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On the Existence of Dark Solitons in a Cubic-Quintic Nonlinear Schrödinger Equation with a Periodic Potential

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Abstract: A proof of the existence of stationary dark soliton solutions of a cubic-quintic nonlinear Schrödinger equation with a periodic potential is given. It is based on the interpretation of the dark soliton as a heteroclinic of the Poincaré map.

1. Introduction

In the present paper we consider stationary solutions of the cubic-quintic nonlinear Schrödinger equation (CQNLS)

$$i\psi_t + \psi_{xx} - V(x)\psi - g_1|\psi|^2\psi - g_2|\psi|^4\psi = 0 \tag{1}$$

with a real, even, and L-periodic potential: $V(x) = V(-x) = V(x+L)$, which is also considered to be bounded. The constants g_1 and g_2 introduced in Eq. (1) are real. More specifically, we are interested in solutions which allow the representation $\psi(t,x) = e^{-i\omega t}\phi(x)$, where ω is a real constant, through the text referred to as the frequency. Then the function ϕ solves the stationary equation

$$\phi_{xx} + \tilde{V}(x)\phi - g_1\phi^3 - g_2\phi^5 = 0 \tag{2}$$

with $\tilde{V}(x) \equiv \omega - V(x)$, subject to the nonzero boundary conditions

$$\phi(x) \to \phi_+(x) \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \phi(x) \to \phi_-(x) \quad \text{as} \quad x \to -\infty \tag{3}$$

with the functions $\phi_\pm(x)$ being real, bounded, sign definite, and L-periodic solutions of (2).

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For the sake of convenience, in Fig. 1 we illustrate the concepts introduced in this paper, using the case of a particular solution considered in detail in Sect. 4. The described solutions $\phi(x)$ will be identified as *dark solitons*.

In a general case a stationary solution of Eq. (1) $\psi(t,x)$ from $\mathbb{C}^2((0,\infty)\times\mathbb{R})$, subjected to the boundary conditions $\lim_{x\to\pm\infty}\psi(x,t) = \phi_\pm(x)e^{-i\omega t}$, can be represented in the form $\psi(t,x) = e^{-i\omega t + i\theta(x)}\phi(x)$, where $\theta(x)$ and $\phi(x)$ are real bounded functions from $\mathbb{C}^2(\mathbb{R})$. Since the potential is real-valued, one can look for a real solution $\phi(x)$, which corresponds to $\theta(x)\equiv$const and is achieved by factoring out the x-independent phase. If however an additional constraint $\phi_+(x)\phi_-(x) < 0$ is imposed, such real solutions (or more precisely solutions with phases independent on x) exhaust all possible bounded and differentiable solutions. Indeed, substituting the representation $\psi(t,x) = e^{-i\omega t+i\theta(x)}\phi(x)$ into Eq. (1), separating real and imaginary parts of the obtained equation, and integrating once the imaginary part of the equation with respect to x, we obtain the link $\phi^2\theta_x = C$, where C is the integration constant. This relation together with the real part of the equation: $\phi_{xx} - \theta_x\phi + \tilde{V}(x)\phi - g_1\phi^3 - g_2\phi^5 = 0$, allows one to conclude that if ϕ acquires zero at some point of the space, then $C = 0$, which in its turn means that the argument θ is a constant. Thus, taking into account that $\phi_+(x)\phi_-(x) < 0$ for all x, a continuous function $\phi(x)$ satisfying the conditions (3) becomes zero in at least one point of the real axis, then one can consider $\phi(x)$ as pure real and satisfying Eq. (2). This fact was taken into account in the passage from (1) to (2).

