $t_0 < t_1$. Indeed, in the case where $t_1 < t_0$ we can consider a function $-v$ instead of v and using the fact that $v \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ we have

$$\int_0^\omega \big[v''(s)\big]_+ \,ds = \int_0^\omega \big[v''(s)\big]_- \,ds = \int_0^\omega \big[-v''(s)\big]_+ \,ds.$$

Put

$$M_1 = \max\big\{v'(t)\colon t \in [0, \omega]\big\}, \qquad m_1 = \min\big\{v'(t)\colon t \in [0, \omega]\big\}.$$

Then, obviously, $M_1 > 0$, $m_1 < 0$ and by the periodicity of v and continuity of v' we have

$$M_v - m_v = \int_0^{t_0} v'(s)\,ds + \int_{t_1}^\omega v'(s)\,ds < M_1(t_0 + \omega - t_1) \tag{2.6}$$

and

$$M_v - m_v = -\int_{t_0}^{t_1} v'(s)\,ds < -m_1(t_1 - t_0). \tag{2.7}$$

On the other hand, we have $M_v - m_v > 0$ and thus the multiplying of the corresponding sides of (2.6) and (2.7) results in

$$(M_v - m_v)^2 < -m_1 M_1 (t_0 + \omega - t_1)(t_1 - t_0). \tag{2.8}$$

Now using the inequality $AB \leqslant \frac{1}{4}(A+B)^2$, from (2.8) we get

$$(M_v - m_v)^2 < \frac{(M_1 - m_1)^2 \omega^2}{16},$$

whence the inequality

$$M_v - m_v < \frac{\omega}{4}(M_1 - m_1) \tag{2.9}$$

follows.

On the other hand, choose $t_2, t_3 \in [0, \omega]$ such that

$$v'(t_2) = M_1, \qquad v'(t_3) = m_1.$$

If $t_2 < t_3$ then by using again that v is ω-periodic we have

$$M_1 - m_1 = M_1 - v'(0) + v'(\omega) - m_1 = \int_0^{t_2} v''(s)\,ds + \int_{t_3}^{\omega} v''(s)\,ds \leqslant \int_0^{\omega} \left[v''(s)\right]_+ ds.$$

If $t_3 < t_2$ then

$$M_1 - m_1 = \int_{t_3}^{t_2} v''(s)\,ds \leqslant \int_0^{\omega} \left[v''(s)\right]_+ ds.$$

Consequently, in both cases $t_2 < t_3$ and $t_3 < t_2$ we have

$$M_1 - m_1 \leqslant \int_0^{\omega} \left[v''(s)\right]_+ ds,$$

which together with (2.9) implies (2.4). □

3. The general case

The following theorems are the main results of the paper.

Theorem 3.1. *Let $H > 0$, $F > 0$, functions $w, \sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ be such that the equalities*

$$w''(t) = Hg(t) - Gh(t) \quad \text{for a.e. } t \in [0, \omega], \tag{3.1}$$

$$\sigma''(t) = -\frac{F}{H}h(t) + f(t) \quad \text{for a.e. } t \in [0, \omega] \tag{3.2}$$

are fulfilled[3] *and let there exist $x_0 \in \,]0, +\infty[$ such that*

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leqslant \left(\frac{H}{x_0GH + F}\right)^{1/\lambda} - \left(\frac{1}{x_0H}\right)^{1/\mu} \quad \text{for } t \in [0, \omega], \tag{3.3}$$

where

$$m_w = \min\big\{w(t)\colon\ t \in [0, \omega]\big\}, \qquad m_\sigma = \min\big\{\sigma(t)\colon\ t \in [0, \omega]\big\}. \tag{3.4}$$

Then the problem (1.1), (1.2) *has at least one positive solution.*

Proof. Put

$$\alpha(t) = \left(\frac{1}{x_0H}\right)^{1/\mu} + x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \quad \text{for } t \in [0, \omega].$$

Then, obviously, $\alpha \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ and in view of (3.1) and (3.2) we have

$$\alpha''(t) = x_0Hg(t) - \left(x_0G + \frac{F}{H}\right)h(t) + f(t) \quad \text{for a.e. } t \in [0, \omega]. \tag{3.5}$$

