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APPLICATIONS OF SCHAUDER'S FIXED POINT THEOREM TO SINGULAR DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we study the existence of positive periodic solutions to second-order singular differential equations. The proof relies on Schauder's fixed point theorem. Our results show that in some situations weak singularities can help create periodic solutions, as pointed out by Torres [*J. Differential Equations* 232 (2007) 277–284].

1. *Introduction*

In this paper, we study the existence of positive periodic solutions of the second-order differential equation

$$x'' + a(t)x = f(t, x) + e(t); \tag{1.1}$$

here, $a(t)$ and $e(t)$ are continuous and 1-periodic functions. The nonlinearity $f(t, x)$ is continuous in (t, x) and 1-periodic in t. We are mainly interested in the case that $f(t, x)$ may be singular at $x = 0$.

Beginning with the paper of Lazer and Solimini [10], the semilinear singular differential equation

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + e(t), \tag{1.2}$$

with $a, b, e \in \mathbb{C}[0, 1]$ and $\lambda > 0$, has attracted the attention of many researchers during the last two decades [2, 4, 7, 13]. Some strong force conditions introduced by Gordon [6] are standard in the related works [4, 12, 17, 19]. This condition corresponds to the case when $\lambda > 1$ in equation (1.2). With a strong singularity, the energy near $x = 0$ becomes infinite, and this fact is very useful for obtaining a priori bounds of periodic solutions. Compared with the case of strong singularities, the study of the existence of periodic solutions under the presence of weak singularities is more recent, and the number of references is much smaller [5, 9, 14–16].

Some classical tools have been used in the literature to study singular equations. These classical tools include the coincidence degree theory of Mawhin [11, 17, 19], the method of upper and lower solutions [1, 3] and some fixed point theorems in cones for completely continuous operators [8, 9, 15]. If the Green function $G(t, s)$ associated with (2.1) and (2.2) is positive, then it has been proved in [9] that equation (1.1) with $e(t) \equiv 0$ has at least two different positive periodic solutions when $f(t, x)$ has a repulsive singularity near $x = 0$ (that is, $\lim_{x\to 0^+} f(t, x) = +\infty$, uniformly in t) and $f(t, x)$ is superlinear near $x = +\infty$ (that is, $\lim_{x\to+\infty} f(t, x)/x = +\infty$, uniformly in t). The proof given in [9] is based on a nonlinear alternative principle of Leray–Schauder and a well-known fixed point theorem in cones.

In this paper, we establish the existence of positive periodic solutions to equation (1.1) through a basic application of Schauder's fixed point theorem, generalizing in several aspects some results contained in [1, 9, 14–16]. Our main motivation is to obtain new existence results

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for positive periodic solutions of the equation

$$x'' + a(t)x = \frac{b(t)}{x^{\alpha}} + \mu c(t)x^{\beta} + e(t), \tag{1.3}$$

with $a, b, c, e \in \mathbb{C}[0, 1]$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ a given parameter.

The rest of the paper is organized as follows. In Section 2, some preliminary results are given. In Sections 3 and 4, we state and prove the main results of the paper, as well as some applications to (1.2) and (1.3).

2. *Preliminaries*

Throughout this paper, we assume that Hill's equation

$$x'' + a(t)x = 0 \tag{2.1}$$

with periodic boundary conditions

$$x(0) = x(1), \qquad x'(0) = x'(1), \tag{2.2}$$

satisfies the following standing hypothesis.

(A) The associated Green function $G(t, s)$ is non-negative for all $(t, s) \in [0, 1] \times [0, 1]$.

