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Solitary Waves for Linearly Coupled Nonlinear Schrödinger Equations with Inhomogeneous Coefficients

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Abstract Motivated by the study of matter waves in Bose–Einstein condensates and coupled nonlinear optical systems, we study a system of two coupled nonlinear Schrödinger equations with inhomogeneous parameters, including a linear coupling. For that system, we prove the existence of two different kinds of homoclinic solutions to the origin describing solitary waves of physical relevance. We use a Krasnoselskii fixed point theorem together with a suitable compactness criterion.

Keywords Nonlinear Schrödinger systems · Solitary waves · Fixed-point theorems in cones

Mathematics Subject Classification (2000) 34C37 · 34C60 · 35Q51 · 35Q55

1 Introduction

Nonlinear Schrödinger (NLS) equations appear in a great array of contexts (Vázquez et al. 1997; Sulem and Sulem 2000) as, for example, in semiconductor electron-

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ics (Brezzi and Markowich 1991; López and Soler 2000), optics in nonlinear media (Kivshar and Agrawal 2003), photonics (Hasegawa 1989), plasmas (Dodd et al. 1982), foundation of quantum mechanics (Rosales and Sánchez-Gómez 1992), dynamics of accelerators (Fedele et al. 1993), mean-field theory of Bose–Einstein condensates (Dalfovo et al. 1999; Lieb and Seiringer 2002; Lieb et al. 2000) or in biomolecule dynamics (Davydov 1985). In some of these fields and many others, the NLS equation appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating in a nonlinear medium (Scott 1999). With respect to the theory of Bose–Einstein condensates, a rigorous derivation of the time-dependent Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensates can be found in Erdős et al. (2006, 2007a, 2007b).

In this paper, we will be interested in systems of two coupled NLS equations in which there is a linear coupling between both components.

A first example which arises in the study of a spinor Bose–Einstein condensate composed of two hyperfine states (for instance, in many experiments, the $|F=1, m_f=-1\rangle$ and $|F=2, m_f=1\rangle$ states of ^{87}Rb atoms are used) and coupled by an optical or r.f. field. These systems have received a lot of attention both experimentally (Matthews et al. 1999a, 1999b; Minardi et al. 2001; Modugno et al. 2002; Maddaloni et al. 2000; Catani et al. 2008) and theoretically (see, e.g., Williams et al. 2000; García-Ripoll et al. 2000, 2002; Deconinck et al. 2004; Kasamatsu and Tsubota 2006; Brazhnyi and Konotop 2005; Merhasin et al. 2005; Nakamura et al. 2007; Saito et al. 2007 and references therein) since they represent the simplest mixture of different ultracold quantum degenerate gases and were studied immediately after the historical achievement of Bose–Einstein condensation in 1995 Dalfovo et al. (1999).

To simplify the treatment yet preserving the spatial aspects, we assume the condensate to be magnetically tightly confined along two of the transverse directions to a single effective dimension (Pérez-García et al. 1998). In the mean-field approximation, the system is described by the Gross–Pitaevskii equations (Matthews et al. 1999a, 1999b)

$$i\frac{\partial \psi_1}{\partial t} = \left(L_1 + U_{11}|\psi_1|^2 + U_{12}|\psi_2|^2\right)\psi_1 + \lambda\psi_2, \tag{1a}$$

$$i\frac{\partial \psi_2}{\partial t} = \left(L_2 + U_{22}|\psi_2|^2 + U_{21}|\psi_1|^2\right)\psi_2 + \lambda\psi_1, \tag{1b}$$

where $L_j = -\partial^2/\partial x^2 + V_j$ with $j = 1, 2$. Equation (1) is written in dimensionless form: the spatial coordinates and time are measured in units of $\ell = \sqrt{\hbar/(2m\omega)}$ and $1/\omega$, respectively, while the energies and frequencies are measured in units of $\hbar\omega$ and ω, ω being the trap frequency in the (x, z)-plane. The dimensionless nonlinear coefficients, for the quasi-one-dimensional condensate, are given by $U_{ij} = U_{ji} = 2Na_{ij}/\ell$ where a_{ij} are the scattering lengths for binary collisions, N is the total number of atoms, and ℓ is the oscillator length in the y-direction. The normalization of the wave function $\psi = \mathrm{col}(\psi_1, \psi_2)$ is then $\int \psi^\dagger \psi \, d^2r = 1$. In many practical situations for spinor condensates, we have $U_{11} \simeq U_{12} = U_{21} \simeq U_{22}$ and it is customary to consider all of the scattering lengths to be equal.

A specific situation which arises also in applications to BEC happens when two different condensates do not coexist, but are spatially separated and weakly coupled.

