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Two inverse problems for analytic potential systems

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Abstract

In this paper, we solve a basic problem about the existence of an analytic potential with a prescribed period function. As an application, it is shown how to extend to the whole phase plane an arbitrary potential defined on a semiplane in order to get isochronicity.

1. Introduction

In general, an inverse problem could be described as a task where the effect is known, but the cause is unknown. In the framework of dynamical systems, one wonders about the existence of a system which exhibits a concrete dynamic response. This paper is devoted to the analysis of two basic inverse problems in the context of analytic potential systems.

Consider the system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = V'(x), \end{cases} \tag{1}$$

where V is an analytic function defined in a neighborhood of the origin. We always assume that the system has a non-degenerate center at 0, that is $V(0) = V'(0) = 0$ and $V''(0) = k > 0$. We

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denote by P the projection in the x-axis of the period annulus of the origin. Clearly P is an open interval containing 0. We will denote its endpoints by x_- and x_+. Thus, $P = (x_-, x_+)$.

For $x \in P \setminus \{0\}$ we denote by $T(x)$ the period of the orbit of the potential system passing through the point $(x, 0)$. It is well known that T is an analytic function on $P \setminus \{0\}$ and that it extends analytically to 0 by $T(0) = \lim_{x\to 0} T(x)$. Also it is well known that $T(0) > 0$ (all these results are direct consequence of Theorem 3 in the next section). Since for each $x \in (0, x_+)$ it exists $y \in (x_-, 0)$ such that $T(x) = T(y)$ it follows that 0 is a local extremum of the function T.

In this paper we first study the following inverse problem. Given an analytic function F having a local extremum at 0, and satisfying $F(0) > 0$, we investigate the existence of some non-degenerate potential V such that F is the period function associated to V. Our main result in this direction is the following theorem:

Theorem 1. *Let F be an analytic function at* 0 *with $F(0) = a > 0$ and assume that F has a local minimum or maximum at* 0*. If F is non-constant then it exists a unique function V analytic at* 0 *such that the period function of the potential system associated to V is F. If F is constant then there exist infinitely many analytic functions V satisfying that the period function of V is F. All these solutions are of the form $\frac{\pi^2}{2a^2}(x - \sigma(x))^2$ where σ is an analytic involution defined in a neighborhood of* 0*.*

Theorem 1 is a direct consequence of Theorem 5 and Corollary 2 in Section 3. We notice that Corollary 2 was already proved in [2].

This paper is closely related with the results which appear in [5,6]. In these classical papers the author considers the analogous inverse problem associated to the energy of the orbits. Consider the energy $H(x, y) = y^2/2 + V(x)$, which is a first integral of our system. Then the orbits can be parameterized by the value of H at the orbit. Set $\mathbb{T}(h)$ the period of the orbit having energy level h. It is well known that $\mathbb{T}$ is also analytic and obviously $\mathbb{T}(H(x, 0)) = T(x)$. In that paper it is proved that given an analytic function $\mathbb{T}$ such that $\mathbb{T}(0) > 0$ there exist infinitely many analytic functions V such that $\mathbb{T}$ is the period function (with respect the energy) of V. A little more general problem was also studied in [3].

Another related reference is [1], where the author considers the half-period function, $\bar{T}$. For each $x \in P$, $x > 0$, $\bar{T}(x)$ is defined as the time that the solution with initial condition $(x, 0)$ spends to intersect for a first time the y-axis. In that paper the author proves an equivalent result to the first part of Theorem 1 for the half-period function. Due to the symmetries in the case when V is even, clearly $T(x) = 4\bar{T}(x)$. So the even case (see Theorem 4) could be obtained from the previous work of Alfawicka. We present here a new proof which is much simpler and can be extended to the general case.

The second inverse problem concerns the way to complete or extend a potential working in the semiplane $y > 0$ to the whole phase plane in order to get an isochronous center at the origin. This problem was raised in [4, Section 3], where the even case was solved. The following result solves completely the question.