The model (1) having a general character, it describes weakly dispersive and weakly nonlinear wave processes, recently attracted considerable attention in connection with its application to the mean-field theory of Bose-Einstein condensates [1]. In this context,

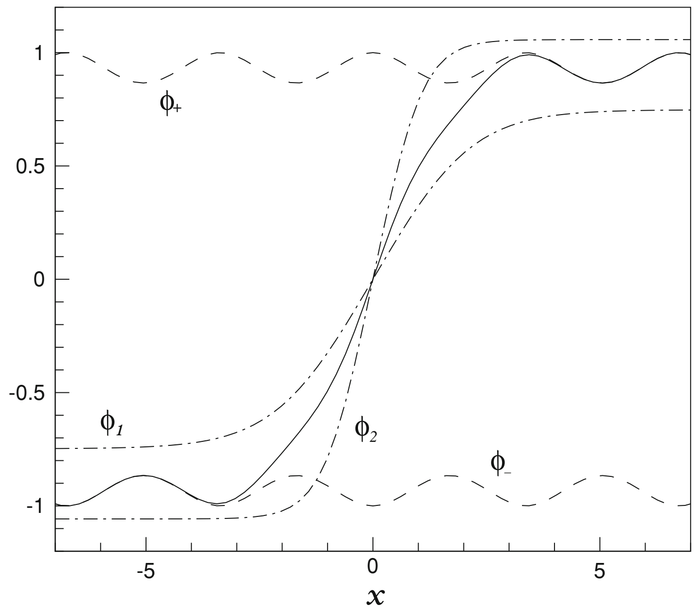

Fig. 1. The dark soliton – heteroclinic (solid line), lower and upper solutions $\rho_{1,2}$ (dashed-dotted line), and the hyperbolic periodic solutions $\phi_\pm(x)$ given by (12), for the potential (11) with the parameters $k = 0.5$ and $\rho = 1$ (see below)

$\psi(x)$ is a macroscopic wave function, ω plays the role of the chemical potential, $|\psi(x)|^2$ describes the linear atomic density, and $V(x)$ is an optical lattice created by standing laser beams [2]. In particular, existence of spatially localized modes (also referred to as bright solitons) has been recently addressed in Refs. [4,5]. In Ref. [5] various families of the solutions were presented and significant differences in behavior of the stationary modes of the standard cubic nonlinear Schrödinger equation (i.e. of Eq. (2) with $g_1 \neq 0$ and $g_2 = 0$) and of the quintic nonlinear Schrödinger equation (i.e. of Eq. (2) with $g_1 = 0$ and $g_2 \neq 0$), have been found. Dark solitons of the NLS equation with a periodic potential have also been discussed in the small amplitude limit [6] and for a general case they have been studied numerically in Ref. [7] (see also [2] and references therein). The approach developed in [7] was based on the numerical study of the Poincaré map generated by Eq. (2) and considered at instants nL. In that approach dark solitons appear as *heteroclinics* of the map. The aim of the present paper is to extend earlier studies, providing for the first time a rigorous proof of the existence of a dark soliton solution of Eq. (1).

From the mathematical point of view, the strategy of proof combines in a novel way several techniques from the classical theory of ODE's (the upper and lower solutions [8] and the truncation arguments, as in Theorem 1 below) and the dynamics of planar homeomorphisms (the topological degree [9–11] and the free homeomorphisms [12]).

We will restrict the consideration to the case $g_2 > 0$ only, which rules out any possibility of blowing up solutions. Then without loss of generality we can set $g_2 = 1$ through the rescaling, which is done in what follows.

To conclude the Introduction, we notice that the method presented in this paper can be seen as a novel general approach working in a more general framework, including some relevant examples like the cubic Schödinger equation with inhomogeneous nonlinearity, i.e. Eq. (2) with $g_2 \equiv 0$ and variable $g_1(x)$. We have limited our analysis to the cubic-quintic Schrödinger equation for the sake of clarity.

2. Existence of Sign Definite Periodic Solutions

We start with a proof of existence of sign definite periodic solutions. To this end we specify the condition of boundness of $V(x)$: there exist constants V_{min}, V_{max} such that $V_{min} < V(x) < V_{max}$, consider

$$\omega > V_{max} \tag{4}$$

and introduce the notations $\lambda_1^2 = \omega - V_{max}$ and $\lambda_2^2 = \omega - V_{min}$. As it is clear $0 < \lambda_1^2 \leq \tilde{V}(x) \leq \lambda_2^2$. Next we consider two stationary equations as follows ($j = 1, 2$):

$$\phi_{j,xx} + \lambda_j^2 \phi_j - g_1 \phi_j^3 - \phi_j^5 = 0. \tag{5}$$

Treating these equations as dynamical systems, one easily finds the (only) two nontrivial real equilibria $\pm\rho_j$, where

