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Existence of Dark Soliton Solutions of the Cubic Nonlinear Schrödinger Equation with Periodic Inhomogeneous Nonlinearity

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Abstract

In this paper, we give a proof of the existence of stationary dark soliton solutions of the cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity, together with an analytical example of a dark soliton.

1 Introduction

Nonlinear Schrödinger (NLS) equations appear in a great array of contexts [1], for example in semiconductor electronics [2, 3], optics in nonlinear media [4], photonics [5], plasmas [6], the fundamentation of quantum mechanics [7], the dynamics of accelerators [8], the mean-field theory of Bose-Einstein condensates [9] or in biomolecule dynamics [10]. In some of these fields and in many others, the NLS equation appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating a nonlinear medium [11].

The study of these equations has served as the catalyzer for the development of new ideas or even mathematical concepts such as solitons [12] or singularities in EDPs [13, 14].

In the recent years there has been an increased interest in a variant of the standard nonlinear Schrödinger equation which is the so called nonlinear Schrödinger equation with inhomogeous nonlinearity, which is

$$i\psi_t = -\frac{1}{2}\Delta\psi + g(x)\,|\psi|^2\,\psi, \tag{1.1}$$

$$\psi(x,0) = \psi_0(x), \tag{1.2}$$

with $x \in \mathbb{R}^d$. This equation arises in different physical contexts such as nonlinear optics and the dynamics of Bose-Einstein condensates with Feschbach resonance management [15, 16, 17, 18, 19, 20, 21, 22] and has received a considerable amount of attention in recent years because of the possibilities for management and control offered by the coefficient function $g(x)$. Various aspects of the dynamics of solitons in these contexts have been studied, such as the emission of solitons [15, 16], the propagation of solitons when the space modulation of the nonlinearity is a random [17], periodic [21], linear [18] or localized function [20] and the construction of localized solutions by means of group-theoretical methods [31, 32].

In [26], the author, motivated by the study of the propagation of electromagnetic waves through a multi-layered optical medium, proved the existence of two different kinds of homoclinic solutions to the origin in a Schrödinger equation with a nonlinear term. In [27], the authors proved the existence of dark solitons for the cubic-quintic nonlinear Schrödinger equation with a periodic potential. In this paper, we will prove the existence of dark solitons for Eq. (1.1) in one spatial dimension for the case of the T-periodic symmetric nonlinear coefficient $g(x)$ such as those which arise when the nonlinear coefficient is managed through an optical lattice [17, 21, 23, 24, 25, 31].

From the mathematical point of view, the strategy of proof combines several techniques from the classical theory of ODE's (upper and lower solutions) and planar homeomorphisms (topological degree and free homeomorphisms) in a novel way.

2 Existence of periodic solutions

In this document, we will study the cubic nonlinear Schrödinger equation with inhomogeneous nonlinearity (INLSE) on $\mathbb{R}$, i.e.

$$i\psi_t = -\frac{1}{2}\psi_{xx} + g(x)\,|\psi|^2\,\psi \tag{2.1}$$

with $g:\mathbb{R}\to\mathbb{R}$ T-periodic and satisfying the following properties:

$$0 < g_{\min} \leqslant g(x) \leqslant g_{\max} \tag{2.2a}$$

$$g(x) = g(-x). \tag{2.2b}$$

The solitary wave solutions of (2.1) are given by $\psi(x,t) = e^{i\lambda t}\phi(x)$, where $\phi(x)$ is a solution of

$$-\frac{1}{2}\phi_{xx} + \lambda\phi + g(x)\phi^3 = 0. \tag{2.3}$$

Such a solution is defined as a dark soliton if it verifies the asymptotic boundary conditions

$$\frac{\phi(x)}{\phi_{\pm}(x)} \to 1, \qquad x \to \pm\infty \tag{2.4}$$

where the functions $\phi_{\pm}(x)$ are sign definite, T-periodic, real solutions of Eq. (2.3).