In that situation, described, for example, in Raghavan et al. (1999), Li and Wang (2008) and references therein the model is equivalent to (1) but with different nonlinearities given by $U_{12} = U_{21} = 0$ (i.e., there is no nonlinear coupling between both equations).

Although the dynamics of coupled Bose–Einstein condensates has attracted a lot of attention in the last years, (1) also arise in other physical contexts, most notably in nonlinear optical models, where they describe coupled optical fibers where the functions ψ_1 and ψ_2 describe the light field within each waveguide (Zafrany et al. 2005). Also, other physical situations in nonlinear optics are described by this type of model equations (Kivshar and Agrawal 2003).

In this paper, we will consider the physically relevant question of the existence of solitary wave solutions to models such as the one given by (1), but in more general scenarios where all the parameters (i.e., the potential, the nonlinear coefficients, and the coupling coefficients) are functions depending on the spatial variables.

The study of the properties of localized solutions and propagating waves in Bose–Einstein condensates with spatially inhomogeneous interactions has been a field of an enormous level of activity in physics in the last few years (Rodas-Verde et al. 2005; Teocharis et al. 2005; Primatarowa et al. 2005; Abdullaev and Garnier 2005; Garnier and Abdullaev 2006; Vázquez-Carpentier et al. 2006; Sakaguchi and Malomed 2006; Belmonte-Beitia et al. 2007; Niarchou et al. 2007; Dong and Hu 2007; Porter et al. 2006; Belmonte-Beitia et al. 2008) motivated by novel experimental ways to control experimentally the interactions (i.e., optical manipulation of the Feschbach resonances 2006, Theis et al. 2004).

In this paper, we will consider theoretically the problem of the existence of solitary wave solutions in two-component condensates when all of the physical parameters are spatially dependent, i.e., the potentials $V_j(x)$, the nonlinear coefficients $U_{ij}(x)$, and the coupling parameter $\lambda(x)$. The latter possibility has not been explored yet, neither experimentally nor theoretically. First, because the analyses of spatially dependent interactions have been focused on the case of a single component (Rodas-Verde et al. 2005; Teocharis et al. 2005; Primatarowa et al. 2005; Abdullaev and Garnier 2005; Garnier and Abdullaev 2006; Vázquez-Carpentier et al. 2006; Sakaguchi and Malomed 2006; Belmonte-Beitia et al. 2007; Niarchou et al. 2007; Dong and Hu 2007; Porter et al. 2006; Belmonte-Beitia et al. 2008). Secondly, because the coupling coefficient λ is usually taken to be spatially homogeneous. However, it is very simple to add a spatial dependence to this parameter in experiments since λ is related to the coupling field and in the optical case this means just using a spatially dependent laser, i.e., a beam with a prescribed profile on the scale of the BEC.

Thus, we will consider the physically relevant problem of studying solitary wave solutions to the problem

$$i\frac{\partial\psi_1}{\partial t} = \left[-\frac{\partial^2}{\partial x^2} + a(x)\right]\psi_1 - b(x)\psi_2 - c(x)F(\psi_1,\psi_2), \tag{2a}$$

$$i\frac{\partial\psi_2}{\partial t} = \left[-\frac{\partial^2}{\partial x^2} + d(x)\right]\psi_2 - e(x)\psi_1 - f(x)H(\psi_1,\psi_2), \tag{2b}$$

where ψ_j are the complex wavefunctions defined on all $\mathbb{R}$ which for solitary waves, i.e., localized solutions, must also decay at infinity and we consider general nonlinearities F and H satisfying certain conditions to be made precise later. In order to look for stationary solutions, we eliminate time as usual by defining

$$\psi_j(x,t) = u_j(x)\exp(i\omega_j t). \tag{3}$$

Since all physically relevant nonlinearities are homogeneous in time and redefining the potentials $a(x), d(x)$ to incorporate the constant factors proportional to $\omega_j u_j$, we get the final equations to be studied in detail in this paper

$$-u_1''(x) + a(x)u_1(x) - b(x)u_2(x) = c(x)F(u_1,u_2)u_1, \tag{4a}$$

$$-u_2''(x) + d(x)u_2(x) - e(x)u_1(x) = f(x)H(u_1,u_2)u_2, \tag{4b}$$

i.e., a set of two coupled stationary linearly coupled nonlinear Schrödinger equations with spatially inhomogeneous coefficients. We will keep in mind the previous discussion from which it is clear that the two most interesting situations from the point of view of applications are