$$\pm \rho_j = \pm\sqrt{\sqrt{g_1^2 + 4\lambda_j^2} - g_1}/\sqrt{2}. \tag{6}$$

These are the hyperbolic points $+\rho_j$ and $-\rho_j$ connected by the heteroclinic orbits, which explicit forms read

$$\phi_j = \frac{\rho_j \alpha_j \tanh(k_j x)}{\sqrt{\rho_j^2 + \alpha_j^2 - \rho_j^2 \tanh^2(k_j x)}}, \tag{7}$$

where

$$\alpha_j = \sqrt{2\rho_j^2 + \frac{3}{2}g_1}, \quad \text{and} \quad k_j = \rho_j\sqrt{\rho_j^2 + \frac{g_1}{2}}, \tag{8}$$

and without loss of generality the condition $\phi_1(0) = \phi_2(0) = 0$ is imposed. Let us call $\rho_{1,2}$ the positive equilibria. As it is clear $\phi_1(x) < \phi_2(x)$ for $x > 0$.

At this point some considerations about the general second order equation

$$\phi_{xx} = f(x, \phi) \tag{9}$$

with $f(x, \phi)$ continuous with respect to both arguments and L-periodic in x, are required. The following definition is classical (see for instance [8] and references therein).

Definition 1. *A function $\alpha : [x_0, +\infty) \to \mathbb{R}$ such that $\alpha_{xx}(x) > f(x, \alpha)$ ($\alpha_{xx}(x) < f(x, \alpha)$) for all $x > x_0$ is called a* **strict lower (upper)** *solution of Eq. (9).*

We notice that the strict inequalities are required for the proper definition of the Brouwer degree in the area between the lower and the upper solutions [8]. Now we can formulate

Proposition 1. *ρ_1 and ρ_2 are respectively strict lower and upper solutions of Eq. (2). Hence there exists an unstable L-periodic solution between them.*

Proof. Let us observe that for $j = 1$,

$$\rho_{1,xx} + \tilde{V}(x)\rho_1 - g_1\rho_1^3 - \rho_1^5 > \lambda_1^2\rho_1 - g_1\rho_1^3 - \rho_1^5 = 0, \tag{10}$$

and similarly for $j = 2$. Hence, $\rho_1 < \rho_2$ are a couple of well-ordered lower and upper solutions respectively; therefore there exists a periodic solution between them [8]. Such a solution is unstable because the associated Brouwer index to the Poincaré map is -1 (see for instance [11,13]). □

Therefore we have a positive L-periodic solution of Eq. (2). Designating it as $\phi_+(x)$, we have that $\rho_1 \le \phi_+(x) \le \rho_2$. By the symmetry of equation we also have a negative solution $\phi_-(x) = -\phi_+(x)$.

To give an example of a periodic solution, we consider the simplified quintic nonlinear Schrödinger model (1) with $g_1 = 0$ and with the potential

$$V(x) = -\rho^4[2 - k^2\mathrm{sn}^2(\rho^2 x, k)]^2, \tag{11}$$

where $\mathrm{sn}(x, k)$ is the Jacobi elliptic function (hereafter we use the standard notations for the Jacobi elliptic functions, which can be found e.g. in [14]), $k \in [0, 1]$ is the elliptic modulus, and $\rho > 0$ (examples of the exact periodic solutions for the cubic nonlinear Schrödinger equation, $g_2 = 0$, and with the potential $V(x) = \mathrm{sn}^2(x, k)$, were obtained in [17,18]). The respective positive definite solution reads

$$\phi_+(x) = \rho\,\mathrm{dn}(\rho^2 x, k) \tag{12}$$

($\mathrm{dn}(x, k)$ is the Jacobi elliptic function, where as above k is the elliptic modulus). The solution (12) corresponds to the frequency $\omega = \rho^4(k^2 - 3)$. To verify the stability of ϕ in the sense of the dynamical system (2) we consider a small deviation $\tilde{\phi}(x) = \phi(x) - \phi_+(x)$ at $x \to \infty$, whose dynamics in the leading order is governed by the equation

$$\tilde{\phi}_{xx} - U(x, k)\tilde{\phi} = 0 \tag{13}$$

with

$$U(x,k) = \rho^4(4 - k^2 - 6k^2 \mathrm{sn}(\rho^2 x, k)^2 + 4k^4 \mathrm{sn}(\rho^2 x, k)^4) \\ \geq \rho^4(\frac{7}{4} - k^2) > 0. \tag{14}$$

Thus, the obtained function $\phi_+(x)$ is a hyperbolic periodic solution of Eq. (2) with the potential (11).