Let us now analyze the range of values of λ for which we can obtain the existence of nontrivial solutions of Eq. (2.3).

Theorem 1. *If $\lambda \geqslant 0$, the only bounded solution of Eq. (2.3) is the trivial one,* $\phi = 0$.

Proof. Let ϕ be a nontrivial solution of Eq. (2.3). We can suppose that such a solution is positive in an interval I (on the contrary, we will take $-\phi$). Note that

$$\phi_{xx}(x_0) = 2\lambda\phi(x_0) + 2g(x_0)\phi^3(x_0) > 0 \tag{2.5}$$

for all $x_0 \in I$. If I is a bounded interval, a contradiction follows easily by simply integrating the equation over I. On the other hand, if I is an unbounded interval, we have a convex and bounded function on an unbounded interval, which is impossible. ■

Therefore, throughout this paper, we will take $\lambda < 0$. As $g_{min} \leqslant g(x) \leqslant g_{max}$, let us consider two auxiliary autonomous equations:

$$-\frac{1}{2}\phi^{(1)}_{xx} + \lambda\phi^{(1)} + g_{min}(\phi^{(1)})^3 = 0 \tag{2.6}$$

$$-\frac{1}{2}\phi^{(2)}_{xx} + \lambda\phi^{(2)} + g_{max}(\phi^{(2)})^3 = 0 \tag{2.7}$$

These equations have two nontrivial equilibria

$$\xi^{(1)} = \pm\sqrt{-\frac{\lambda}{g_{min}}} \tag{2.8}$$

$$\xi^{(2)} = \pm\sqrt{-\frac{\lambda}{g_{max}}} \tag{2.9}$$

These are hyperbolic points (saddle points). We denote $\xi^{(i)}$ for the positive equilibria points, for $i = 1,2$. We note that $\xi^{(1)} > \xi^{(2)}$.

Before continuing, we will give the results of the second order differential equation. These results are known [28], and they will be very helpful to us.

Let the following second order differential equation be

$$u_{xx} = f(x,u) \tag{2.10}$$

with f continuous with respect to both arguments and T-periodic in x.

Definition 1. (i) We say that $\overline{u} : [a,+\infty) \to \mathbb{R}$ is a lower solution of (2.10) if

$$\overline{u}_{xx} > f(x,\overline{u}) \tag{2.11}$$

for all $x > a$.

(ii) Similarly, $\underline{u} : [a,+\infty) \to \mathbb{R}$ is an upper solution of (2.10) provided that

$$\underline{u}_{xx} < f(x,\underline{u}) \tag{2.12}$$

for all $x > a$.

We shall now prove the existence of a T-periodic and an unstable solution between both points $\xi^{(1)}$ and $\xi^{(2)}$.

Proposition 1. *The points $\xi^{(1)}$ and $\xi^{(2)}$, which were previously calculated are, respectively, constant upper and lower solutions of Eq (2.3). Moreover, an unstable periodic solution exists between them.*

Proof. By using Eq. (2.6), we obtain:

$$-\frac{1}{2}\xi^{(1)}_{xx}+\lambda\xi^{(1)}+g(x)(\xi^{(1)})^3>\lambda\xi^{(1)}+g_{min}(\xi^{(1)})^3=0 \tag{2.13}$$

and, similarly, for the Eq. (2.7):

$$-\frac{1}{2}\xi^{(2)}_{xx}+\lambda\xi^{(2)}+g(x)(\xi^{(2)})^3<\lambda\xi^{(2)}+g_{max}(\xi^{(2)})^3=0 \tag{2.14}$$

Thus, by the latter definition, $\xi^{(1)}$ and $\xi^{(2)}$ are upper and lower solutions, respectively. Following [28], we obtain that a T-periodic solution exists between them. As the Brouwer index associated to the Poincaré map is -1, (see, for example [29]), such a solution is unstable. ■

We therefore have a positive and T-periodic solution of Eq. (2.3), $\phi_+(x)$, satisfying $\xi^{(2)}\leqslant\phi_+(x)\leqslant\xi^{(1)}$. Owing to the symmetry of the equation we also have a negative solution $\phi_-(x)=-\phi_+(x)$.