Moreover, according to (3.3) and (3.4),

$$\left(\frac{1}{x_0H}\right)^{1/\mu} \leqslant \alpha(t) \leqslant \left(\frac{H}{x_0GH + F}\right)^{1/\lambda} \quad \text{for } t \in [0, \omega]. \tag{3.6}$$

Now (3.5) and (3.6) imply

$$\alpha''(t) \geqslant \frac{g(t)}{\alpha^\mu(t)} - \frac{h(t)}{\alpha^\lambda(t)} + f(t) \quad \text{for a.e. } t \in [0, \omega].$$

Consequently, α is a lower function to (1.1), (1.2).

Further, we can choose $x_1 \in \,]0, x_0]$ such that

$$x_1\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leqslant \left(\frac{1}{x_1H}\right)^{1/\mu} - \left(\frac{H}{x_1GH + F}\right)^{1/\lambda} \quad \text{for } t \in [0, \omega] \tag{3.7}$$

and put

[3] See Remark 3.1 below.

$$\beta(t) = \left(\frac{H}{x_1 GH + F}\right)^{1/\lambda} + x_1\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \quad \text{for } t \in [0, \omega].$$

Then, $\beta \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ and in view of (3.1) and (3.2) we have

$$\beta''(t) = x_1 Hg(t) - \left(x_1 G + \frac{F}{H}\right)h(t) + f(t) \quad \text{for a.e. } t \in [0, \omega]. \tag{3.8}$$

Moreover, according to (3.4) and (3.7),

$$\left(\frac{H}{x_1 GH + F}\right)^{1/\lambda} \leqslant \beta(t) \leqslant \left(\frac{1}{x_1 H}\right)^{1/\mu} \quad \text{for } t \in [0, \omega]. \tag{3.9}$$

Now (3.8) and (3.9) imply

$$\beta''(t) \leqslant \frac{g(t)}{\beta^\mu(t)} - \frac{h(t)}{\beta^\lambda(t)} + f(t) \quad \text{for a.e. } t \in [0, \omega].$$

Consequently, β is an upper function to (1.1), (1.2).

Moreover, (3.6) and (3.9) imply

$$\alpha(t) \leqslant \beta(t) \quad \text{for } t \in [0, \omega]. \tag{3.10}$$

Thus the assertion follows from Lemma 2.1. □

Remark 3.1. Note that for every $q \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ such that $\int_0^\omega q(t)\,dt = 0$, the periodic solution v of the equation

$$v''(t) = q(t) \quad \text{for a.e. } t \in [0, \omega]$$

is given by the Green formula

$$v(t) = -\frac{1}{\omega}\left((\omega - t)\int_0^t sq(s)\,ds + t\int_t^\omega (\omega - s)q(s)\,ds\right) + c \quad \text{for } t \in [0, \omega], \tag{3.11}$$

where $c \in \mathbb{R}$. Therefore, the periodic functions w and σ with properties (3.1) and (3.2) exist and, moreover, are unique up to a constant term, the value of which has no influence on the validity of the condition (3.3). A similar observation can be made in relation to the formulations of the theorems given below.

Theorem 3.2. *Let $\lambda > \mu$, $H > 0$, $G > 0$, $F = 0$, functions $w, \sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ be such that the equalities* (3.1) *and*

$$\sigma''(t) = f(t) \quad \textit{for a.e. } t \in [0, \omega] \tag{3.12}$$

are fulfilled, and let there exist $x_0 \in]0, +\infty[$ such that

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leqslant \left(\frac{1}{x_0 G}\right)^{1/\lambda} - \left(\frac{1}{x_0 H}\right)^{1/\mu} \quad \textit{for } t \in [0, \omega], \tag{3.13}$$

where m_w and m_σ are defined by (3.4)*. Then the problem* (1.1), (1.2) *has at least one positive solution.*

Proof. Note that the inequality $\lambda > \mu$ implies

$$\lim_{x\to 0_+}\left(\frac{1}{xH}\right)^{1/\mu} - \left(\frac{1}{xG}\right)^{1/\lambda} = +\infty.$$