$$F(u_1,u_2) = u_1^2, \qquad H(u_1,u_2) = u_2^2 \tag{5a}$$

for weakly coupled Bose–Einstein condensates or optical fibers and

$$F(u_1,u_2) = H(u_1,u_2) = u_1^2 + u_2^2, \tag{5b}$$

for multicomponent spinor Bose–Einstein condensates coupled by an optical field (Williams et al. 2000). The purpose of this paper is to study the existence of localized positive solutions $\mathbf{u} = (u_1, u_2)$ of (4) satisfying

$$\lim_{|x|\to\infty} u_i(x) = 0, \qquad \lim_{|x|\to\infty} u_i'(x) = 0, \qquad i = 1,2, \tag{6}$$

where $a,b,c,d,e,f \in L^\infty(\mathbb{R})$, in (4), are nonnegative almost everywhere and $F(u_1,u_2)$ and $H(u_1,u_2)$ are continuous functions. Such solution $\mathbf{u}$ will have finite energy that mathematically means that it belongs to $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

Between the different solutions of (4) those with a more direct physical interest are the so-called ground states, which are positive real solutions of that system. Although we will be mostly interested on those solutions, we will also consider the existence of other families of solutions which are relevant for applications. The existence of positive solutions for different types of vector nonlinear Schrödinger equations has attracted a lot of interest in recent years from the mathematical point of view. For instance, in Ambrosetti et al. (2007, 2008), Ambrosetti and Colorado (2006, 2007), the existence of positive solutions was proven using either critical point theory or variational approaches.

In that context, in this paper, we complement previous studies with the analysis of the richer system given by (4), which describes physical situations beyond those studied previously. To do so, we will also develop a new theoretical approach to the problem based on the use of a fixed point theorem due to Krasnoselskii for completely continuous operators defined in cones of a Banach space together with a suitable study of the Green's function for the linear part of the problem. This method has been employed successfully for scalar problems on the real line (Torres 2006) and in some problems on bounded domains (Chu et al. 2007). Using this technique, we will be able to prove the existence of positive solutions of (4) under the conditions (6). We think that beyond its own interest, this topological approach appears to be complementary to the variational approach. In fact, these equations cannot be studied by using a variational approach, since that the linear coefficients $b(x)$ and $e(x)$ are different. In this way, we have an example where a topological approach provides more information than a variational approach. Finally, let us remark that the extension of our study to higher dimensions can be done without difficulty with the logical changes. We have limited the exposition to the 2D case for the sake of brevity.

The rest of the paper is organized as follows. In Sect. 2, some preliminary results are collected. Section 3 contains the main result about the existence of a homoclinic orbit to the origin (positive solution). In Sect. 4, we prove that if in addition to the hypotheses of the main result of Sect. 3 we assume a set of hypotheses on the functions a, b, c, d, e, f, then there exists a second wave which is odd. Finally, Sect. 5 contains a result about the study of branches of solutions dependent on a parameter.

We use the notation $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$. For a given $a \in L^\infty(\mathbb{R})$, the norm of the supremum in denoted by $\|a\|_\infty$ and the essential infimum is a_*. The support of a given function a is denoted by $\text{Supp}(a)$. The limit value of a given function u in $+\infty$ (or $-\infty$) is written simply as $u(+\infty)$ (or $u(-\infty)$).

2 Preliminaries

The proof of the main results is based on a well-known fixed-point theorem in cones for a completely continuous operator defined on a Banach space, due to Krasnoselskii (1964). We recall the statement of this result below, after introducing the definition of a cone (see, for example, Granas and Dugundji 2003).

Definition 1 Let X be a Banach space and P be a closed, nonempty subset of X. P is a cone if

(1) $\lambda x + \mu y \in P$ $\forall x, y \in P$ and $\forall \lambda \geq 0, \mu \geq 0$.
(2) $x, -x \in P$ implies $x = 0$.

We also recall that a given operator is completely continuous if the image of a bounded set is relatively compact.

Theorem 1 *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ and let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that one of the following conditions is satisfied*:

(1) $\|Tu\| \leq \|u\|$, *if* $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, *if* $u \in P \cap \partial\Omega_2$.
(2) $\|Tu\| \geq \|u\|$, *if* $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, *if* $u \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

This result has been extensively employed in the study of nonlinear equations (Torres 2006; Zima 2001) and also in the study of boundary value nonlinear systems (Chu et al. 2007; Jiang et al. 2002). However, for problems defined in noncompact intervals such as ours, there is the difficulty that the Ascoli–Arzela theorem is not sufficient for proving the complete continuity of the operator. We shall employ the following compactness criterion (reminiscent of a result by Zima 2001) to show that the operator is completely continuous.