3 Existence of a dark soliton.

In this section, we prove the existence of a heteroclinic orbit connecting the periodic solutions ϕ_- and ϕ_+. This heteroclinic orbit may also be called a ”dark soliton”.

The following theorem is the key to our results. It was proved in [27] by using some ideas from [30].

Theorem 2. *Let bounded functions be $u,v:[a,+\infty)\to\mathbb{R}$ verifying*

1. $u(x)<v(x),\quad\forall x>a$
2. $u_{xx}(x)>f(x,u)$ *and* $v_{xx}(x)<f(x,v),\quad\forall x>a.$

   A solution $\phi(x)$ of (2.10) therefore exists such that

$$u(x)<\phi(x)<v(x) \tag{3.1}$$

   If moreover, x exists such that

3. $\min_{\substack{x\in[0,T]\\ u\in[\inf_{x\geqslant a}u(x),\sup_{x\geqslant a}v(x)]}}\frac{\partial f(x,u)}{\partial u}>0$

then an T-periodic solution $\rho(x)$ exists such that

$$\lim_{x\to+\infty}(|\phi(x)-\rho(x)|+|\phi_x(x)-\rho_x(x)|)=0 \tag{3.2}$$

Moreover, $\rho(x)$ is the unique T-periodic solution in the interval $[\inf_{x\geqslant x_0}u(x),\sup_{x\geqslant x_0}v(x)]$

We shall apply this theorem to our model. We have that

$$f(x,\phi)=2\lambda\phi(x)+2g(x)\phi^3(x). \tag{3.3}$$

Moreover, as g is symmetric, it can consider $x\geqslant 0$ and extend the obtained solution $\phi(x)$ as an odd function to $x<0$. The solutions of Eqs (2.6) and (2.7), $\phi^{(1)}$ and $\phi^{(2)}$, which are heteroclinic orbits joining $-\xi^{(1)}$ with $\xi^{(1)}$ and $-\xi^{(2)}$ with $\xi^{(2)}$, respectively, satisfy the conditions (1) and (2) of Theorem 2, with $v(x)=\phi^{(1)}(x)$ and $u(x)=\phi^{(2)}(x)$. We thus have a bounded solution $\phi(x)$ of Eq. (2.3) such that

$$\phi^{(2)}(x)<\phi(x)<\phi^{(1)}(x) \tag{3.4}$$

Now that we have the above mentioned solution $\phi(x)$ tends to $\phi_+(x)$ and $\phi_-(x)$, found in Proposition 1, as $x\to\pm\infty$. Hence, it must verify condition (3) of Theorem 2.

As a can be taken as being arbitrarily large, condition (3) is equivalent to

$$\min_{\substack{x\in[0,T]\\ u\in[\xi^{(2)},\xi^{(1)}]}} [2\lambda+6g(x)u^2]>0 \tag{3.5}$$

This last inequality is equivalent to $2\lambda+6g_{\min}(\xi^{(2)})^2>0$. Using Eq. (2.9) and the fact that $\lambda<0$, we obtain a connection between $g_{\min}$ and $g_{\max}$:

$$g_{min}>\frac{g_{\max}}{3}. \tag{3.6}$$

So, if relation (3.6) is verified, we obtain the existence of dark solitons in the nonlinear Schödinger equation with inhomogeneous nonlinearity.