Theorem 2. *Let V be an analytic function defined in a neighborhood of* 0 *satisfying $V(0) = V'(0) = 0$ and $V''(0) = k > 0$. Then, for any $A > \frac{\pi}{\sqrt{2k}}$, there exists a unique analytic function V^* defined in a neighborhood of* 0 *satisfying $V^*(0) = (V^*)'(0) = 0$ and $(V^*)''(0) > 0$ verifying that the system*

$$\dot{x} = -y,$$
$$\dot{y} = \begin{cases} V'(x), & \text{if } y > 0, \\ (V^*)'(x), & \text{if } y < 0, \end{cases}$$

has an isochronous center at the origin with period A.

The paper is structured as follows. In Section 2, a symmetrization argument is provided in such a way that any arbitrary potential V has an equivalent even potential. Section 3 contains the proof of Theorem 1 while in Section 4 we give the proof of Theorem 2.

2. Symmetrization of a potential system

Our first purpose is to associate to system (1) a symmetric potential system given by an even function $\tilde{V}$. To do this, observe that since V has a local minimum at 0, we can define an involution by

$$V\big(\sigma(x)\big) = V(x),$$
$$x\sigma(x) \leqslant 0.$$

Note that σ is well defined in P. Set

$$g(x) = x\sqrt{\frac{V(x)}{x^2}} = \operatorname{sign}(x)\sqrt{V(x)}.$$

Writing $V(x) = kx^2 + \cdots$, where the dots mean the higher order terms, it is clear that g is analytic in P, $g(0) = 0$ and $g'(0) = \sqrt{k}$. An easy computation shows that

$$\sigma(x) = g^{-1}\big(-g(x)\big).$$

So, σ is analytic in P. Now set $h(x) = \frac{x-\sigma(x)}{2}$. Since $\sigma'(x) < 0$ we get that $h'(x) > 0$, and hence it is a diffeomorphism from P to $h(P) = (\frac{x_- - x_+}{2}, \frac{x_+ - x_-}{2})$. We define $\tilde{V} = V \circ h^{-1}$. The next lemma states some basic properties of h and $\tilde{V}$.

Lemma 1. *The following assertions hold*:

(1) $h(\sigma(x)) = -h(x)$;
(2) $V(x) = \tilde{V}(h(x))$ *and* $\tilde{V}$ *is an even function on* $(\frac{x_- - x_+}{2}, \frac{x_+ - x_-}{2})$;
(3) $h^{-1}(x) - h^{-1}(-x) = 2x$.

Proof. The property (1) follows by direct computations

$$h\big(\sigma(x)\big) = \frac{\sigma(x) - \sigma^2(x)}{2} = -h(x).$$

The first assertion of (2) follows directly from the definition of $\tilde{V}$. Also we have

$$\tilde{V}\bigl(-h(x)\bigr) = \tilde{V}\bigl(h\bigl(\sigma(x)\bigr)\bigr) = V\bigl(\sigma(x)\bigr) = V(x) = \tilde{V}\bigl(h(x)\bigr).$$

Since h is a diffeomorphism it follows that $\tilde{V}$ is even.

From (1) we get

$$\sigma(x) = h^{-1}\bigl(-h(x)\bigr)$$

and

$$\sigma\bigl(h^{-1}(x)\bigr) = h^{-1}(-x).$$

Then,

$$h^{-1}(x) - h^{-1}(-x) = h^{-1}(x) - \sigma\bigl(h^{-1}(x)\bigr) = 2h\bigl(h^{-1}(x)\bigr) = 2x. \qquad \square$$

Our next purpose is to compare the potential systems associated to V and $\tilde{V}$. To do this we define

$$\tilde{g}(x) = x\sqrt{\frac{\tilde{V}(x)}{x^2}}.$$

An easy computation shows that $\tilde{g}$ is an analytic odd diffeomorphism defined in $h(P)$ and

$$g = \tilde{g} \circ h.$$

For $x \in P$ let $T_V(x)$ be the period of the orbit of the potential system passing through the point $(x, 0)$. It is well known that T is also analytic at P. In the same way we denote by $T_{\tilde{V}}$ the period function of the potential system associated to $\tilde{V}$. The next theorem relates these two period functions.

