# Lie Symmetries and Solitons in Nonlinear Systems with Spatially Inhomogeneous Nonlinearities

Juan Belmonte-Beitia, Víctor M. Pérez-García, and Vadym Vekslerchik

*Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales, and Instituto de Matemática Aplicada a la Ciencia y la Ingeniería (IMACI), Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain*

Pedro J. Torres

*Departamento de Matemática Aplicada, Universidad de Granada, Campus de Fuentenueva s/n, 18071 Granada, Spain*

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Using Lie group theory and canonical transformations, we construct explicit solutions of nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities. We present the general theory, use it to show that localized nonlinearities can support bound states with an arbitrary number solitons, and discuss other applications of interest to the field of nonlinear matter waves.

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*Introduction.*—Solitons are self-localized nonlinear waves which are sustained by an equilibrium between dispersion and nonlinearity and appear in a great variety of physical contexts [1]. In particular, these nonlinear structures have been generated recently in ultracold atomic bosonic gases cooled down below the Bose-Einstein transition temperature [2–4]. In those systems, the effective nonlinear interactions are a result of the elastic two-body collisions between the condensed atoms.

These interactions can be controlled by the so-called Feschbach resonance (FR) management [5], which has been used to generate bright solitons [3,6], induce collapse [7], etc. Many different nonlinear structures induced by time variations of the condensate scattering length have been theoretically predicted, such as periodic waves [8], shock waves [9], stabilized solitons [10], etc.

Interactions can be made spatially dependent through the spatial dependence of either the magnetic field or the laser intensity (in the case of optical control of FR [11]) acting on the Feschbach resonances. This fact has motivated recently many theoretical studies on nonlinear waves in Bose-Einstein condensates (BECs) with spatially inhomogeneous interactions including solitonic emission [12] and the dynamics of solitons when the modulation of the nonlinearity is a random [13], linear [14], periodic [15], or localized function [16]. The existence and stability of solutions has been studied in Ref. [17].

In this Letter, we construct general classes of nonlinearity modulations and external potentials for which explicit solutions can be constructed. To do so, we use Lie group theory and canonical transformations connecting problems with inhomogeneous nonlinearities with simpler ones having an homogeneous nonlinearity. These mathematical methods will be used to show that localized nonlinearities can support bound states with an arbitrary number of solitons without any additional external potential. This is an interesting result with physical implications and a peculiarity of inhomogeneous nonlinearities. Our focus will be on applications to matter waves in BECs, but our ideas can be applied to nonlinear optical systems [18].

*General theory.*—In this Letter, we consider physical systems ruled by the nonlinear Schrödinger equation with a spatially inhomogeneous nonlinearity (INLSE), i.e.,

$$i\psi_t = -\psi_{xx} + V(x)\psi + g(x)|\psi|^2\psi, \tag{1}$$

where $V(x)$ is an external potential and $g(x)$ describes the spatial modulation of the nonlinearity. Stationary solutions of the INLSE are of the form $\psi = \phi e^{-i\lambda t}$, where

$$-\phi_{xx} + V(x)\phi + g(x)\phi^3 = \lambda\phi, \qquad \phi(\pm\infty) = 0. \tag{2}$$

A second-order differential equation $A(x, u, u', u'') = 0$ possesses a Lie point symmetry [19,20] of the form $M = \xi(x, u)\partial/\partial x + \eta(x, u)\partial/\partial u$ if

$$\begin{aligned} M^{(2)}A(x, u, u', u'') = \Big[ \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} + \eta_1(x, u) \\ \times \frac{\partial}{\partial u'} + \eta_2(x, u)\frac{\partial}{\partial u''}\Big]A(x, u, u', u'') \\ = 0. \end{aligned} \tag{3}$$