4 An example of a dark soliton

In this section, we shall consider an example of a dark (black) soliton from Eq. (2.3), which illustrates the concepts introduced in the study. For this example, we shall take the periodic nonlinearity $g(x)$ as

$$g(x)=\frac{g_0}{(1+\alpha\cos(\omega x))^3} \tag{4.1}$$

with $\omega=2\sqrt{|\lambda|}$ and g_0, and $\alpha<1$ positive constants. To satisfy connection (3.6), α must fulfil the constraint $\alpha<(3^{1/3}-1)/(3^{1/3}+1)$.

The boundary condition ϕ_+ is

$$\phi_+=\frac{\omega}{2}\sqrt{\frac{(1-\alpha^2)(1+\alpha\cos\omega x)}{g_0}} \tag{4.2}$$

and $\phi_-=-\phi_+$.

Following [31, 32], the solution of Eq. (2.3) with the boundary conditions (2.4) is

$$\phi(x)=\frac{\omega}{2}\sqrt{\frac{1-\alpha^2}{g_0}}\sqrt{1+\alpha\cos\omega x}\tanh\left[\frac{\omega}{2}\sqrt{\frac{1-\alpha^2}{2}}\,X(x)\right] \tag{4.3}$$

with $X(t)$ given by

$$\tan\left(\frac{\omega}{2}\sqrt{1-\alpha^2}\,X(x)\right)=\sqrt{\frac{1-\alpha}{1+\alpha}}\,\tan\frac{\omega x}{2}. \tag{4.4}$$

This solution is depicted in Fig. 1.

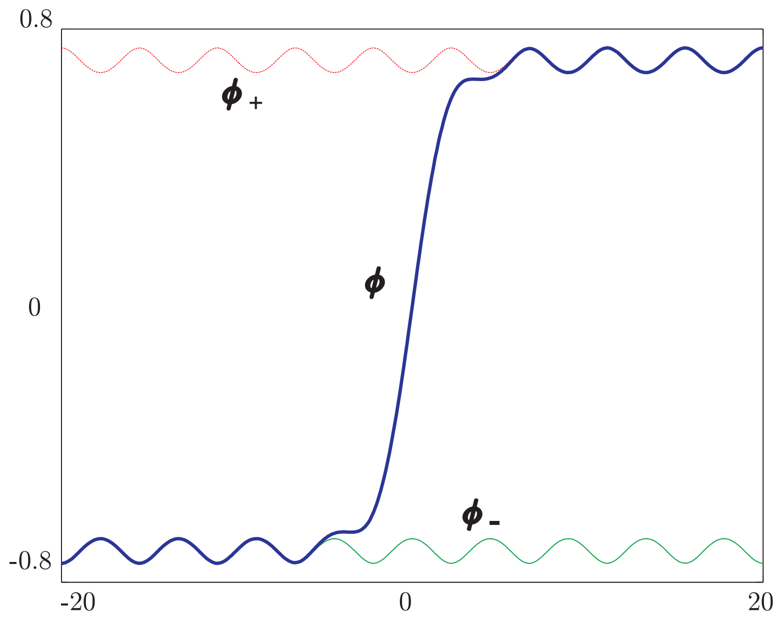

Figure 1: [Color Online] . The dark soliton $\phi(x)$ (solid blue line) and the hyperbolic periodic solutions $\phi_{\pm}(x)$ (red and green dashed line), for the parameters $\lambda = -0.5$, $g_0 = 1$, $\alpha = 0.1$.

5 Conclusions

In this paper, we have studied the existence of dark solitons or heteroclinic orbits of the INLSE. The method of proof begins with a standard separation of variables and relies on classical results of the qualitative theory of ordinary differential equations that require some concepts such as upper and lower solutions, topological degree and free homeomorphisms. As an example, we have constructed an analytical black soliton-solution when the coefficient of the nonlinear term $g(x)$ is periodic. Clearly, we are looking for a particular type of solutions. Of course, it is still possible to wonder about the presence of dark solitons with a more complex structure and not coming from a separation of variables. This is an interesting and difficult problem to be considered in the future.

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