Theorem 3.

$$T_V(x) = 2\sqrt{2}\int_0^{\pi/2} \bigl(\tilde{g}^{-1}\bigr)'\bigl(\tilde{g}\bigl(h(x)\bigr)\sin\theta\bigr)\,d\theta = T_{\tilde{V}}\bigl(h(x)\bigr).$$

Proof. For $x > 0$, we have

$$T_V(x) = \sqrt{2}\int_{\sigma(x)}^{x} \frac{dy}{\sqrt{V(x) - V(y)}}.$$

Putting $y = g^{-1}(g(x)\sin\theta)$ and taking into account that $V = g^2$, we obtain

$$\begin{aligned} T_V(x) &= \sqrt{2}\int_{-\pi/2}^{\pi/2} \big(g^{-1}\big)'\big(g(x)\sin\theta\big)\,d\theta \\ &= \sqrt{2}\int_{-\pi/2}^{0} \big(g^{-1}\big)'\big(g(x)\sin\theta\big)\,d\theta + \sqrt{2}\int_{0}^{\pi/2} \big(g^{-1}\big)'\big(g(x)\sin\theta\big)\,d\theta \\ &= \sqrt{2}\int_{0}^{\pi/2} \big[\big(g^{-1}\big)'\big(g(x)\sin\theta\big) + \big(g^{-1}\big)'\big(-g(x)\sin\theta\big)\big]\,d\theta. \end{aligned}$$

Now we focus our attention to the expression appearing in the integral. Putting $z = g(x)\sin\theta$, taking into account that $g = \tilde{g}\circ h$, and that $\tilde{g}$ is odd, we get

$$\begin{aligned} \big(g^{-1}\big)'(z) + \big(g^{-1}\big)'(-z) &= \big(h^{-1}\big)'\big(\tilde{g}^{-1}(z)\big)\big(\tilde{g}^{-1}\big)'(z) + \big(h^{-1}\big)'\big(\tilde{g}^{-1}(-z)\big)\big(\tilde{g}^{-1}\big)'(-z) \\ &= \big(\tilde{g}^{-1}\big)'(z)\big[\big(h^{-1}\big)'\big(\tilde{g}^{-1}(z)\big) + \big(h^{-1}\big)'\big(-\tilde{g}^{-1}(z)\big)\big]. \end{aligned}$$

From Lemma 1(3) we obtain

$$\big(h^{-1}\big)'\big(\tilde{g}^{-1}(z)\big) + \big(h^{-1}\big)'\big(-\tilde{g}^{-1}(z)\big) = 2.$$

Thus, we get

$$T_V(x) = 2\sqrt{2}\int_{0}^{\pi/2} \big(\tilde{g}^{-1}\big)'\big(g(x)\sin\theta\big)\,d\theta = 2\sqrt{2}\int_{0}^{\pi/2} \big(\tilde{g}^{-1}\big)'\big(\tilde{g}\big(h(x)\big)\sin\theta\big)\,d\theta$$

and the first equality holds for $x > 0$. One can check easily that the same expression holds for $x < 0$. The same computations for $T_{\tilde{V}}$ gives

$$T_{\tilde{V}}(x) = 2\sqrt{2}\int_{0}^{\pi/2} \big(\tilde{g}^{-1}\big)'\big(\tilde{g}(x)\sin\theta\big)\,d\theta.$$

Hence, we get

$$T(x) = T_{\tilde{V}}\big(h(x)\big). \qquad \square$$

3. The proof of Theorem 1

Taking advantage of the latter section, we easily obtain the following well-known results. Corollary 1 was originally proved in [5] and Corollary 2 in [2].