In our case, $A(x, \phi, \phi_x, \phi_{xx})$ is given by Eq. (2), and the action of the operator $M^{(2)}$ on it leads to

$$\xi_{\phi\phi} = 0, \tag{4a}$$

$$\eta_{\phi\phi} - 2\xi_{\phi x} = 0, \tag{4b}$$

$$2\eta_{x\phi} - \xi_{xx} - 3f\xi_\phi = 0, \tag{4c}$$

$$\eta_{xx} - \xi f_x - \eta f_\phi + \eta_\phi f - 2\xi_x f = 0, \tag{4d}$$

where $f(x, \phi) = V(x)\phi + g(x)\phi^3 - \lambda\phi$. Solving the previous equations, we find that the only Lie point symmetries of Eq. (2) are of the form

$$M = b(x)\frac{\partial}{\partial x} + c(x)\phi\frac{\partial}{\partial \phi}, \tag{5}$$

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where

$$g(x) = \frac{g_0}{b(x)^3} \exp\left[-2K \int_0^x \frac{1}{b(s)} ds\right], \tag{6a}$$

$$c(x) = \tfrac{1}{2} b'(x) + K, \tag{6b}$$

$$0 = c''(x) - b(x)V'(x) - 2b'(x)[V(x) - \lambda], \tag{6c}$$

for any constant $K$. Equations (6) allow us to construct pairs $\{V(x), g(x)\}$ for which a Lie point symmetry exists.

*Conservation laws and canonical transformations.*—It is known [21] that the invariance of the energy is associated to the translational invariance. The generator of such a transformation is of the form $M = \partial/\partial x$. To use this fact, we define the transformation

$$X = f(x), \qquad U = n(x)\phi, \tag{7}$$

where $f(x)$ and $n(x)$ will be determined by requiring that a conservation law of energy type $M = \partial/\partial X$ exists in the canonical variables. Using Eqs. (5) and (7), we get

$$f(x) = \int_0^x \frac{1}{b(s)} ds, \tag{8a}$$

$$n(x) = \frac{1}{b(x)^{1/2}} \exp\left[-K \int_0^x \frac{1}{b(s)} ds\right]. \tag{8b}$$

When $K = 0$, the transformations preserve the Hamiltonian structure, and Eq. (2) in terms of $U = b^{-1/2}(x)\phi$ and $X = \int_0^x [1/b(s)] ds$ becomes

$$-\frac{d^2U}{dX^2} + g_0 U^3 = EU, \tag{9}$$

where $E = [\lambda - V(x)]b(x)^2 - \frac{1}{4}b'(x)^2 + \frac{1}{2}b(x)b''(x)$ is a constant. This means that in the new variables we obtain the nonlinear Schrödinger equation (NLSE) without an external potential and with an homogeneous nonlinearity. Of course, not any choice of $V(x)$ and $g(x)$ leads to the existence of a Lie symmetry or an appropriate canonical transformation [e.g., the function $b(x)$ must be sign definite for $U$ and $X$ to be properly defined].

*Connection between the NLSE and INLSE via the LSE.*—We can use all of the known solutions of the NLSE (9), e.g., solitons, plane waves, and cnoidal waves, to construct solutions to Eq. (2). Setting $K = 0$ and eliminating $c(x)$ in Eqs. (6), we get

$$g(x) = g_0/b(x)^3 \tag{10}$$

and an equation relating $b(x)$ and $V(x)$:

$$b'''(x) - 2b(x)V'(x) + 4b'(x)\lambda - 4b'(x)V(x) = 0. \tag{11}$$

Although we can eliminate $b(x)$ and obtain a nonlinear equation for the pairs $g(x)$ and $V(x)$ for which there is a Lie symmetry, it is more convenient to work with (11), which is a linear equation. Alternatively, we can define $\rho(x) = b^{1/2}(x)$ and get an Ermakov-Pinney equation [22]

$$\rho_{xx} + [\lambda - V(x)]\rho = E/\rho^3, \tag{12}$$

the solutions of which can be constructed as

$$\rho = (\alpha\varphi_1^2 + 2\beta\varphi_1\varphi_2 + \gamma\varphi_2^2)^{1/2}, \tag{13}$$

with $\alpha$, $\beta$, and $\gamma$ constant and $\varphi_j(x)$ being two linearly independent solutions of the Schrödinger equation

$$-\varphi_{xx} + V(x)\varphi = \lambda\varphi. \tag{14}$$

This choice leads to $E = \Delta W^2$, with $\Delta = \alpha\gamma - \beta^2$ and $W$ being the (constant) Wronskian $W = \varphi_1'\varphi_2 - \varphi_1\varphi_2'$. Thus, given any arbitrary solution of the linear Schrödinger equation (14), we can construct solutions of the nonlinear spatially inhomogeneous problem Eq. (2) from the known solutions of Eq. (9). Thus, using the huge amount of knowledge on the linear Schrödinger equation, we can get potentials $V(x)$ for which $\varphi_1$ and $\varphi_2$ are known and construct $b(x)$, the canonical transformations $f(x)$ and $n(x)$, the nonlinearity $g(x)$, and the explicit solutions $\phi(x)$.