Proposition 1 *Let $\Omega \subset BC(\mathbb{R})$. Let us assume that the functions $u \in \Omega$ are equicontinuous in each compact interval of $\mathbb{R}$ and that for all $u \in \Omega$ we have*

$$\left|u(x)\right| \le \xi(x), \quad \forall x \in \mathbb{R} \tag{7}$$

where $\xi \in BC(\mathbb{R})$ verifies

$$\lim_{|x|\to+\infty} \xi(x) = 0. \tag{8}$$

Then Ω is relatively compact.

Proof Given $\{u_n\}_n$ a sequence of functions of Ω, we have to prove that there exists a partial sequence which is uniformly convergent to a certain u. Note that the elements of Ω are uniformly bounded by $\|\xi\|_\infty$ and equicontinuous on compact intervals by hypothesis, therefore, the Ascoli–Arzela theorem provides partial sequence (call it again $\{u_n\}_n$) which is uniformly convergent to a certain u on compact intervals. Of course, u satisfies also (7). Now, we have to prove that

$$\forall \varepsilon > 0,\ \exists n_0 \quad \text{s.t. } n \ge n_0 \quad \Longrightarrow \quad \|u_n - u\|_\infty < \varepsilon.$$

By using (8), fix $k > 0$ such that $\max_{x\in\mathbb{R}\setminus]-k,k[} |\xi(x)| < \varepsilon/4$. On the other hand, by using the uniform convergence on compact intervals, there exists n_0 such that $\max_{x\in[-k,k]} |u_n(x) - u(x)| < \varepsilon/2$ for all $n \ge n_0$. Then

$$\begin{aligned}\|u_n - u\|_\infty &\le \max_{x\in[-k,k]}\left|u_n(x) - u(x)\right| + \max_{x\in\mathbb{R}\setminus]-k,k[}\left|u_n(x) - u(x)\right| \\ &< \frac{\varepsilon}{2} + 2\max_{x\in\mathbb{R}\setminus]-k,k[}\left|\xi(x)\right| < \varepsilon,\end{aligned}$$

and the proof is complete. □

In order to apply Theorem 1, we need some information about the properties of the Green's function. Let $a \in L^\infty(\mathbb{R})$, $a_* > 0$. For the homogeneous problem

$$-\phi'' + a(x)\phi = 0, \tag{9a}$$

$$\phi(-\infty) = 0, \qquad \phi(\infty) = 0, \tag{9b}$$

the associated Green's function is given by

$$G_1(x,s) = \begin{cases} \phi_1(x)\phi_2(s), & -\infty < x \le s < +\infty, \\ \phi_1(s)\phi_2(x), & -\infty < s \le x < +\infty \end{cases} \tag{10}$$

where ϕ_1, ϕ_2 are solutions such that $\phi_1(-\infty) = 0, \phi_2(+\infty) = 0$. Moreover, ϕ_1, ϕ_2 can be chosen as positive increasing and positive decreasing functions, respectively. For a given $h(x) \in L^1(\mathbb{R})$, the function $u(x) = \int_{\mathbb{R}} G(x,s)h(s)\,ds$ is the unique solution of the equation $-\phi'' + a(x)\phi = h(x)$ in the Sobolev space $H^1(\mathbb{R})$. In particular, u and u' vanish at $\pm\infty$. See, for instance (Stuart 1993).

Note that ϕ_1, ϕ_2 intersect in a unique point x_0. So, we can define a function $p_1 \in BC(\mathbb{R})$ by

$$p_1(x) = \begin{cases} 1/\phi_2(x), & x \le x_0, \\ 1/\phi_1(x), & x > x_0. \end{cases} \tag{11}$$

The following result was proven in Torres (2006).

Proposition 2 *The following properties for the Green's function defined by* (10) *hold.*

(P1) $G_1(x,s) > 0$ *for every* $(x,s) \in \mathbb{R} \times \mathbb{R}$.
(P2) $G_1(x,s) \le G_1(s,s)$ *for every* $(x,s) \in \mathbb{R} \times \mathbb{R}$.
(P3) *Given a nonempty compact subset* $P \subset \mathbb{R}$, *we define*

$$m_1(P) = \min\big(\phi_1(\inf P), \phi_2(\sup P)\big). \tag{12}$$

Then

$$G_1(x,s) \ge m_1(P)p_1(s)G_1(s,s) \quad \textit{for all } (x,s) \in P \times \mathbb{R}. \tag{13}$$

Obviously, the linear operator $L_2[u] \equiv -u'' + d(x)u$ with $d_* > 0$ defines a second Green's function $G_2(x,s)$ with analogous properties involving properly defined $p_2(x), m_2(P)$.