Corollary 1. *If V is an even isochronous potential with associated constant period $a > 0$, then $V(x) = kx^2$ with $k = \frac{2\pi^2}{a^2} > 0$.*

Proof. Assume that $V = kx^2 + \cdots$ is an even isochronous potential and let $g(x) = x\sqrt{\frac{V(x)}{x^2}}$. Since V is even, g is odd. So $(g^{-1})'$ is even. Let

$$\left(g^{-1}\right)'(x) = \sum_{i=0}^{\infty} c_{2i} x^{2i}$$

be the power expansion of $(g^{-1})'$ at the origin. Clearly $c_0 = \frac{1}{\sqrt{k}}$. Since the period function $T(x)$ is constant, we get that $T(g^{-1}(x))$ is also the same constant a. From Theorem 3 we have

$$a = T\left(g^{-1}(x)\right) = \sqrt{2} \int_{-\pi/2}^{\pi/2} \left(g^{-1}\right)'(x \sin\theta)\, d\theta.$$

Hence we obtain that

$$a = \sum_{i=0}^{\infty} c_{2i} I_{2i} x^{2i}, \tag{2}$$

where $I_{2i} = 2\sqrt{2}\int_{-\pi/2}^{\pi/2} \sin^{2i}\theta\, d\theta$. This implies that $c_{2i} = 0$ for all $i > 0$. Thus $(g^{-1})'(x) = \frac{1}{\sqrt{k}}$, and since $g(0) = 0$ we obtain $g(x) = \sqrt{k}x$. Therefore, $V(x) = kx^2$. Besides, from (2) we get $a = c_0 I_0$, taking into account that $c_0 = \frac{1}{\sqrt{k}}$, one easily finds $k = \frac{2\pi^2}{a^2}$. □

Corollary 2. *If V is an isochronous potential with associated constant period $a > 0$, then $V(x) = k(x - \sigma(x))^2$ for some analytic involution σ and $k = \frac{\pi^2}{2a^2}$.*

Proof. Let $\tilde{V}$ be the even potential associated to V. By Theorem 3 it follows that $\tilde{V}$ is also isochronous (with the same associated period) and by the previous corollary we obtain that $\tilde{V}(x) = \frac{2\pi^2}{a^2}x^2$. The result follows from the fact that $V(x) = \tilde{V}(h(x))$. □

This previous result is the second part of Theorem 1. It remains to prove the first part. As we remarked in the Introduction, although the even case can be deduced from [1], we give here a simpler proof.

Theorem 4. *Let T be a non-constant even function analytic at 0 with $T(0) = a > 0$. Then there exists a unique function V analytic at 0, such that the period function of the potential system associated to V is T. Moreover, V is even.*

Proof. In view of Theorem 3 we look for an analytic odd function g with $g'(0) \neq 0$ satisfying

$$T(x) = 2\sqrt{2} \int_0^{\pi/2} \left(g^{-1}\right)'\left(g(x)\sin\theta\right) d\theta \tag{3}$$

or equivalently

$$T\left(g^{-1}(x)\right) = 2\sqrt{2} \int_0^{\pi/2} \left(g^{-1}\right)'(x\sin\theta)\, d\theta. \tag{4}$$

If we put

$$T(x) = \sum_{i=0}^{\infty} t_{2i} x^{2i}$$

and

$$g^{-1}(x) = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1},$$

we obtain $a_1 = t_0/I_0 = a/I_0 > 0$ and the following recurrence

$$a_{2n+1} = \frac{1}{(2n+1)I_{2n}} \sum_{i=1}^{n} t_{2i} \left(\sum_{j_1+\cdots+j_{2i}=2n} a_{j_1} \cdots a_{j_{2i}} \right), \tag{5}$$

where $I_k = 2\sqrt{2}\int_0^{\pi/2} \sin^k\theta\, d\theta$. Note also that since T is an analytic function at 0 it exists $d > 0$ such that $|t_{2n}| < d^{2n}$ for all $n > 0$. From this recurrence we obtain a unique formal power series satisfying Eq. (3). Since $V = g^2$ this implies that if there exists an analytic potential with the prescribed period function it is unique. To solve the existence problem it suffices to show that the power series $\sum_{i=0}^{\infty} a_{2i+1}x^{2i+1}$ with coefficients satisfying (5) and $a_1 = a/I_0 > 0$, has positive convergence radius.

To do this consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = d^2y^3 - (a_1 - 1)d^2xy^2 + a_1x - y$$

which is analytic and satisfies $(\frac{\partial F}{\partial y})_{(0,0)} = -1$. Thus from the Implicit Function Theorem it follows that there exists a neighborhood of 0, V in $\mathbb{R}$ and an analytic function $G : V \to \mathbb{R}$ satisfying $G(0) = 0$ and $F(x, G(x)) = 0$. Notice that $F(x, -G(-x)) = -F(-x, G(-x)) = 0$. So from the Implicit Function Theorem $-G(-x) = G(x)$ and G is odd.