Considering now $\psi_+(x,t) = \phi_+(x)\exp(i\rho^2(3-k^2)t)$ as a solution of Eq. (1), performing the stability analysis as in Ref. [17], only slightly modified due to the presence of quintic nonlinearity, and taking into account that $\phi_+(x) > 0$, one verifies that $\psi_+(x,t)$ is linearly stable in the sense of the evolution problem (1).

More sophisticated models allowing exact sign definite periodic solutions can be constructed using a kind of "reverse engineering" (i.e. by obtaining potentials starting with given periodic solutions) as it is explained in [2].

3. Existence of a Dark Soliton

In this section we prove the existence of a heteroclinic orbit connecting the periodic solutions ϕ_- and ϕ_+. A battery of preparatory lemmas is necessary.

Lemma 1. *If $\phi : [x_0, +\infty) \to \mathbb{R}$ is a bounded solution of Eq. (9), then the derivative ϕ_x is also bounded in $[x_0, +\infty)$.*

Proof. By the hypothesis $|\phi(x)| < M$, where M is a constant, for all $x \geq x_0$. Then, by the mean value theorem there exists $x_n \in (nL, (n+1)L)$ such that $\phi((n+1)L) - \phi(nL) = \phi_x(x_n)L$ for $n > n_0$. From here $\phi_x(x_n) < 2M/L$ for all n. Applying the mean value theorem one more time one obtains

$$|\phi_x(x) - \phi_x(x_n)| < L \max_{|\phi| \leq M} |f(x,\phi)|, \quad \forall x \in (nL, (n+1)L) \tag{15}$$

because $|\phi_{xx}(x)| < \max_{|\phi|\leq M} |f(x,\phi)|$. Then

$$|\phi_x(x)| < L \max_{|\phi| \leq M} |f(x,\phi)| + \frac{2M}{L}, \quad \forall x \geq x_0. \tag{16}$$

□

Let us denote by $\gamma\{I - P, p_0\}$ the local index associated to the Brouwer degree of p_0 as a fixed point of the homeomorphism P (see [9, 10] for a rigorous definition of the Brouwer degree). The following lemma is a key ingredient in our main result.

Lemma 2 ([15]). *Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism with a unique fixed point p_L such that $\gamma\{I - P, p_L\} \neq 1$. Then for any $p_0 \in \mathbb{R}^2$ one of the following possibilities holds*

i) $P^n(p_0) \to p_L$ *as* $n \to +\infty$,
ii) $\|P^n(p_0)\| \to \infty$ *as* $n \to +\infty$.

The proof relies on a basic property of free homeomorphisms exposed in [12], namely if an orientation preserving homeomorphism has a unique fixed point p_L and has index different from 1, then it is a free homeomorphism. Then, by [12, Theorem 5.3] the ω-limit set of a given point has to be a connected set of the fixed point set.

Lemma 3. *If $f(x, y)$ is strictly increasing in y, there exists atmost one L-periodic solution of (9).*

Proof. By contradiction, let us assume that y_1, y_2 are two different L-periodic solutions of (9). First, let us suppose that y_1, y_2 intersect among themselves, that is, there should be x_1, x_2 such that $z(x) = y_1(x) - y_2(x)$ verifies $z(x_1) = 0 = z(x_2)$ and $z(t) > 0$ for $t \in (x_1, x_2)$. However, by subtracting the corresponding equations and using that f is strictly increasing, we get that z should be convex in (x_1, x_2), which is a contradiction. Therefore, y_1, y_2 do not intersect and we assume without loss of generality that $y_1(x) > y_2(x)$ for all x. Again, z should be convex in the whole real line, but this is impossible because it is periodic. □

With the help of these auxiliary lemmas we are able to prove an abstract convergence result.