3 Existence of Positive Bound States

From now on, we will assume that $M = \mathrm{Supp}(b) \cup \mathrm{Supp}(c) \cup \mathrm{Supp}(e) \cup \mathrm{Supp}(f)$ is a nonempty compact set. In order to apply Theorem 1, we take the Banach space $X = BC(\mathbb{R}) \times BC(\mathbb{R})$ with the norm $\|\mathbf{u}\| = \max_{i=1,2} \|u_i\|_\infty$, for $\mathbf{u} = (u_1, u_2) \in X$. Let us define the cone

$$P = \Big\{\mathbf{u} = (u_1, u_2) \in X : u_1(x), u_2(x) \ge 0\ \forall x,\ \min_{x\in M} u_i(x) \ge m_i p_0^i \|u_i\|_\infty\Big\} \tag{14}$$

where $p_0^1 = \inf_M p_1(x)$, $p_0^2 = \inf_M p_2(x)$ and the constants $m_1 \equiv m_1(M)$ and $m_2 \equiv m_2(M)$ are defined by property (P3). Note that the compactness of M implies that $p_0^1 > 0$ and $p_0^2 > 0$. Moreover, as from (P3), it is easy to see that $m_i p_0^i < 1$, for $i = 1, 2$. Thus, one can easily verify that P is a cone in X.

Let $T : P \to X$ be a map with components (T_1, T_2) defined by

$$T_1(\mathbf{u})(x) = \int_{\mathbb{R}} G_1(x,s)\big(b(s)u_2(s) + c(s)F(u_1,u_2)u_1(s)\big)\,ds \tag{15a}$$

$$= \int_M G_1(x,s)\big(b(s)u_2(s) + c(s)F(u_1,u_2)u_2(s)\big)\,ds. \tag{15b}$$

$$T_2(\mathbf{u})(x) = \int_{\mathbb{R}} G_2(x,s)\big(e(s)u_1(s) + f(s)H(u_1,u_2)u_2(s)\big)\,ds \tag{15c}$$

$$= \int_M G_2(x,s)\big(e(s)u_1(s) + f(s)H(u_1,u_2)u_2(s)\big)\,ds. \tag{15d}$$

A fixed point of T is a solution of system (4) which belongs to $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, and, therefore, verifies the boundary conditions (6).

Lemma 1 *Let us assume that*

$$F(u_1,u_2),\ H(u_1,u_2) \geq 0 \quad \textit{for every } u_1,u_2 \geq 0.$$

Then $T(P) \subset P$.

Proof Take $\mathbf{u} = (u_1,u_2) \in P$. The property (P1) of the Green's function together with $b_*, c_* \geq 0$ implies $T_1\mathbf{u}(x) \geq 0$ for all x. Let us call x_m the point where $\min_{x\in M} T_1\mathbf{u}$ is attained. Then for all $x \in \mathbb{R}$,

$$\begin{aligned}
T_1(\mathbf{u})(x_m) = T_1(u_1,u_2)(x_m) &= \int_M G_1(x_m,s)\big(b(s)u_2(s) + c(s)F(u_1,u_2)u_1(s)\big)\,ds \\
&\geq m_1 \int_M p(s)G_1(s,s)\big(b(s)u_2(s) + c(s)F(u_1,u_2)u_1(s)\big)\,ds \\
&\geq m_1 p_0^1 \int_M G_1(x,s)\big(b(s)u_2(s) + c(s)F(u_1,u_2)u_1(s)\big)\,ds \\
&= m_1 p_0^1 T_1(u_1,u_2)(x) = m_1 p_0^1 T_1(\mathbf{u})(x)
\end{aligned} \tag{16}$$

where we have used (P2) and (P3). In a similar way, we can prove that $T_2(\mathbf{u})(x_m) \geq m_2 p_0^2 T_2(\mathbf{u})(x)$. This completes the proof. □

Lemma 2 *$T : P \to P$ is continuous and completely continuous.*

Proof The continuity is trivial. Let us prove that the components of T are completely continuous. Let $\Omega \subset P$ be a bounded set, with $C > 0$ a uniform bound for its elements. The functions of $T_1(\Omega)$ are equicontinuous on each compact interval (in fact, the derivative is bounded in compacts). On the other hand, for any $\mathbf{u} \in \Omega$,

$$\big|T_1(\mathbf{u})(x)\big| \leq C \int_{\mathbb{R}} G_1(x,s)b(s)\,ds + C \max_{\|\mathbf{u}\|\leq C} F(u_1,u_2) \int_{\mathbb{R}} G_1(x,s)c(s)\,ds =: \xi(x).$$

Since the supports of b, c are compact, $\xi \in BC(\mathbb{R}) \cap L^1(\mathbb{R})$, therefore, $T_1(\Omega)$ is relatively compact by Proposition 1. The proof for T_2 is analogous. □

The following one is the main result in this section.

