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Periodic solutions of second order non-autonomous singular dynamical systems

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Abstract

In this paper, we establish two different existence results of positive periodic solutions for second order non-autonomous singular dynamical systems. The first one is based on a nonlinear alternative principle of Leray–Schauder and the result is applicable to the case of a strong singularity as well as the case of a weak singularity. The second one is based on Schauder's fixed point theorem and the result sheds some new light on problems with weak singularities and proves that in some situations weak singularities may help create periodic solutions. Recent results in the literature are generalized and significantly improved.

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Keywords: Periodic solutions; Non-autonomous; Dynamical systems; Strong singularity; Weak singularity; Leray–Schauder alternative principle; Schauder's fixed point theorem

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1. Introduction

The main purpose of this paper is to study the existence of positive T-periodic solutions of the second order non-autonomous dynamical system

$$\ddot{x} + a(t)x = f(t, x) + e(t), \tag{1.1}$$

or

$$-\ddot{x} + a(t)x = f(t, x) + e(t), \tag{1.2}$$

where $a(t), e(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$, $f(t, x) \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$. As usual, by a T-periodic positive solution, we mean a function $x(t) = (x_1(t), \dots, x_N(t)) \in \mathbb{C}^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ solving (1.1) (or (1.2)) and such that $x_i(t) > 0$ for all t, $i = 1, 2, \dots, N$. In particular, a T-periodic positive solution is also a non-collision solution, which means that a function $x \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ solving (1.1) (or (1.2)) and such that $x(t) \neq 0$ for all t.

In this paper, we are mainly interested in equations with a singularity in $x = 0$, which means

$$\lim_{x \to 0^+} f_i(t, x) = +\infty \quad \text{uniformly in } t,\ i = 1, 2, \dots, N.$$

Then (1.1) presents a singularity of repulsive type whereas (1.2) has an attractive singularity. Electrostatic or gravitational forces are the most important examples of singular interactions.

The question of existence of collisionless periodic orbits for Lagrangian systems with singularities has attracted much attention of many mathematicians and physicists over many years, such as [1,8,10,21,22,32]. There are two main lines of research in this area. The first one is the variational approach [2,23–25]. In the attractive case, it is necessary some condition on the action functional near the singularity to guarantee that its critical points have no collisions with the singularity. An example is the well-known strong force condition, which was first introduced with this name by Gordon in [12], although the idea goes back at least to Poincaré [19]. This condition has been widely used for avoiding collisions in the attractive case (see [2] and the references therein). For example, if we consider the system

$$\ddot{x} + \nabla_x V(t, x) = f(t) \tag{1.3}$$

with $V(t, x) = -\frac{1}{|x|^\alpha}$, the strong force condition corresponds to the case $\alpha \geqslant 2$. In the repulsive case and dimension higher than 1, the possibility of the solutions to wind around the singularity enables to avoid the strong force condition [23].

Besides the variational approach, topological methods have been widely applied, starting with the pioneering paper of Lazer and Solimini [16]. In particular, the method of upper and lower solutions, degree theory, some fixed point theorems in cones for completely continuous operators and Schauder's fixed point theorem are the most relevant tools [3,10,11,14,27]. Here we remark that, even in the scalar case, the existence of periodic solutions for singular problems has commanded much attention in recent years [4,6,7,9,13,20,31]. Contrasting with the variational setting, the strong force condition plays here a different role linked to repulsive singularities. A counterexample in the paper of Lazer and Solimini [16, Theorem 4.1] shows that a strong force assumption (unboundedness of the potential near the singularity) is necessary in some sense for the existence of positive periodic solutions in the scalar case. Compared with the case of strong singularities, the study of the existence of periodic solutions under the presence of weak singularities by topological methods is more recent and the number of references is much smaller. Here we refer the reader to [5,11,21,28].

This paper is mainly motivated by the recent papers [15,28], in which the scalar periodic singular problems have been studied by Leray–Schauder alternative principle, a well-known fixed point theorem in cones, and Schauder's fixed point theorem, respectively. Some results in [28] prove that in some situations weak singularities may help create periodic solutions. We remark here that there are examples in the literature of techniques that work well for the scalar case without assuming the strong force condition, but need to consider it to deal with higher-dimensional systems, see Remark 1 in [29]. However, as we will see, that is not our case.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by employing a nonlinear alternative principle of Leray–Schauder, we state and prove the first existence result for (1.1) under the positiveness of the Green's function associated with (2.1)–(2.2). The result is applicable to the case of a strong singularity as well as the case of a weak singularity. Analogous results still remain valid for the system with an attractive singularity since the proof relies essentially on the positiveness of the Green's function of the linear part.

In Section 4, by using Schauder's fixed point theorem, we state and prove the second existence result for (1.1), assuming that the Green's function associated with (2.1)–(2.2) is non-negative. Our view point sheds some new light on problems with weak force potentials and we prove that in some situations weak singularities may stimulate the existence of periodic solutions, just as pointed out in [28] for the scalar case.