Theorem 1. *Let $\phi : [x_0, +\infty) \to \mathbb{R}$ be a bounded solution of (9). Let us assume that*

$$\min_{\substack{x\in[0,L] \\ y\in[\inf_{x\geq x_0}\phi(x), \sup_{x\geq x_0}\phi(x)]}} \frac{\partial f(x, y)}{\partial y} > 0. \tag{17}$$

Then there exists an L-periodic solution $\varphi(x)$ such that

$$\lim_{x\to+\infty} (|\phi(x) - \varphi(x)| + |\phi_x(x) - \varphi_x(x)|) = 0. \tag{18}$$

Proof. Let us define $m = \inf_{x\geq x_0} \phi(x)$ and $M = \sup_{x\geq x_0} \phi(x)$, as well as the truncated function

$$\tilde{f}(x, y) = \begin{cases} f(x, y), & \forall y \in [m, M] \\ f(x, M) + f_y(x, M)(y - M), & \forall y > M \\ f(x, m) + f_y(x, M)(y - m), & \forall y < m \end{cases}. \tag{19}$$

$\tilde{f}$ is strictly increasing in y. Note also that ϕ is a solution of the truncated equation

$$\phi_{xx} = \tilde{f}(x, \phi). \tag{20}$$

Obviously, $\lim_{y\to\pm\infty} \tilde{f}(x, y) = \pm\infty$ uniformly in x. Hence, there exist constants α and β, such that $\alpha < \beta$ and $\tilde{f}(x, \alpha) < 0 < \tilde{f}(x, \beta)$ for all x. Such α and β is a well-ordered couple of L-periodic lower and upper solutions. Hence there exists an L-periodic solution of (20) between them [8]. Like in Proposition 1, this solution is unstable and the associated Brouwer index to the Poincaré map is -1. By the previous lemma, this solution is unique. Then, by Lemma 2, $\phi(x)$ must converge to $\varphi(x)$ since Lemma 1 excludes the possibility ii) of Lemma 2. As $\varphi(x) \in [m, M]$, it is a solution of (9). □

Theorem 2. *Let us consider bounded functions $\phi_1, \phi_2 : [x_0, +\infty) \to \mathbb{R}$ verifying*

1) $\phi_1(x) < \phi_2(x)$, $\forall x > x_0$,
2) $\phi_{1xx}(x) > f(x, \phi_1)$ *and* $\phi_{2xx}(x) < f(x, \phi_2)$, $\forall x > x_0$.
 Then there exists a solution $\phi(x)$ of (9) such that

$$\phi_1(x) < \phi(x) < \phi_2(x). \tag{21}$$

If moreover, there exists x such that

3) $$\min_{\substack{x\in[0,L]\\ y\in[\inf_{x\geq x_0}\phi_1(x),\sup_{x\geq x_0}\phi_2(x)]}} \frac{\partial f(x,y)}{\partial y} > 0,$$

then there exists an L-periodic solution $\varphi(x)$ such that

$$\lim_{x\to+\infty}(|\phi(x)-\varphi(x)|+|\phi_x(x)-\varphi_x(x)|)=0. \tag{22}$$

Besides, $\varphi(x)$ is the unique L-periodic solution in the interval $[\inf_{x\geq x_0}\phi_1(x), \sup_{x\geq x_0}\phi_2(x)]$.

Proof. The first assertion (21) is the classical result due to Opial [16]. The second conclusion (22) is a corollary of Theorem 1. □

In order to apply the above results to our model (2), we observe that now i) $f(x,y)\equiv y^5+g_1y^3-\tilde{V}(x)y$, ii) due to parity of the potential one can consider $x\geq 0$ and extend the obtained solution $\phi(x)$ as an odd function to $x\leq 0$, iii) the functions $\phi_{1,2}(x)$ given by (7) satisfy the conditions 1) and 2) of the Theorem 2. Hence, in order to prove that there exists a solution $\phi(x)$ of (1) converging to $\phi_\pm(x)$, found in Proposition 1, as $x\to\pm\infty$, one has to verify the condition 3) of Theorem 2.