Now we put $G(x)=\sum_{i=0}^{\infty}b_{2i+1}x^{2i+1}$. Note that since G is analytic at 0 this power series has positive radius of convergence. On the other hand substituting this power series in the equation $F(x,G(x))=0$ we obtain

$$\frac{1}{x}\left(\sum_{i=0}^{\infty}b_{2i+1}x^{2i+1}\right)=(a_1-1)+\frac{1}{1-d^2(\sum_{i=0}^{\infty}b_{2i+1}x^{2i+1})^2}. \tag{6}$$

Then $b_1=a_1$ and

$$b_{2n+1}=\sum_{i=1}^{n}d^{2i}\left(\sum_{j_1+\cdots+j_{2i}=2n}b_{j_1}\cdots b_{j_{2i}}\right). \tag{7}$$

We claim that $|a_{2i+1}|\leqslant b_{2i+1}$ for all $i\geqslant 0$. Since $b_1=a_1>0$ the claim holds for $i=0$. Now assume that $|a_{2i+1}|\leqslant b_{2i+1}$ for all $i<n$. Taking into account that

$$(2n+1)I_{2n}=\frac{\sqrt{2}\pi(2n+1)!!}{(2n)!!}>1$$

for all $n\geqslant 0$, we have

$$\begin{aligned}|a_{2n+1}|&=\left|\frac{1}{(2n+1)I_{2n}}\sum_{i=1}^{n}t_{2i}\left(\sum_{j_1+\cdots+j_{2i}=2n}a_{j_1}\cdots a_{j_{2i}}\right)\right|\\&\leqslant\sum_{i=1}^{n}d^{2i}\left(\sum_{j_1+\cdots+j_{2i}=2n}|a_{j_1}|\cdots|a_{j_{2i}}|\right)\\&\leqslant\sum_{i=1}^{n}d^{2i}\left(\sum_{j_1+\cdots+j_{2i}=2n}b_{j_1}\cdots b_{j_{2i}}\right)\\&=b_{2n+1},\end{aligned}$$

and the claim is proved. Thus, since the power series $\sum_{i=0}^{\infty}b_{2i+1}x^{2i+1}$ has positive radius of convergence the same holds for the power series $\sum_{i=0}^{\infty}a_{2i+1}x^{2i+1}$. □

Now we are able to extend this theorem to the general case.

Theorem 5. *Let F be an analytic function at 0 with $F(0)=a>0$. Assume that F has a local minimum or maximum at 0 and that it is not constant. Then there exists a unique function V analytic at 0 such that the period function of the potential system associated to V is F.*

Proof. Since F has a local extremum and it is not constant it has an associated analytic involution σ defined by

$$F\big(\sigma(x)\big)=F(x),$$
$$x\sigma(x)\leqslant 0.$$

In fact σ can be obtained in the following way. From our hypotheses on F there exists $k > 0$ in such a way $F(x) = a + a_{2k}x^{2k} + \cdots$ with $a_{2k} \neq 0$. Assume for instance that $a_{2k} > 0$. Set

$$f(x) = x\left(\frac{F(x) - a}{x^{2k}}\right)^{\frac{1}{2k}} = \operatorname{sign}(x)\big(F(x) - a\big)^{\frac{1}{2k}}.$$

Clearly f is analytic at 0, $f(0) = 0$ and $f'(0) \neq 0$. Then $\sigma(x) = f^{-1}(-f(x))$.

We also define $h(x) = \frac{x-\sigma(x)}{2}$. Thus we get $F(x) = \tilde{F}(h(x))$, where $\tilde{F} = F \circ h^{-1}$ is an analytic even function. Applying Theorem 4 to $\tilde{F}$ we obtain an analytic even function $\tilde{V}$ having $\tilde{F}$ as a period function. Now let $V = \tilde{V} \circ h$. From Theorem 3 we obtain that the period function associated to V satisfies $T(x) = \tilde{F}(h(x)) = F(x)$. This proves the existence of a potential satisfying the required conditions.