Therefore, analogously to the proof of Theorem 3.1, one can show that there exist lower and upper functions α, β satisfying (3.10). Consequently, the assertion follows from Lemma 2.1. □

Corollary 3.1. *Let $\lambda > \mu$, $H > 0$, $G > 0$, and let $w \in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ be such that* (3.1) *is fulfilled. Let, moreover,*

$$M_w - m_w \leqslant \frac{H^{\frac{1+\lambda}{\lambda-\mu}}}{G^{\frac{1+\mu}{\lambda-\mu}}}\left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda}\right)^{\frac{(1+\lambda)\mu}{\lambda-\mu}}\frac{\lambda-\mu}{(1+\mu)\lambda}, \tag{3.14}$$

where m_w is given by (3.4) *and*

$$M_w = \max\{w(t):\ t \in [0,\omega]\}. \tag{3.15}$$

Then the problem (1.3), (1.2) *has at least one positive solution.*

Proof. In order to apply Theorem 3.2, put $f \equiv 0$, then $\sigma \equiv 0$. Take

$$x_0 = \left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{\frac{\lambda\mu}{\lambda-\mu}}\frac{G^{\frac{\mu}{\lambda-\mu}}}{H^{\frac{\lambda}{\lambda-\mu}}}.$$

Then (3.14) implies (3.13), and thus the assertion follows from Theorem 3.2. □

At this stage, Lemma 2.4 enables us to give a first concrete existence criterion.

Corollary 3.2. *Let $\lambda > \mu$, $H > 0$, and $G > 0$. Let, moreover,*

$$\frac{G^{1+\lambda}}{H^{1+\mu}} \leqslant \left(\frac{4}{\omega}\right)^{\lambda-\mu}\left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda}\right)^{(1+\lambda)\mu}\left(\frac{\lambda-\mu}{(1+\mu)\lambda}\right)^{\lambda-\mu}. \tag{3.16}$$

Then the problem (1.3), (1.2) *has at least one positive solution.*

Proof. By Lemma 2.4, it is easy to verify that

$$M_w - m_w \leqslant \frac{\omega}{4}GH.$$

Now the assertion follows directly from Corollary 3.1. □

To illustrate this latter result, we have selected a concrete physical model studied in [8, Section 5]. The dynamics of a trapless 3D Bose–Einstein condensate with variable scattering length is ruled by the equation

$$u''(t) = \frac{Q_1}{u^3} + \frac{a(t)Q_2}{u^4}, \tag{3.17}$$

where Q_1, Q_2 are positive parameters and $a(t)$ models the s-wave scattering length, which is assumed to vary ω-periodically in time. A negative $a(t)$ corresponds to attractive interactions between the elementary particles. Then the existence of a positive periodic solution of (3.17) is interpreted as a bound state of the condensate without external trap. Eq. (3.17) is a particular case of (1.3) with $\mu = 3$, $\lambda = 4$. Then, a direct consequence of Corollary 3.2 is the existence of ω-periodic solution of (3.17) for any $a \in L(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$, $a(t) \leqslant 0$ for a.e. t, such that

$$\left(\int_0^\omega a(t)\,dt\right)^4 \geqslant \left(\frac{16}{15}\right)^{15} \frac{4Q_1^5\omega^6}{Q_2^4} \simeq 10.5315\frac{Q_1^5\omega^6}{Q_2^4}.$$

The following results are devoted to the remaining cases $F < 0$ and $\mu > \lambda$. We are compelled to construct upper and lower functions on the reversed order.