As x_0 can be taken arbitrarily large, this last condition is equivalent to

$$\min_{\substack{y\in[\rho_1,\rho_2]\\ x\in[0,L]}}\left\{5y^4+3g_1y^2-\tilde{V}(x)\right\}>0. \tag{23}$$

Starting with the case $g_1\geq 0$ we observe that (23) is now equivalent to $5\rho_1^4+3g_1\rho_1^2-\lambda_2^2>0$. The straightforward analysis of this last inequality, which takes into account the link (6), the definition of $\lambda_{1,2}$, and the requirement (4) necessary for $\lambda_1^2>0$, gives the following estimate for the frequency:

$$\omega-g_1\sqrt{g_1^2+4\omega-4V_{max}}>-V_{min}+5V_{max}. \tag{24}$$

Thus (24) is a sufficient condition for the existence of a dark soliton at the non-negative g_1. After some cumbersome but straightforward computations, one realizes that (24) is equivalent to the explicit bound

$$\omega>\omega_0=5V_{max}-V_{min}+2g_1+\sqrt{g_1^3+4g_1^2+4g_1(4V_{max}-V_{min})}. \tag{25}$$

Considering now $g_1<0$ the constraint (23) is reduced to $5\rho_1^4-3|g_1|\rho_2^2-\lambda_2^2>0$ and subsequently to the following inequality constraint to the frequency

$$\begin{aligned}&8\omega+2g_1^2-10V_{max}+2V_{min}-5g_1\sqrt{4\omega+g_1^2-4V_{max}}\\&+3g_1\sqrt{4\omega+g_1^2-4V_{min}}>0\end{aligned} \tag{26}$$

which must be satisfied simultaneously with (4). In this case, we omit the explicit bound for ω.

In both cases considered above, the conclusion is that there exists an explicitly computable ω_0 such that the CQNLS has a dark soliton for any $\omega>\omega_0$.

4. Concluding Remarks

First of all we emphasize that ω is not a parameter of the original Eq. (1), but just the temporal frequency of a stationary solution. The obtained condition (26) is needed for technical reasons. Clearly, $\omega > \omega_0$ implies that the origin of Eq. (2) is a center, so that the possible heteroclinics must connect non-trivial solutions. Otherwise, when the origin is a hyperbolic point one can expect the presence of homoclinics (this is well known) connecting the origin with itself, corresponding to bright solitons of Eq. (1) (for numerical examples of such homoclinics in cubic and quintic nonlinear Schrödinger equations we refer to [2,7] and to [5], respectively).

The other imposed condition (4) guarantees the existence of the two nontrivial hyperbolic points (6) (we also notice that in physical terms, condition (4) means that the chemical potential is bigger than the amplitude of the periodic potential) implying the existence of sign definite periodic solutions $\phi_\pm$. This naturally leaves an open question about the existence of dark solitons whose asymptotics at $x \pm \infty$ are given by sign-alternating nonlinear waves (such a possibility is suggested by the numerical simulations [2,6,7]).

Finally, we consider a particular example illustrating a dark soliton, as well as other concepts introduced in the paper. To this end we recall the potential (11) and construct a dark soliton which tends to ϕ_+ given by (12) (we thus consider now $g_1 = 0$). This cannot be done analytically, and that is why we employ numerics. An example is shown in Fig. 1.

To numerically obtain the dark soliton, we use the shooting method [7]. To this end we observe that (13) is a Hill's equation and thus taking into account (14), from Floquet's theorem we obtain that $\phi(x) \to \tilde{\phi}(x) = P(x)\exp(-\alpha x)$ as $x \to +\infty$, where $P(x) = P(x+2\mathrm{K}(k)/\rho^2)$ is a periodic function with the period $L = 2\mathrm{K}(k)/\rho^2$ and $\mathrm{K}(k)$ is the complete elliptic integral of the first kind. For given parameters k and ρ one can easily compute the respective Floquet exponent α. In particular, for our choice of $k = 0.5$ and $\rho = 1$, we obtain $\alpha \approx 2.014298$. Thus, starting with a point $x_{ini} = 2n\mathrm{K}(k)/\rho^2$, where n is an integer, for which the equalities $\phi(x_{ini}) = \phi_+(0) - C$ and $\phi_x(x_{ini}) = -C\alpha$ with a positive constant C are verified, we compute $\phi(0)$ by varying the parameter C. We start with $C = 0$ and increase C until we meet the condition $\phi(0) = 0$. The respective smallest positive value of C corresponds to the dark soliton we are looking for. An example of numerical implementation of this procedure is shown in Fig. 1.