Theorem 2 *Let us assume the following hypotheses*:

(i) $a_*, d_* > 0, b_*, c_*, e_*, f_* \geq 0$.
(ii) M *is a nonempty compact set*.
(iii) $F(u_1, u_2), H(u_1, u_2) \geq 0$ *for every* $u_1, u_2 \geq 0$.
(iv) *There exist* $r_0 > 0$ *and* $\gamma, k > 0$ *such that given* $0 < r < r_0$,

$$F(u_1, u_2), H(u_1, u_2) < kr^\gamma, \quad \textit{for all } \|\mathbf{u}\| < r.$$

(v) *There exist* $R_0 > 0$ *and* $\delta, K > 0$ *such that given* $R > R_0$,

$$F(u_1, u_2), H(u_1, u_2) > KR^\delta, \quad \textit{if } u_1 \in [m_1 p_0^1 R, R] \textit{ or } u_2 \in [m_2 p_0^2 R, R].$$

(vi) $\int_M G_1(x, s)b(s)\,ds, \int_M G_2(x, s)e(s)\,ds < 1$ *for all* $x \in \mathbb{R}$.

Then there exists a nontrivial solution $\mathbf{u} \in X$ *of the system* (4)–(6).

Proof We define the open sets Ω_1 and Ω_2 as the open balls in X centered in the origin and with radius r and R, respectively, to be fixed later. Let us take $\mathbf{u} \in P \cap \partial\Omega_1$. Thus,

$$\begin{aligned}
\|T\mathbf{u}\| &= \max_{x\in\mathbb{R}}\biggl(\int_M G_1(x, s)\bigl(c(s)F(u_1, u_2)u_1(s) + b(s)u_2(s)\bigr)\,ds, \\
&\qquad \int_M G_2(x, s)\bigl(f(s)H(u_1, u_2)u_2(s) + e(s)u_1(s)\bigr)\,ds\biggr) \\
&\leq kr^{\gamma+1}\max_{x\in\mathbb{R}}\biggl(\int_M G_1(x, s)c(s)\,ds, \int_M G_2(x, s)f(s)\,ds\biggr) \\
&\quad + r\max_{x\in\mathbb{R}}\biggl(\int_M G_1(x, s)b(s)\,ds, \int_M G_2(x, s)e(s)\,ds\biggr) \leq r = \|\mathbf{u}\| \qquad (17)
\end{aligned}$$

for r sufficiently small, where we have used the hypothesis (iv) and (vi).

On the other hand, let us take $\mathbf{u} \in P \cap \partial\Omega_2$. Thus, at least a component of $\mathbf{u}$, say u_1, satisfies $\|u_1\|_\infty = R$ (the case $\|u_2\|_\infty = R$ is similar). Then $m_1 p_0^1 R \leq u_1(x) \leq R$ for all $x \in M$. Therefore, by using hypothesis (v), we get

$$\begin{aligned}
\|T\mathbf{u}\| &= \max_{i=1,2,x\in\mathbb{R}}\bigl(|T\mathbf{u}|\bigr)_i \geq \max_{x\in M}\int_M G_1(x, s)\bigl(c(s)F(u_1, u_2)u_1(s) + b(s)u_2(s)\bigr)\,ds \\
&\geq KR^{\delta+1}\max_{x\in M}\int_M G_1(x, s)c(s)\,ds \geq R = \|\mathbf{u}\| \qquad (18)
\end{aligned}$$

for R sufficiently large. For the case $\|u_2\|_\infty = R$, everything works in the same way.

Now, Theorem 1 guarantees that T has a fixed point $\mathbf{u} \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Thus, $r \leq \|\mathbf{u}\| \leq R$ so it is a nontrivial solution. □

Although the hypothesis (vi) appears to be quite technical, its meaning can be clarified by realizing that the function $\int_M G_1(x, s)b(s)\,ds$ (resp. $\int_M G_2(x, s)e(s)\,ds$)

can be interpreted as the unique solution of the equation $-u'' + a(x)u = b(x)$ (resp. $-u'' + d(x)u = e(x)$) with boundary conditions $u(-\infty) = 0 = u(+\infty)$. Having this idea in mind, we can formulate the following corollary.