In other words, the anti-maximum principle holds. Under this assumption, let us define the function

$$\gamma(t) = \int_0^1 G(t, s)e(s)\, ds,$$

which is the unique 1-periodic solution of the linear equation

$$x'' + a(t)x = e(t).$$

Now we make condition (A) clear. When $a(t) = k^2$, condition (A) is equivalent to saying that $0 < k^2 \leqslant \mu_1 = \pi^2$. Note that μ_1 is the first eigenvalue of the linear problem with Dirichlet conditions $x(0) = 0 = x(1)$. For a non-constant function $a(t)$, there is an L^p-criterion proved in [15]. Let $K(q)$ denote the best Sobolev constant in the following inequality:

$$C\|u\|_q^2 \leqslant \|u'\|_2^2, \qquad \text{for all } u \in H_0^1(0, 1).$$

The explicit formula for $K(q)$ is

$$K(q) = \begin{cases} \dfrac{2\pi}{q}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2 & \text{if } 1 \leqslant q < \infty, \\ 4 & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function.

From now on, let us denote by p^* and p_*, respectively, the essential supremum and infimum of a given function $p \in L^1[0, 1]$, if they exist. Also, we write $p \succ 0$ if $p \geqslant 0$ for almost every $t \in [0, 1]$ and is positive in a set of positive measure.

Lemma 2.1 [15]. *Assume that $a(t) \succ 0$ and $a \in L^p[0, 1]$ for some $1 \leqslant p \leqslant \infty$. If*

$$\|a\|_p \leqslant K(2p^*),$$

then the standing hypothesis (A) *holds.*

Remark 2.2. In [9, 15], the existence results are based on the positivity of $G(t, s)$, which plays a very important role in employing some fixed point theorems in cones for completely continuous operators. Our assumption (A) needs only that $G(t, s)$ be non-negative, and therefore our results cover the critical case, which was not covered in the above papers (see [**9**, Theorems 3.1 and 3.3] and [**15**, Theorem 3.2]).

3. *Main results I*

In this section, we establish the existence of positive periodic solutions for equation (1.1). The following is the main result in this section.

THEOREM 3.1. *Suppose that $a(t)$ satisfies* (A). *Furthermore, assume that the following conditions hold.*

(H_1) *For each $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that $f(t,x) \geqslant \phi_L(t)$ for all $(t,x) \in [0,1] \times (0,L]$.*

(H_2) *There exist continuous, non-negative functions $g(x)$, $h(x)$ and $k(t)$, such that*

$$0 \leqslant f(t,x) \leqslant k(t)\{g(x)+h(x)\} \qquad \text{for all } (t,x) \in [0,1] \times (0,\infty),$$

and $g(x) > 0$ is non-increasing and $h(x)/g(x)$ is non-decreasing in $x \in (0,\infty)$.

(H_3) *There exists a positive constant $R > 0$ such that $R > \Phi_{R*} + \gamma_* > 0$ and*

$$R \geqslant g(\Phi_{R*} + \gamma_*)\left(1 + \frac{h(R)}{g(R)}\right)K^* + \gamma^*;$$

here,

$$\gamma(t) = \int_0^1 G(t,s)e(s)\,ds,$$
$$\Phi_R(t) = \int_0^1 G(t,s)\phi_R(s)\,ds,$$
$$K(t) = \int_0^1 G(t,s)k(s)\,ds.$$

Then equation (1.1) *has at least one positive periodic solution.*

Proof. A periodic solution of equation (1.1) is just a fixed point of the completely continuous map $T : \mathbb{C}_1 \to \mathbb{C}_1$ defined by

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)[f(s,x(s)) + e(s)]\,ds \\ &= \int_0^1 G(t,s)f(s,x(s))\,ds + \gamma(t).\end{aligned} \tag{3.1}$$

Here, $\mathbb{C}_1$ denotes the set of all continuous 1-periodic functions. Let R be the positive constant satisfying (H_3) and

$$r = \Phi_{R*} + \gamma_*.$$

Then we have $R > r > 0$. Now we define the set

$$\mathcal{A} = \{x \in \mathbb{C}_1 : r \leqslant x(t) \leqslant R \text{ for all } t \in [0,1]\}. \tag{3.2}$$

Obviously, $\mathcal{A}$ is a closed convex set. Next we prove that $T(\mathcal{A}) \subset \mathcal{A}$.