*Systems without external potential [$V(x) = 0$].*—As a first application of our ideas, we choose $V(x) = 0$; then Eq. (11) becomes $b'''(x) + 4b'(x)\lambda = 0$ and its solution is

$$b(x) = C_1 \sin\omega x + C_2 \cos\omega x + C_3 (\lambda > 0), \tag{15a}$$

$$b(x) = C_1 e^{\omega x} + C_2 e^{-\omega x} + C_3 (\lambda < 0), \tag{15b}$$

where $\omega = 2\sqrt{|\lambda|}$. Using Eq. (10), we see that Eq. (15b) corresponds to an exponentially localized nonlinearity [Fig. 1(a)] and Eq. (15a) leads to a periodic one:

$$g(x) = g_0(1 + \alpha\cos\omega x)^{-3} \tag{16}$$

[Figs. 1(b) and 1(d)]. For small $\alpha$, this nonlinearity is approximately harmonic [Fig. 1(b)]

$$g(x) \simeq g_0(1 - 3\alpha\cos\omega x), \qquad \alpha \ll 1. \tag{17}$$

We can construct our canonical transformation by using Eqs. (7) and obtain

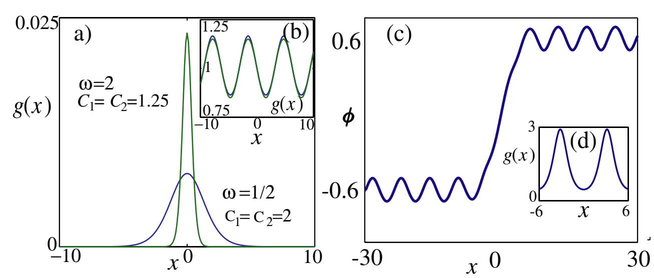


FIG. 1 (color online). (a) Examples of exponentially localized nonlinearities given by Eq. (15b) with $C_1 = C_2 = 2$, $C_3 = 0$, $\omega = 1/2$ (blue line), and $C_1 = C_2 = 1.25$, $C_3 = 0$, $\omega = 2$ (green line). (b) Comparison of $g(x)$ given by Eq. (16) for $g_0 = \omega = 1$, $\alpha = 0.05$ (blue line) and its harmonic approximation (17) (green line). (c) Example of a black soliton solution of Eq. (2) with $V = 0$ and an inhomogeneous nonlinearity given by Eq. (16) with $\omega = 1$, $\alpha = 0.3$, and $g_0 = 1$. (d) Inhomogeneous nonlinearity given by Eq. (16) with $\omega = 1$, $\alpha = 0.3$, and $g_0 = 1$ used to calculate the black soliton solution.

$$\tan\left[\frac{\omega}{2}\sqrt{1-\alpha^2}X(x)\right] = \sqrt{\frac{1-\alpha}{1+\alpha}}\tan\frac{\omega x}{2}. \tag{18}$$

Using any solution of Eq. (9) with $E = \frac{1}{4}\omega^2(1-\alpha^2)$, this transformation provides solutions of Eq. (2) with $g(x)$ given by (16). For example, when $g_0 > 0$ we can obtain black soliton solutions of Eq. (2) of the form

$$\phi(x) = \frac{\omega}{2}\sqrt{\frac{1-\alpha^2}{g}(1+\alpha\cos\omega x)} \times \tanh\left[\frac{\omega}{2}\sqrt{\frac{1-\alpha^2}{2}}X(x)\right], \tag{19}$$

where $X(x)$ is defined by Eq. (18) [Fig. 1(c)]. We emphasize that this is only a simple example of the many possible solutions that can be constructed in such a way.