Theorem 3.3. *Let $G > 0$, $F < 0$, functions $w, \sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ be such that the equalities* (3.1) *and*

$$\sigma''(t) = \frac{|F|}{G} g(t) + f(t) \quad \textit{for a.e. } t \in [0,\omega] \tag{3.18}$$

are fulfilled, and let there exist $x_0 \in\,]0,+\infty[$ such that

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leqslant \left(\frac{G}{x_0GH + |F|}\right)^{1/\mu} - \left(\frac{1}{x_0G}\right)^{1/\lambda} \quad \textit{for } t \in [0,\omega], \tag{3.19}$$

where m_w and m_σ are defined by (3.4)*. Moreover, let us define*

$$\beta(t) = \left(\frac{1}{x_0G}\right)^{1/\lambda} + x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \quad \textit{for } t \in [0,\omega] \tag{3.20}$$

and assume that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ verifies the property (P)*. Then the problem* (1.1), (1.2) *has at least one positive solution.*

Proof. Let β be defined by (3.20). Then, $\beta \in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ and in view of (3.1) and (3.18) we have

$$\beta''(t) = \left(x_0H + \frac{|F|}{G}\right) g(t) - x_0Gh(t) + f(t) \quad \text{for a.e. } t \in [0,\omega]. \tag{3.21}$$

Moreover, according to (3.4) and (3.19),

$$\left(\frac{1}{x_0G}\right)^{1/\lambda} \leqslant \beta(t) \leqslant \left(\frac{G}{x_0GH + |F|}\right)^{1/\mu} \quad \text{for } t \in [0,\omega]. \tag{3.22}$$

Now (3.21) and (3.22) imply

$$\beta''(t) \leqslant \frac{g(t)}{\beta^\mu(t)} - \frac{h(t)}{\beta^\lambda(t)} + f(t) \quad \text{for a.e. } t \in [0,\omega].$$

Consequently, β is an upper function to (1.1), (1.2).

Further, we can choose $x_1 \in\,]0, x_0]$ such that

$$x_1\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leqslant \left(\frac{1}{x_1G}\right)^{1/\lambda} - \left(\frac{G}{x_1GH + |F|}\right)^{1/\mu} \quad \text{for } t \in [0,\omega] \tag{3.23}$$

and put

$$\alpha(t) = \left(\frac{G}{x_1 GH + |F|}\right)^{1/\mu} + x_1\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \quad \text{for } t \in [0, \omega].$$

Then, $\alpha \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ and in view of (3.1) and (3.18) we have

$$\alpha''(t) = \left(x_1 H + \frac{|F|}{G}\right) g(t) - x_1 G h(t) + f(t) \quad \text{for a.e. } t \in [0, \omega]. \tag{3.24}$$

Moreover, according to (3.4) and (3.23),

$$\left(\frac{G}{x_1 GH + |F|}\right)^{1/\mu} \leqslant \alpha(t) \leqslant \left(\frac{1}{x_1 G}\right)^{1/\lambda} \quad \text{for } t \in [0, \omega]. \tag{3.25}$$

Now (3.24) and (3.25) imply

$$\alpha''(t) \geqslant \frac{g(t)}{\alpha^\mu(t)} - \frac{h(t)}{\alpha^\lambda(t)} + f(t) \quad \text{for a.e. } t \in [0, \omega].$$

Consequently, α is a lower function to (1.1), (1.2) and according to (3.22) and (3.25) we have

$$\beta(t) \leqslant \alpha(t) \quad \text{for } t \in [0, \omega]. \tag{3.26}$$

Furthermore, note that a function

$$\psi(y) = \frac{\mu}{\beta^{1+\mu}} y + \frac{1}{y^\mu}$$

is nondecreasing for $y \geqslant \beta$. Therefore we have

$$g(t)\left(\frac{\mu}{\beta^{1+\mu}(t)} u(t) + \frac{1}{u^\mu(t)}\right) - \frac{h(t)}{u^\lambda(t)} \leqslant g(t)\left(\frac{\mu}{\beta^{1+\mu}(t)} v(t) + \frac{1}{v^\mu(t)}\right) - \frac{h(t)}{v^\lambda(t)} \quad \text{for a.e. } t \in [0, \omega],$$

whenever $\beta(t) \leqslant u(t) \leqslant v(t)$ for $t \in [0, \omega]$, whence we get

$$\frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} - \left(\frac{g(t)}{v^\mu(t)} - \frac{h(t)}{v^\lambda(t)}\right) \leqslant \frac{\mu g(t)}{\beta^{1+\mu}(t)}\big(v(t) - u(t)\big) \quad \text{for a.e. } t \in [0, \omega].$$