Corollary 1 *The later result still holds if* (vi) *is replaced by*

$$\text{(vi}'\text{)} \qquad \max_{x\in\mathbb{R}} \frac{b(x)}{a(x)} < 1, \qquad \max_{x\in\mathbb{R}} \frac{e(x)}{d(x)} < 1.$$

Proof Defining $u(x) = \int_M G_1(x,s)b(s)\,ds$, we need to prove that $\|u\|_\infty < 1$. Take x_0 such that $u(x_0) = \|u\|_\infty$, then u is convex in a neighborhood of x_0, so from the equation $-u'' + a(x)u = b(x)$ we easily get

$$u(x_0) \le \frac{b(x_0)}{a(x_0)} < 1.$$

In the same way, $\int_M G_2(x,s)e(s)\,ds < 1$ and Theorem 2 applies. □

4 Odd Solitary Waves in the Equation with Symmetric Coefficients

The aim of this section is to extend some ideas presented in (Torres 2006, Sect. 4) for a scalar NLS equation to the vectorial case. In the previous section, we have obtained positive solutions of the problem under consideration, which is the case considered in the all related papers known to the authors. In this section, we prove the existence of a new kind of solution when the coefficients are even. In the following result, $p_0^1 = \inf_{M\cap\mathbb{R}^+} p_1(x)$, $p_0^2 = \inf_{M\cap\mathbb{R}^+} p_2(x)$ and $m_1 \equiv m_1(M\cap\mathbb{R}^+)$, $m_2 \equiv m_2(M\cap\mathbb{R}^+)$.

Theorem 3 *Under the conditions of Theorem* 2, *if moreover* a, b, c, d, e, f *are even functions*, $0 \notin M$ *and* F *and* H *are even functions*, *then there exists an odd nontrivial solution* $\mathbf{u} \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *of* (4) *such that* $r \le \|\mathbf{u}\| \le R$.

Proof The proof mimics the steps of the proof of Theorem 2, working now with the Green's functions for the problem on the half-line $\mathbb{R}_+$, with boundary conditions $u_i(0) = 0 = u_i(+\infty)$. The operator $T : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \to BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is defined by

$$T_1(\mathbf{u})(x) = \int_{\mathbb{R}_+} G_1(x,s)\big(b(s)u_2(s) + c(s)F(u_1,u_2)u_1(s)\big)\,ds, \tag{19a}$$

$$T_2(\mathbf{u})(x) = \int_{\mathbb{R}_+} G_2(x,s)\big(e(s)u_1(s) + f(s)H(u_1,u_2)u_2(s)\big)\,ds. \tag{19b}$$

The adequate cone is

$$P = \Big\{\mathbf{u} \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) : u_1(0) = u_2(0) = 0,$$
$$u_1(x), u_2(x) \ge 0,\ \forall x \in \mathbb{R}_+,\ \min_{x\in M\cap\mathbb{R}_+} u_i(x) \ge m_i p_0^i \|u_i\|\Big\}. \tag{20}$$

Then $T(P) \subset P$ and T is a continuous and completely continuous operator, since Proposition 1 can be applied to functions u defined only in $\mathbb{R}_+$ and such that $u(0)=0$ by simply extending as the zero constant function on the negative axis. Finally, the sets Ω_1 and Ω_2 are defined again as the open balls of radius r and R, respectively. Everything works in the same way so the repetitive details are omitted. In conclusion, we obtain a positive nontrivial solution $\mathbf{u} \in H^1(\mathbb{R}_+) \times H^1(\mathbb{R}_+)$ such that $u_1(0) = u_2(0) = 0$ and the odd extension gives the desired solution. □

Note that the assumptions $0 \notin M$ is necessary in order to have $m_1 \neq 0$, $m_2 \neq 0$, which is a key point in the proper definition of the cone.

Finally, let us note that an analogous of Corollary 1 also holds for this case.

5 Branches of Solutions in Situations of Physical Interest

As it was said in the Introduction, there are two situations arising in physical applications described by (5a) and (5b). Our results can also provide some information on the localization of the solutions that can be of interest in the study of branches of solutions in systems controlled by parameters. As a basic example, we consider the system (4) with the nonlinear contribution given by (5a), i.e.,

$$-u_1''(x) + a(x)u_1(x) = b(x)u_2(x) + \lambda c(x)u_1^3(x), \tag{21a}$$

$$-u_2''(x) + d(x)u_2(x) = e(x)u_1(x) + \lambda f(x)u_2^3(x) \tag{21b}$$

where $\lambda > 0$.

Corollary 2 *Let us assume the conditions of Theorem 2. Then for all $\lambda > 0$, there exists a positive solution $\mathbf{u}_\lambda = (u_{1\lambda}, u_{2\lambda}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of (21a). Moreover,*

$$\lim_{\lambda \to 0^+} \|\mathbf{u}_\lambda\| = +\infty, \qquad \lim_{\lambda \to +\infty} \|\mathbf{u}_\lambda\| = 0. \tag{22}$$

If moreover a, b, c, d, e, f are even functions and $0 \notin M$, there exists a second branch of odd solutions $\tilde{\mathbf{u}}_\lambda \in H^1(\mathbb{R})$ with the same limiting properties.