Now assume that V_1 and V_2 are analytic potentials satisfying that its period function is F. Then all the functions F, V_1, V_2 have associated the same analytic involution σ and the same diffeomorphism $h(x) = \frac{x-\sigma(x)}{2}$. Thus $\tilde{F} = F \circ h^{-1}$, $\tilde{V}_1 = V_1 \circ h^{-1}$ and $\tilde{V}_2 = V_2 \circ h^{-1}$ are analytic and even. Also from Theorem 3 both $\tilde{V}_2$ and $\tilde{V}_1$ have the same period function $\tilde{F}$ and from Theorem 4 we get $\tilde{V}_1 = \tilde{V}_2$. Since h is a diffeomorphism this implies $V_1 = V_2$. □

With this result the proof of Theorem 1 is completed.

4. The proof of Theorem 2

Given an analytic potential V that has a non-degenerate center at the origin, we will denote by σ_V the analytic involution implicitly defined in a neighborhood of 0 by

$$V(x) = V\big(\sigma_V(x)\big), \qquad x\sigma_V(x) \leqslant 0.$$

We also denote by T_V the period function associated to the potential system given by V. We stress the fact that when T_V is not constant, σ_V is also determined by the equation

$$T_V\big(\sigma_V(x)\big) = T_V(x), \qquad x\sigma_V(x) \leqslant 0.$$

The next proposition was proved in [4].

Proposition 1. *Let V_1 and V_2 be analytic functions defined in a neighborhood of the origin satisfying that $V_1(0) = V_2(0) = V_1'(0) = V_2'(0) = 0$ and $V_1''(0), V_2''(0) > 0$. Then the following statements are equivalent*:

- *the system*

$$\dot{x} = -y,$$
$$\dot{y} = \begin{cases} V_1'(x), & \textit{if } y > 0, \\ V_2'(x), & \textit{if } y < 0, \end{cases}$$

  has a center at the origin;
- $\sigma_{V_1} = \sigma_{V_2}$;
- *there exists an analytic diffeomorphism g such that $V_1 = g(V_2)$ and $g(0) = 0$.*

Proof of Theorem 2. Let T_V be the period function associated to V. Denote by T_V^+ the time expended by the solution beginning at $(x,0)$ to come to $(\sigma_V(x),0)$. Clearly $T_V^+ = \frac{T_V}{2}$. In particular $T_V^+(0) = \frac{\pi}{\sqrt{2k}}$. So, in view of Proposition 1 we look for an analytic function V^* satisfying $V^*(0) = (V^*)'(0) = 0$, $(V^*)'' > 0$, $\sigma_V = \sigma_{V^*}$ and $T_V^+ + T_{V^*}^+ = A$. Note that this last equality is equivalent to $T_{V^*} = 2A - T_V$. Note also that $2A - T_V(0) = 2T - \frac{2\pi}{\sqrt{2k}} > 0$.

Consider first the case when T_V is constant. This implies that T_{V^*} must also be constant, namely $T_{V^*} = 2A - \frac{2\pi}{\sqrt{2k}}$. In this case from Theorem 1 we have that $V = \frac{k}{4}(x - \sigma_V(x))^2$ and also $V^* = \frac{B}{4}(x - \sigma(x))^2$ for some $B > 0$. In view of Proposition 1, σ must be exactly σ_V and B is determined from the relation $T_{V^*} = 2A - \frac{2\pi}{\sqrt{2B}}$. This ends the proof of the theorem in this case.

Now assume that T_V is not constant. Now the analytic function $2A - T_V$ is not constant, is positive at 0 and has a local minimum or maximum at 0. From Theorem 1 it follows that there exists one and only one analytic function V^* such that $V^*(0) = (V^*)'(0) = 0$, $(V^*)''(0) > 0$ and $T_{V^*} = 2A - T_V$. Moreover since T_V and $2A - T_V$ define the same involutions it follows that $\sigma_V = \sigma_{V^*}$. This ends the proof of the theorem. □

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