Concerning the case given by Eq. (15b), we would like to discuss it in more detail since we will get an interesting physical phenomenon from its analysis. In order to simplify the following formulas (without losing any significant features), we restrict ourselves to a particular choice of the constants $\omega = 1$, $C_1 = C_2 = 1/2$, and $C_3 = 0$ in Eq. (15b). Thus, $b(x) = \cosh x$ and Eq. (2) with $g(x)$ given by Eq. (10) and $\lambda = -1/4$,

$$-\phi_{xx} + \frac{1}{4}\phi + \frac{g_0}{\cosh^3 x}\phi^3 = 0, \tag{20}$$

in terms of $U$ and $X$ can be written as Eq. (9) with $E = 1/4$ being $\cos X(x) = -\tanh x$; thus, $0 \le X \le \pi$, and to meet the boundary conditions $\phi(\pm\infty) = 0$ one has to impose $U(0) = U(\pi) = 0$. This means that the original infinite domain in Eq. (20) is mapped into a bounded domain for Eq. (9). It is easy to check that, when $g_0 < 0$,

$$U(X) = \eta\frac{\mathrm{sn}(\mu X, k)}{\mathrm{dn}(\mu X, k)} \tag{21}$$

solves Eq. (9) provided $\mu^2 = 1/[4(1-2k^2)]$ and $\eta^2 = k^2(1-k^2)/[2|g_0|(1-2k^2)]$. The function $U(X)$ satisfies $U(0) = 0$, and, in order to meet $U(\pi) = 0$, the condition $\mu\pi = 2nK(k)$, where $K(k)$ is the elliptic integral $K(k) = \int_0^{\pi}(1-k^2\sin^2\varphi)^{-1/2}d\varphi$, must hold. Thus, to satisfy the boundary conditions, $k$ must be chosen to satisfy $4nK(k)\times\sqrt{1-2k^2} = \pi$ for $n = 1, 2, \ldots$. It can be shown that for every integer number $n$ this algebraic equation has only a solution $k_n$, which means that there are an infinite number of solutions of Eq. (20) of the form given by Eq. (21). Moreover, each of those solutions has exactly $n-1$ zeros. In Fig. 2, we plot some of them corresponding to $n = 1, 2, 3$. These solutions can be seen as "bound states" of several ($n$) solitons with alternating phases, and their existence is remarkable. When the nonlinearity is homogeneous, $g(x) = g_0 < 0$, Eq. (2) has only one localized solution for each $\lambda$, the cosh-type soliton; in other words, there are no bound states of several solitons. However, when $g(x)$ is modulated and decays exponentially as given in Eq. (20), we get an infinite number of localized solutions labeled by their finite number of nodes. This is a novel and interesting feature of localized nonlinearities.

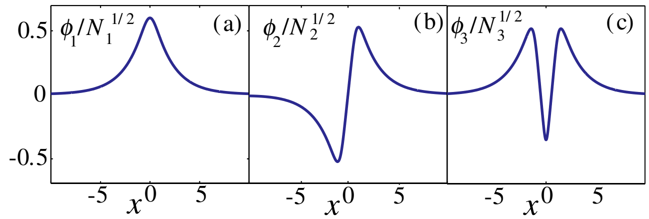


FIG. 2 (color online). Solutions of Eq. (20) with $g_0 = 1$ corresponding to (a) $n = 1$, $k_1 = 0.634\,493$, (b) $n = 2$, $k_2 = 0.690\,727$, and (c) $n = 3$, $k_3 = 0.699\,957$.

*Systems with quadratic potentials $V(x) = x^2$.*—Any potential for which explicit solutions of Eqs. (14) are known can be used to find nontrivial nonlinearities for which solutions can be constructed. Out of many possibilities, we discuss only an example of interest for the applications to nonlinear matter waves in Bose-Einstein condensates, which is $V(x) \propto x^2$.