In fact, for each $x \in \mathcal{A}$ and for all $t \in [0,1]$, using the fact that $G(t,s) \geqslant 0$ for all $(t,s) \in [0,1] \times [0,1]$, together with condition (H_1), we have

$$(Tx)(t) \geqslant \int_0^1 G(t,s)\phi_R(s)\,ds + \gamma(t) \geqslant \Phi_{R*} + \gamma_* = r > 0.$$

On the other hand, by conditions (H_2) and (H_3), we have

$$(Tx)(t) \leqslant \int_0^1 G(t,s)k(s)(g(x(s))+h(x(s)))\,ds+\gamma(t)$$
$$\leqslant g(r)\left(1+\frac{h(R)}{g(R)}\right)K^*+\gamma^*\leqslant R.$$

In conclusion, $T(\mathcal{A})\subset\mathcal{A}$. By a direct application of Schauder's fixed point theorem, the proof is complete. □

As an application of Theorem 3.1, we consider the case $\gamma_*=0$. The following corollary is a direct result of Theorem 3.1.

Corollary 3.2. *Suppose that $a(t)$ satisfies* (A) *and $f(t,x)$ satisfies conditions* (H_1) *and* (H_2). *Furthermore, assume that the following condition holds.*

(H_3^*) *There exists a positive constant $R>0$ such that $R>\Phi_{R*}$ and*

$$g(\Phi_{R*})\left(1+\frac{h(R)}{g(R)}\right)K^*+\gamma^*\leqslant R.$$

If $\gamma_*=0$, then equation (1.1) *has at least one positive periodic solution.*

As a particular case of Corollary 3.2, we recover results proved in [**16**, Theorem 2].

Example 3.3. *Suppose that $a(t)$ satisfies* (A) *and $b\succ 0$, $0<\lambda<1$. If $\gamma_*=0$, then equation* (1.2) *has at least one positive periodic solution.*

Proof. We will apply Corollary 3.2. To this end, we take

$$\phi_L(t)=\frac{b(t)}{L^\lambda},\qquad k(t)=b(t),\qquad g(x)=\frac{1}{x^\lambda},\qquad h(x)\equiv 0;$$

then conditions (H_1) and (H_2) are satisfied and the existence condition (H_3^*) becomes

$$\left(\frac{R^\lambda}{\beta_*}\right)^\lambda\beta^*+\gamma^*\leqslant R,\qquad R>\frac{\beta_*}{R^\lambda},\tag{3.3}$$

for some $R>0$. Here

$$\beta(t)=\int_0^1 G(t,s)b(s)\,ds.$$

Note that $\beta_*>0$ as a consequence of (A). Since $0<\lambda<1$, we can choose $R>0$ large enough so that (3.3) is satisfied and the proof is complete. □

Remark 3.4. The validity of Corollary 3.2 under strong force conditions was posed as an open problem in [**16**]. Such an open problem has not yet been solved.

The following result generalizes the previous one when $b(t)\equiv 1$.

Example 3.5. *Let the nonlinearity in* (1.1) *be*

$$f(t,x)=x^{-\alpha}+\mu x^\beta,\tag{3.4}$$

where $0<\alpha<1$, $\beta\geqslant 0$ and $\mu\geqslant 0$ is a non-negative parameter. For each $e(t)$ with $\gamma_*=0$,

(i) *if $\alpha+\beta<1-\alpha^2$, then* (1.1) *has at least one positive periodic solution for each $\mu\geqslant 0$;*

(ii) *if $\alpha+\beta\geqslant 1-\alpha^2$, then* (1.1) *has at least one positive periodic solution for each $0\leqslant\mu<\mu_1$, where μ_1 is some positive constant.*

Proof. We apply Corollary 3.2. To this end, we take

$$\phi_L(t) = L^{-\alpha}, \qquad g(x) = x^{-\alpha}, \qquad h(x) = \mu x^{\beta}, \qquad k(t) = 1.$$