Thus the assertion follows from Lemma 2.2. □

Theorem 3.4. *Let $\mu > \lambda$, $H > 0$, $G > 0$, $F = 0$, functions $w, \sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ be such that the equalities* (3.1) *and* (3.12) *are fulfilled, and let there exist $x_0 \in]0, +\infty[$ such that*

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leqslant \left(\frac{1}{x_0 H}\right)^{1/\mu} - \left(\frac{1}{x_0 G}\right)^{1/\lambda} \quad \text{for } t \in [0, \omega], \tag{3.27}$$

where m_w and m_σ are defined by (3.4)*. Moreover, assume that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ verifies the property* (P)*, where β is given by* (3.20)*. Then the problem* (1.1), (1.2) *has at least one positive solution.*

Proof. Note that the inequality $\mu > \lambda$ implies

$$\lim_{x\to 0_+}\left(\frac{1}{xG}\right)^{1/\lambda} - \left(\frac{1}{xH}\right)^{1/\mu} = +\infty.$$

Therefore, analogously to the proof of Theorem 3.3, one can show that there exist lower and upper functions α, β satisfying (3.26). Consequently, the assertion follows from Lemma 2.2 with $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$. $\square$

Corollary 3.3. *Let $\mu > \lambda$, $H > 0$, $G > 0$, $w \in AC^1(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ be such that (3.1) is fulfilled, and let*

$$M_w - m_w \leqslant \frac{G^{\frac{1+\mu}{\mu-\lambda}}}{H^{\frac{1+\lambda}{\mu-\lambda}}}\left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{\frac{(1+\mu)\lambda}{\mu-\lambda}}\frac{\mu-\lambda}{(1+\lambda)\mu}, \tag{3.28}$$

where m_w and M_w are given by (3.4) and (3.15), respectively. Moreover, let us define

$$\beta(t) = \left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{\frac{\mu}{\mu-\lambda}}\left(\frac{G}{H}\right)^{\frac{1}{\mu-\lambda}} + \left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda}\right)^{\frac{\lambda\mu}{\mu-\lambda}}\frac{H^{\frac{\lambda}{\mu-\lambda}}}{G^{\frac{\mu}{\mu-\lambda}}}\big(w(t)-m_w\big) \quad \text{for } t \in [0,\omega] \tag{3.29}$$

and assume that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ verifies the property (P). Then the problem (1.3), (1.2) has at least one positive solution.

Proof. Put $f \equiv 0$ and

$$x_0 = \left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda}\right)^{\frac{\lambda\mu}{\mu-\lambda}}\frac{H^{\frac{\lambda}{\mu-\lambda}}}{G^{\frac{\mu}{\mu-\lambda}}}.$$

Then the assertion follows from Theorem 3.4. $\square$

Corollary 3.4. *Let $\mu > \lambda$, $H > 0$, and $G > 0$. Let, moreover,*

$$\frac{H^{1+\mu}}{G^{1+\lambda}} \leqslant \left(\frac{4}{\omega}\right)^{\mu-\lambda}\left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{(1+\mu)\lambda}\left(\frac{\mu-\lambda}{(1+\lambda)\mu}\right)^{\mu-\lambda} \times \min\left\{1, \left(\frac{1+\lambda}{\mu-\lambda}\right)^{\mu-\lambda}\left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{(\mu-\lambda)(1+\mu)}\right\}. \tag{3.30}$$

Then the problem (1.3), (1.2) has at least one solution.

Proof. According to Lemma 2.4, the inequality (3.30) implies (3.28) and moreover, after some tedious computations one has

$$\int_0^\omega \varphi(s)\,ds = \mu\int_0^\omega \frac{g(s)}{\beta^{1+\mu}(s)}\,ds \leqslant \frac{4}{\omega},$$

with β defined by (3.29). Consequently, by Lemma 2.3, $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ verifies the property (P) and the assertion follows from Corollary 3.3. □

To finish this section, we remark that our approach does not cover the case $\lambda = \mu$, $F = 0$ which is of particular interest for applications (see the introduction of [1]). The following problem is unsolved.