Let us choose $b(x) = e^{x^2}$, which leads to a quadratic trapping potential $V(x) = x^2$ and a Gaussian nonlinearity such as the one generated by controlling the Feschbach resonances optically using a Gaussian beam (see, e.g., [12]); thus,

$$g(x) = g_0\exp(-3x^2), \qquad V(x) = x^2. \tag{22}$$

Our canonical transformation is given by $X(x) = \int_0^x dt\exp(-t^2) = (\sqrt{\pi}/2)\mathrm{erf}x$. In this case, Eq. (2) is transformed into

$$-U_{XX} + g_0U^3 = 0. \tag{23}$$

Note that the range of $X$ is again finite since $-\sqrt{\pi}/2 \le X \le \sqrt{\pi}/2$, and, hence, we can again construct many *localized* solutions to Eq. (2) starting from solutions of (23) which satisfy the boundary conditions $U(\pm\sqrt{\pi}/2) = 0$. This can be done noting that, for $g_0 < 0$ and any $\mu$, the functions

$$U^{(1)}(X) = \frac{\mu}{\sqrt{|g_0|}}\mathrm{cn}(\mu X, k_*) \tag{24}$$

and

$$U^{(2)}(X) = \frac{\mu}{\sqrt{2|g_0|}}\frac{\mathrm{sn}(\mu X, k_*)}{\mathrm{dn}(\mu X, k_*)}, \tag{25}$$

with $k_* = 1/\sqrt{2}$, solve Eq. (23) and that $U^{(1)}(X)$ and $U^{(2)}(X)$ vanish when $\mu X = (2n+1)K(k_*)$ and $\mu X = 2nK(k_*)$ correspondingly. Thus, we come to an infinite number of solutions of Eq. (23) under zero boundary conditions on the new finite interval, which correspond to different values of $\mu$. Finally, localized solutions of the NLS equation (2) are given by

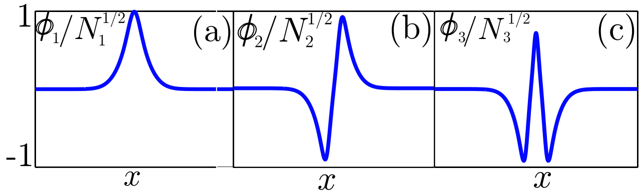


FIG. 3 (color online). Different solutions of Eq. (2) for potential and nonlinearities given by Eq. (22). (a) $n = 1$, (b) $n = 2$, and (c) $n = 3$. The spatial region shown corresponds to $x \in [-6, 6]$.

$$\phi_n(x) = \begin{cases} \frac{2nK(k_*)}{\sqrt{\pi|g_0|}} e^{x^2/2} \mathrm{cn}(\theta_n(x), k_*) & n = 1, 3, \ldots, \\ \frac{2nK(k_*)}{\sqrt{2\pi|g_0|}} e^{x^2/2} \frac{\mathrm{sn}(\theta_n(x), k_*)}{\mathrm{dn}(\theta_n(x), k_*)} & n = 2, 4, \ldots, \end{cases} \tag{26}$$

with

$$\theta_n(x) = nK(k_*)\mathrm{erf}x. \tag{27}$$

It can be shown by simple asymptotic analysis that the last factors in Eq. (26) tend to zero as $x \to \pm\infty$ faster than $\exp(-x^2/2)$ and that these are indeed *localized* solutions of our problem as can be seen in the ones plotted in Fig. 3, with different numbers of zeros [$\phi_n(x)$ possesses $n - 1$ zeros].

In conclusion, we have used Lie symmetries and canonical transformations to construct explicit solutions of the nonlinear Schrödinger equation with a spatially inhomogeneous nonlinearity from those of the homogeneous nonlinear Schrödinger equation. The range of nonlinearities and potentials for which this can be done is very wide. We have studied in detail the case $V = 0$ with localized and periodic nonlinearities. In the former case, we have used our theory to construct an infinite number of multisoliton bound states, something which is not possible in the case of spatially homogeneous nonlinearities. Finally, we have presented an example of physical interest (harmonic trap and Gaussian nonlinearity) for which exact solutions can be constructed. The ideas contained in this Letter could also be applied to study time-dependent problems, higher-dimensional situations, multicomponent systems, etc. We hope that this Letter will stimulate further research on those topics and help to understand the behavior of nonlinear waves in systems with spatially inhomogeneous nonlinearities.

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