Then (H_1) and (H_2) are satisfied. Let $\omega(t) = \int_0^1 G(t,s)\,ds$. Now the existence condition (H_3^*) becomes

$$\mu < \frac{R^{1-\alpha^2}\omega_*^{\alpha} - \gamma^* \omega_*^{\alpha} R^{-\alpha^2} - \omega^*}{\omega^* R^{\alpha+\beta}}$$

for some $R > 0$ with $R^{1+\alpha} > \omega_*$. So (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu_1 = \sup_{R > \omega_*^{1/1+\alpha}} \frac{R^{1-\alpha^2}\omega_*^{\alpha} - \gamma^* \omega_*^{\alpha} R^{-\alpha^2} - \omega^*}{\omega^* R^{\alpha+\beta}}.$$

Note that $\mu_1 = \infty$ if $\alpha + \beta < 1 - \alpha^2$ and $\mu_* < \infty$ if $\alpha + \beta \geqslant 1 - \alpha^2$. We have the desired results (i) and (ii). □

The next results explore the case when $\gamma_* > 0$.

THEOREM 3.6. *Suppose that $a(t)$ satisfies* (A) *and $f(t,x)$ satisfies* (H_2). *Furthermore, assume that the following condition holds.*

(H_4) *there exists $R > 0$ such that*

$$g(\gamma_*)\left(1 + \frac{h(R)}{g(R)}\right)K^* + \gamma^* \leqslant R.$$

If $\gamma_* > 0$, then equation (1.1) *has at least one positive periodic solution.*

Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. Let R be the positive constant satisfying (H_4) and let $r = \gamma_*$; then $R > r > 0$ since $R > \gamma^*$. Next we prove that $T(\mathcal{A}) \subset \mathcal{A}$.

For each $x \in \mathcal{A}$ and for all $t \in [0,1]$, by the non-negative sign of $G(t,s)$ and $f(t,x)$ we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s) f(s, x(s))\,ds + \gamma(t)\\ &\geqslant \gamma_* = r > 0.\end{aligned}$$

On the other hand, by (H_2) and (H_4), we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)k(s)(g(x(s)) + h(x(s)))\,ds + \gamma(t)\\ &= g(r)\left(1 + \frac{h(R)}{g(R)}\right)K^* + \gamma^*\\ &\leqslant R.\end{aligned}$$

In conclusion, $T(\mathcal{A}) \subset \mathcal{A}$ and the proof is complete, by Schauder's fixed point theorem. □

EXAMPLE 3.7. *Suppose that $a(t)$ satisfies* (A), *and that $b \succ 0$, $\lambda > 0$. If $\gamma_* > 0$, then equation* (1.2) *has at least one positive periodic solution.*

Proof. We apply Theorem 3.6. Take $k(t)$, $g(x)$ and $h(x)$ as in the proof of Example 3.5. Then (H_2) is satisfied and the existence condition (H_4) is satisfied if we take $R > 0$ with

$$R \geqslant \frac{\beta^*}{\gamma_*^{\lambda}} + \gamma^*.$$

□

Example 3.8. *Let the nonlinearity in* (1.1) *be* (3.4) *with* $\alpha > 0$ *and* $\beta \geqslant 0$. *For each* $e(t)$ *with* $\gamma_* > 0$,

(i) *if* $\alpha + \beta < 1$, *then* (1.1) *has at least one positive periodic solution for each* $\mu \geqslant 0$;
(ii) *if* $\alpha + \beta \geqslant 1$, *then* (1.1) *has at least one positive periodic solution for each* $0 \leqslant \mu < \mu_2$, *where* μ_2 *is some positive constant.*

Proof. We apply Theorem 3.6. To this end, we take $g(x)$, $h(x)$, $k(t)$ as in the proof of Example 3.3; then (H_2) is satisfied and the existence condition (H_4) becomes

$$\mu < \frac{R\gamma_*^{\alpha} - \gamma^* \gamma_*^{\alpha} - \omega^*}{\omega^* R^{\beta+\beta}}$$

for some $R > 0$. So (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu_2 = \sup_{R>0} \frac{R\gamma_*^{\alpha} - \gamma^* \gamma_*^{\alpha} - \omega^*}{\omega^* R^{\alpha+\beta}}.$$