Open problem 3.1. If $\lambda = \mu$, we know that $H > G > 0$ is a necessary condition for the existence of a positive solution of problem (1.3), (1.2). Prove that it is also sufficient.

4. The attractive case

In this section we focus on the equation with a pure attractive singularity, that is, the case when $g \equiv 0$.

Corollary 4.1. *Let $H > 0$, $F > 0$, $\sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ be such that* (3.2) *is fulfilled, and let*

$$(M_\sigma - m_\sigma)^\lambda F < H, \tag{4.1}$$

where m_σ is defined by (3.4) *and*

$$M_\sigma = \max\{\sigma(t)\colon t \in [0, \omega]\}. \tag{4.2}$$

Then the problem (1.4), (1.2) *has at least one positive solution.*

Proof. The assertion follows from Theorem 3.1 with $G = 0$. □

Corollary 4.2. *Let $H > 0$, $F > 0$, and let*

$$\left(\frac{\omega}{4}F_+\right)^\lambda F \leqslant H. \tag{4.3}$$

Then the problem (1.4), (1.2) *has at least one positive solution.*

Proof. By Lemma 2.4, in view of $F > 0$, we have

$$M_\sigma - m_\sigma < \frac{\omega}{4}F_+.$$

Now the assertion follows from Corollary 4.1 in a trivial way. □

The latter result is new even for the original equation posed by Lazer and Solimini,

$$u''(t) = -\frac{1}{u^\lambda(t)} + f(t). \tag{4.4}$$

In [7], it is proved that if f is continuous and ω-periodic, then $F > 0$ is a necessary and sufficient condition for the existence of a positive ω-periodic solution. Here we are extending partially this result to the case when $f \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$. On the other hand, even if f is continuous, then $F > 0$ is not sufficient condition for the existence of a positive ω-periodic solution to Eq. (1.4) in the case, when h is possibly zero on the set of a positive measure, as shown in the following example.

Counter-example 4.1. Let $\varepsilon \in]0, \omega/4[$ and put

$$f(t) = \begin{cases} 0 & \text{for } t \in [0, \frac{\omega}{2} - \varepsilon[\cup]\frac{\omega}{2} + \varepsilon, \omega], \\ \frac{2}{\varepsilon}(t - \frac{\omega}{2} + \varepsilon) & \text{for } t \in]\frac{\omega}{2} - \varepsilon, \frac{\omega}{2}[, \\ \frac{2}{\varepsilon}(\frac{\omega}{2} + \varepsilon - t) & \text{for } t \in [\frac{\omega}{2}, \frac{\omega}{2} + \varepsilon[, \end{cases}$$

$$h(t) = \begin{cases} -\frac{t^2}{2} + \varepsilon(\frac{\omega}{2} - \varepsilon) & \text{for } t \in [0, \varepsilon[, \\ 0 & \text{for } t \in [\varepsilon, \omega - \varepsilon[, \\ -\frac{(\omega - t)^2}{2} + \varepsilon(\frac{\omega}{2} - \varepsilon) & \text{for } t \in [\omega - \varepsilon, \omega], \end{cases}$$

$$v(t) = \begin{cases} -\frac{t^2}{2} + \varepsilon(\frac{\omega}{2} - \varepsilon) & \text{for } t \in [0, \varepsilon[, \\ \varepsilon(\frac{\omega}{2} - t) - \frac{\varepsilon^2}{2} & \text{for } t \in [\varepsilon, \frac{\omega}{2} - \varepsilon[, \\ \frac{(t - \frac{\omega}{2} + \varepsilon)^3}{3\varepsilon} + \varepsilon(\frac{\omega}{2} - t) - \frac{\varepsilon^2}{2} & \text{for } t \in [\frac{\omega}{2} - \varepsilon, \frac{\omega}{2}[, \\ \frac{(\frac{\omega}{2} + \varepsilon - t)^3}{3\varepsilon} + \varepsilon(t - \frac{\omega}{2}) - \frac{\varepsilon^2}{2} & \text{for } t \in [\frac{\omega}{2}, \frac{\omega}{2} + \varepsilon[, \\ \varepsilon(t - \frac{\omega}{2}) - \frac{\varepsilon^2}{2} & \text{for } t \in [\frac{\omega}{2} + \varepsilon, \omega - \varepsilon[, \\ -\frac{(\omega - t)^2}{2} + \varepsilon(\frac{\omega}{2} - \varepsilon) & \text{for } t \in [\omega - \varepsilon, \omega], \end{cases}$$