Proof The application of Theorem 2 requires the existence of r_λ, R_λ such that

$$\begin{aligned} r_\lambda^2 &\le \left[1 - \max_{x\in M}\left(\int_M G_1(x,s)b(s)\,ds, \int_M G_2(x,s)e(s)\,ds\right)\right] \\ &\quad \times \left[\lambda \max_{x\in M}\left(\int_M G_1(x,s)c(s)\,ds, \int_M G_2(x,s)f(s)\,ds\right)\right]^{-1} \\ &\le \left[\lambda m_0 p_0^1 \max_{x\in M}\int_M G_1(x,s)c(s)\,ds\right]^{-1} \le R_\lambda^2. \end{aligned} \tag{23}$$

Since $\int_M G_1(x,s)b(s)\,ds, \int_M G_2(x,s)e(s)\,ds < 1$ for all $x \in \mathbb{R}$, such r_λ, R_λ exist and can be chosen so that

$$\lim_{\lambda \to 0^+} r_\lambda = +\infty, \qquad \lim_{\lambda \to +\infty} R_\lambda = 0. \tag{24}$$

Hence, we obtain a branch of solutions $\mathbf{u}_\lambda$ such that $r_\lambda \le \|\mathbf{u}_\lambda\| \le R_\lambda$, and now a passing to the limit finishes the proof. The arguments for the branch of odd solutions are analogous. □

Thus, this theorem implies that we get a bifurcation from infinity when $\lambda \to 0^+$. We can also include the linear part under the effect of the parameter λ, i.e.,

$$-u_1''(x) + a(x)u_1(x) = \lambda\big(b(x)u_2(x) + c(x)u_1^3(x)\big), \tag{25a}$$

$$-u_2''(x) + d(x)u_2(x) = \lambda\big(e(x)u_1(x) + f(x)u_2^3(x)\big) \tag{25b}$$

where now $\lambda \in (0, m)$ is a positive parameter with

$$m = \left[\max_{x\in M}\left(\int_M G_1(x,s)b(s)\,ds, \int_M G_2(x,s)e(s)\,ds\right)\right]^{-1}. \tag{26}$$

Then we obtain the following result.

Corollary 3 *Let us assume the conditions of Theorem 2. Then for all $\lambda \in (0, m)$, there exists a positive solution $\mathbf{u}_\lambda = (u_{1\lambda}, u_{2\lambda}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of (25a). Moreover,*

$$\lim_{\lambda\to 0^+} \|\mathbf{u}_\lambda\| = +\infty. \tag{27}$$

If moreover a, b, c, d, e, f are even functions and $0 \notin M$, there exists a second branch of odd solutions $\tilde{\mathbf{u}}_\lambda \in H^1(\mathbb{R})$ with the same limiting properties.

By using the same argument as in Corollary 1, it is possible to obtain an explicit lower bound for m.

6 Conclusions

In this paper, we have studied a system of two coupled nonlinear Schrödinger equation with inhomogeneous parameters, including a linear coupling arising in several fields of application.

We have proven the existence of two different kinds of homoclinic solutions to the origin describing solitary waves of physical relevance, specifically the so-called ground state (positive nodeless solution) and the first excited state (odd solution with a zero). To complete this goal, we have used a Krasnoselskii fixed-point theorem together with a suitable compactness criterion. Finally, we have studied branches of solutions corresponding to different systems of physical interest.

In comparison with functional or variational methods, topological techniques as the one presented in this paper have been scarcely exploited in the framework of infinite-dimensional extended systems. This opens the doorway to extend our study to many other models arising in mathematical physics, the key point being getting a suitable fixed-point formulation. When such a formulation is possible, the possibilities are as wide as the number of more or less sophisticated fixed point theorems available in the literature. For each particular case, the investigation of the properties of the resolvent of the invertible linear differential operator would be necessary, which from the mathematical point of view would certainly be a research of relevance on its own. Other assumptions such as the compact support condition or the limitation to one spatial dimension are mostly technical and do not have an essential nature, hence they could be eventually overcome in a more skilled treatment. Anyway, the compact support condition is a good approximation to the relevant case of Gaussian coefficients, which is the typical form of the modulation of the coefficients arising in the specific applications studied here.

In conclusion, we believe that our method could be applied to a variety of models with space-dependent coefficients, that eventually arise in applications when properties of the medium such as wave propagation speeds, densities, etc., are spatially dependent. The range of models for which such an analysis could be accomplished includes stationary coupled nonlinear Klein–Gordon systems, coupled nonlinear Schrödinger equations without linear coupling, inhomogeneous coupled Korteweg-de-Vries equations and many others.

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