References

1. Pitaevskii, L.P., Stringari, S.: *Bose-Einstein condensation*. Oxford: Oxford University Press, 2003
2. Brazhnyi, V.A., Konotop, V.V.: Theory of nonlinear matter waves in optical lattices. Mod. Phys. Lett. B **18**, 627 (2004)
3. Brazhnyi, V.A., Konotop, V.V., Pitaevskii, L.P.: Dark solitons as quasiparticles in trapped condensates. Phys. Rev. A **73**, 053601 (2006)
4. Abdullaev, F.K., Salerno, M.: Gap-Townes solitons and localized excitations in low-dimensional Bose-Einstein condensates in optical lattices. Phys. Rev. A **72**, 033617 (2005)
5. Alfimov, G.L., Konotop, V.V., Pacciani, P.: Stationary localized modes in the quintic nonlinear Schrodinger equation with a periodic potential. Phys. Rev. A **75**, 023624 (2007)
6. Konotop, V.V., Salerno, M.: Modulational instability in Bose-Einstein condensates in optical lattices. Phys. Rev. A **65**, 021602(R) (2002)
7. Alfimov, G.L., Konotop, V.V., Salerno, M.: Matter solitons in Bose-Einstein condensates with optical lattices. Europhys. Lett. **58**, 7 (2002)
8. De Coster, C., Habets, P.: Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results. In: *Nonlinear Analysis and Boundary Value Problems for Ordinary Differential Equations*, ed. F. Zanolin, CISM-ICMS Courses and Lectures **371**, New York: Springer Verlag, 1996
9. Lloyd, N.G.: *Degree Theory*. Cambridge: Cambridge University Press, 1978

10. Mawhin, J.: Topological degree and boundary value problems for nonlinear differential equations. In: "*Topological Methods for Ordinary Differential Equations*", (Montecatini Terme, 1991), M. Furi, P. Zecca eds., Lecture Notes in Mathematics **1537**, Berlin: Springer-Verlag 1993, pp. 74–142
11. Ortega, R.: Some applications of the topological degree to stability theory. In: "*Topological methods in differential equations and inclusions*", NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **472**, Dordrecht: Kluwer Acad. Publ., 1995, pp. 377–409
12. Brown, M.: Homeomorphisms of two-dimensional manifolds. Houston J. of Math. **11**, 455–469 (1985)
13. Dancer, E.N., Ortega, R.: The index of Lyapunov stable fixed points in two dimensions. J. Dyn. Differ. Eqs. **6**, 631 (1994)
14. Lawden, D.K.: *Elliptic functions and applications*. New York: Springer-Verlag, 1989
15. Campos, J., Torres, P.: On the structure of the set of bounded solutions on a periodic Lienard equation. Proc. Amer. Math. Soc. **127**, 1453 (1999)
16. Opial, Z.: Sur les intégrales bornees de l'equation $u'' = f(t, u, u')$. Ann. Polonici Math. **4**, 314–324 (1958)
17. Bronski, J.C., Carr, L.D., Deconinck, B., Kutz, J.N., Promislow, K.: Stability of repulsive Bose-Einstein condensates in a periodic potential. Phys. Rev. E **63**, 036612 (2001)
18. Bronski, J.C., Carr, L.D., Carratero-Gonzalez, R., Deconinck, B., Kutz, J.N., Promislow, K.: Stability of attractive Bose-Einstein condensates in a periodic potential. Phys. Rev. E **64**, 056615 (2001)
19. Sulem, C., Sulem, P.: *The nonlinear Schrödinger equation: Self-focusing and wave collapse*. Berlin: Springer 2000
20. Rybin, A.V., Varzugin, G.G., Lindberg, M., Timonen, J., Bullough, R.K.: Similarity solutions and collapse in the attractive Gross-Pitaevskii equation. Phys. Rev. E **62**, 6224 (2000)
21. Krasnosel'skii, M.A., Burd, V.Sh., Kolesov, Yu.S.: *Nonlinear almost periodic oscillations*. New York: John Wiley & Sons, 1973

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