and

$$\sigma(t) = -\frac{1}{\omega}\left[(\omega - t)\int_0^t s\left(-\frac{F}{H}h(s) + f(s)\right)ds + t\int_t^\omega (\omega - s)\left(-\frac{F}{H}h(s) + f(s)\right)ds\right],$$

where

$$H = \int_0^\omega h(s)\,ds = 2\varepsilon^2\left(\frac{\omega}{2} - \varepsilon\right) - \frac{\varepsilon^3}{3}, \qquad F = F_+ = \int_0^\omega f(s)\,ds = 2\varepsilon.$$

Obviously, f is continuous, $v, \sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$, $\sigma(t) = \sigma(\omega - t)$ for $t \in [0, \omega]$, and consequently, $\sigma'(t) = -\sigma'(\omega - t)$ for $t \in [0, \omega]$. Therefore,

$$\sigma'(\omega) = \sigma'(0) = -\sigma'(\omega), \qquad \sigma'(\omega/2) = -\sigma'(\omega/2),$$

which implies $\sigma'(0) = 0$, $\sigma'(\omega/2) = 0$. Moreover, now it can be easily verified that

$$\max\{\sigma(t)\colon t \in [0, \omega]\} = \sigma(0), \qquad \min\{\sigma(t)\colon t \in [0, \omega]\} = \sigma(\omega/2).$$

Thus

$$M_\sigma - m_\sigma = \sigma(0) - \sigma(\omega/2) = \varepsilon\left(\frac{\omega}{2} - \varepsilon\right) + \frac{\varepsilon^2}{6} + \frac{\varepsilon^3}{12\omega - 28\varepsilon},$$

$$\frac{H}{F} = \varepsilon\left(\frac{\omega}{2} - \varepsilon\right) - \frac{\varepsilon^2}{6}, \qquad \frac{\omega}{4}F_+ = \frac{\omega}{2}\varepsilon.$$

We will show that the problem

$$u'' = -\frac{h(t)}{u} + f(t); \qquad u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{4.5}$$

has no positive solution. Suppose on the contrary, that there exists a positive solution u to (4.5). Put $w(t) = v(t) - u(t)$ for $t \in [0, \omega]$. Then

$$w'' = p(t)w; \qquad w(0) = w(\omega), \qquad w'(0) = w'(\omega)$$

with

$$p(t) = \begin{cases} \frac{1}{u(t)} & \text{for } t \in [0, \varepsilon[\cup [\omega - \varepsilon, \omega], \\ 0 & \text{for } t \in [\varepsilon, \omega - \varepsilon[. \end{cases}$$

Consequently $w \equiv 0$, i.e. $u \equiv v$. However, $v(\omega/2) = -\varepsilon^2/6 < 0$, which contradicts our assumption.

This example shows that the inequalities (4.1) and (4.3) in Corollaries 4.1 and 4.2 are optimal in a certain sense and cannot be improved. In particular, the condition (4.1), respectively (4.3), cannot be replaced by the condition

$$(M_\sigma - m_\sigma)^\lambda F \leqslant H + \varepsilon,$$

respectively

$$\left(\frac{\omega}{4} F_+\right)^\lambda F \leqslant H + \varepsilon,$$

no matter how small ε is.

We finish the section with two open questions.

Open problem 4.1. Let us assume $f \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$, and $\lambda > 0$. Prove or disprove that $F > 0$ is a necessary and sufficient condition for the existence of an ω-periodic positive solution of the Lazer–Solimini equation (4.4).

Open problem 4.2. Let us assume $h \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R}_+)$, $f \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$, $\lambda > 0$. Assume, moreover, that

$$h(t) > 0 \quad \text{for a.e. } t \in [0, \omega],$$

and there does not exist $\varepsilon_0 > 0$ such that

$$h(t) \geqslant \varepsilon_0 \quad \text{for a.e. } t \in [0, \omega].$$

Find a condition different from (4.1) (respectively (4.3)) sufficient for the existence of a positive solution of problem (1.4), (1.2).