Note that $\mu_2 = \infty$ if $\alpha + \beta < 1$ and $\mu_2 < \infty$ if $\alpha + \beta \geqslant 1$. We have the desired results (i) and (ii). □

Remark 3.9. It is easy to find results analogous to Examples 3.3 and 3.8 for the general equation (1.3) with $b, c \succ 0$, but the notation becomes cumbersome. Here we consider only (3.4) for simplicity.

Remark 3.10. By a direct application of Theorem 3.1, we can consider the case that $\gamma_* \leqslant 0$. In this case, the existence condition is satisfied if γ_* and γ^* satisfy some inequalities. For example, we can obtain the same results as in [**16**, Theorem 4]. Here we omit the details and leave them to the reader.

Remark 3.11. We emphasize that in our results e does not need to be positive. It is interesting to compare our results with those of [**9**].

4. *Main results II*

In the main results of the previous section, condition (H_2) implies in particular that the nonlinearity $f(t, x)$ is non-negative for all values (t, x), which is quite a hard restriction. In this section, we show how to avoid this restriction for $\gamma_* > 0$.

Theorem 4.1. *Suppose that* $a(t)$ *satisfies* (A). *Furthermore, assume that the following conditions hold.*

($\tilde{H}_1$) *There exist continuous, non-negative functions* $g(x)$ *and* $k(t)$, *such that*

$$f(t, x) \leqslant k(t)g(x) \qquad \text{for all } (t, x) \in [0, 1] \times (0, \infty),$$

and $g(x) > 0$ *is non-increasing in* $x \in (0, \infty)$.

($\tilde{H}_2$) *Let us define*

$$R := g(\gamma_*)K^* + \gamma^*,$$

and assume that $f(t, x) \geqslant 0$ *for all* $(t, x) \in [0, 1] \times (0, R]$.

If $\gamma_* > 0$, *then equation* (1.1) *has at least one positive periodic solution.*

Proof. We again use Schauder's fixed point theorem. Let R be the positive constant satisfying ($\tilde{H}_2$) and $r = \gamma_*$; then $R > r > 0$ since $R > \gamma^*$. By again using the method of Section 3, it is easy to prove that $T(\mathcal{A}) \subset \mathcal{A}$. We omit the details. □

As an example of the applications of this result, we have selected the equation

$$x'' + a(t)x + \mu x^{\beta} = \frac{1}{x^{\alpha}} + e(t), \tag{4.1}$$

with $a, e \in \mathbb{C}[0,1]$, $\alpha, \beta, \mu > 0$. Such an equation is not covered by the results contained in the references mentioned above.

EXAMPLE 4.2. *Suppose that $a(t)$ satisfies* (A). *If $\gamma_* > 0$, then equation* (4.1) *has at least one positive periodic solution for each $0 \leqslant \mu < \mu_3$, where μ_3 is some positive constant.*

Proof. The nonlinearity is

$$f(t,x) = \frac{1}{x^{\alpha}} - \mu x^{\beta},$$

and therefore ($\tilde{\mathrm{H}}_1$) holds with $k(t) = 1$, $g(x) = 1/x^{\alpha}$. Let

$$\omega(t) = \int_0^1 G(t,s)\,ds.$$

Then R as defined in ($\tilde{\mathrm{H}}_2$) is just $R = w^*/\gamma_*^{\alpha} + \gamma^*$. Note that $f(t,x) \geqslant 0$ if and only if $x^{\alpha+\beta} \leqslant 1/\mu$. Therefore, ($\tilde{\mathrm{H}}_2$) is verified for any $\mu < R^{-\alpha-\beta}$. As a consequence, the result holds for

$$\mu_3 = \left(\frac{w^*}{\gamma_*^{\alpha}} + \gamma^*\right)^{-\alpha-\beta}.$$

□

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