5. The repulsive case

Finally, we analyze the equation with a pure repulsive singularity, that is, the case when $h \equiv 0$.

Corollary 5.1. *Let $G > 0$, $F < 0$, $\sigma \in AC^1(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ be such that* (3.18) *is fulfilled, and let*

$$(M_\sigma - m_\sigma)^\mu |F| < G, \tag{5.1}$$

where m_σ and M_σ are defined by (3.4) *and* (4.2)*, respectively. Let, moreover, either*

$$\frac{\mu |F|^{\frac{1+\mu}{\mu}} g(t)}{(G^{1/\mu} - |F|^{1/\mu}(M_\sigma - \sigma(t)))^{1+\mu}} \leqslant \left(\frac{\pi}{\omega}\right)^2 \quad \textit{for a.e. } t \in [0, \omega], \tag{5.2}$$

or

$$\mu |F|^{\frac{1+\mu}{\mu}} \int_0^\omega \frac{g(s)}{(G^{1/\mu} - |F|^{1/\mu}(M_\sigma - \sigma(s)))^{1+\mu}}\, ds \leqslant \frac{4}{\omega}. \tag{5.3}$$

Then the problem (1.5), (1.2) *has at least one positive solution.*

Proof. Put $H = 0$,

$$x_0 = \frac{|F|^{\lambda/\mu}}{G(G^{1/\mu} - |F|^{1/\mu}(M_\sigma - m_\sigma))^\lambda},$$

and define a function β by (3.20). After some algebra,

$$\beta(t) = \left(\frac{G}{|F|}\right)^{1/\mu} - M_\sigma + \sigma(t) \quad \text{for } t \in [0, \omega]$$

and each of (5.2) and (5.3) guarantees that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ satisfies the property (P). Moreover, (5.1) yields (3.19). Therefore the assertion follows from Theorem 3.3. □

Corollary 5.2. *Let* $G > 0$ *and* $F < 0$*. Let, moreover,*

$$\left(\frac{\omega}{4}\mu G\right)^{\frac{1}{1+\mu}} |F|^{1/\mu} + \frac{\omega}{4} F_- |F|^{1/\mu} \leqslant G^{1/\mu}. \tag{5.4}$$

Then the problem (1.5), (1.2) *has at least one positive solution.*

Proof. According to Lemma 2.4,

$$M_\sigma - m_\sigma \leqslant \frac{\omega}{4} F_-.$$

Then, the inequality (5.4) implies both (5.1) and (5.3). Consequently, the assertion follows from Corollary 5.1. □

Again, this result is new even for the original equation posed by Lazer and Solimini,

$$u''(t) = \frac{1}{u^\mu(t)} + f(t). \tag{5.5}$$

In [7], it is proved that if $f \in L(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R})$ and $\mu \geqslant 1$ (strong force assumption), then $F < 0$ is a necessary and sufficient condition for the existence of a positive ω-periodic solution. Moreover, it is shown with a counter-example that the strong force assumption cannot be dropped without additional conditions. Later, in [11] the authors proved that (5.5) with $\mu < 1$ has a positive ω-periodic solution if $F < 0$ and

$$f(t) \geqslant -\left(\frac{\pi^2}{\omega^2 \mu}\right)^{\frac{\mu}{\mu+1}} (\mu + 1) \quad \text{for a.e. } t \in [0, \omega].$$

Therefore, a uniform bound from below is required. The importance of Corollary 5.2 in this context relies in that it provides for the first time a sufficient existence condition for a truly $f \in L(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$ (possibly unbounded). Of course, the main question remains open.

Open problem 5.1. Let us assume $g \in L(\mathbb{R}/\omega\mathbb{Z};\mathbb{R}_+)$, $g \not\equiv 0$, $f \in L(\mathbb{R}/\omega\mathbb{Z};\mathbb{R})$, and $\mu > 0$. Find a necessary and sufficient condition over f, g for the existence of a positive solution of problem (1.5), (1.2).

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