To illustrate our results, in both Sections 3 and 4, we have selected the system

$$\begin{cases} \ddot{x} + a_1(t)x = \sqrt{\left(x^2+y^2\right)^{-\alpha}} + \mu\sqrt{\left(x^2+y^2\right)^{\beta}} + e_1(t), \\ \ddot{y} + a_2(t)y = \sqrt{\left(x^2+y^2\right)^{-\alpha}} + \mu\sqrt{\left(x^2+y^2\right)^{\beta}} + e_2(t) \end{cases} \tag{1.4}$$

with $a_1, a_2, e_1, e_2 \in \mathbb{C}[0,T]$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ is a given parameter. Here we emphasize that in the new results e_1, e_2 does not need to be positive. Therefore we generalize and improve some results contained in [11,17] and even for the scalar cases in [4,15].

In this paper, we will use the notation $\mathbb{R}_+^N = \{x \in \mathbb{R}^N\colon\ x_i \geqslant 0 \text{ for each } i = 1,2,\dots,N\}$ with the norm $|x| = \max_i |x_i|$. For $x = (x_1,\dots,x_N)$, $y = (y_1,\dots,y_N)$, we write $x \geqslant y$, if $x - y = (x_1 - y_1,\dots,x_N - y_N) \in \mathbb{R}_+^N$. We say that a function $\varphi:\mathbb{R}^N \to \mathbb{R}$ is non-decreasing if $\varphi(x) \geqslant \varphi(y)$ for $x, y \in \mathbb{R}^N$ with $x \geqslant y$. Given $\psi \in L^1[0,T]$, we write $\psi \succ 0$ if $\psi \geqslant 0$ for a.e. $t \in [0,T]$ and it is positive in a set of positive measure. For a given function $p \in L^1[0,T]$, we denote the essential supremum and infimum by p^* and p_*, if they exist. The usual L^p-norm is denoted by $\|\cdot\|_p$. The conjugate exponent of p is denoted by $\tilde{p}$: $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$.

2. Preliminaries

We denote by $a_1, a_2, \dots, a_N$ and $e_1, e_2, \dots, e_N$ the components of given functions $a(t), e(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$, respectively. For each $i = 1,2,\dots,N$, we consider the scalar equation

$$x'' + a_i(t)x = e_i(t) \tag{2.1}$$

with periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{2.2}$$

In Section 3, we assume that the following standing hypothesis is satisfied:

(A) the Green function $G_i(t,s)$, associated with (2.1)–(2.2), is positive for all $(t,s) \in [0,T] \times [0,T]$, $i = 1,2,\ldots,N$.

In Section 4, we assume that

(B) the Green function $G_i(t,s)$, associated with (2.1)–(2.2), is non-negative for all $(t,s) \in [0,T] \times [0,T]$, $i = 1,2,\ldots,N$.

In other words, the (strict) anti-maximum principle holds for (2.1)–(2.2). Under the condition (A) or (B), the solution of (2.1)–(2.2) is given by

$$x(t) = \int_0^T G_i(t,s) e_i(s)\, ds.$$

When $a_i(t) = k^2$, condition (A) is equivalent to $0 < k^2 < \lambda_1 = (\frac{\pi}{T})^2$ and condition (B) is equivalent to $0 < k^2 \leqslant \lambda_1$. Note that λ_1 is the first eigenvalue of the linear problem with Dirichlet conditions $x(0) = x(T) = 0$. In this case, we have

$$G_i(t,s) = \begin{cases} \frac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leqslant s \leqslant t \leqslant T, \\ \frac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leqslant t \leqslant s \leqslant T, \end{cases}$$

and

$$\frac{1}{2k} \cot \frac{kT}{2} \leqslant G_i(t,s) \leqslant \frac{1}{2k \sin \frac{kT}{2}}.$$

See [10,26].

For a non-constant function $a(t)$, there is an L^p-criterion proved in [26], which is given in the following lemma for the sake of completeness. Let $\mathbf{K}(q)$ denote the best Sobolev constant in the following inequality:

$$C \|u\|_q^2 \leqslant \|u'\|_2^2, \quad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \left(\frac{2}{2+q}\right)^{1-2/q} \left(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})}\right)^2 & \text{if } 1 \leqslant q < \infty, \\ \frac{4}{T} & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function. See [30].

Lemma 2.1. *For each $i = 1, 2, \dots, N$, assume that $a_i(t) \succ 0$ and $a_i \in L^p[0, T]$ for some $1 \leqslant p \leqslant \infty$. If*

$$\|a_i\|_p < \mathbf{K}(2\tilde{p}),$$

then the standing hypothesis (A) *holds. Moreover, condition* (B) *holds if*

$$\|a_i\|_p \leqslant \mathbf{K}(2\tilde{p}).$$

Under hypothesis (A), we always denote

$$m_i = \min_{0 \leqslant s,t \leqslant T} G_i(t, s), \qquad M_i = \max_{0 \leqslant s,t \leqslant T} G_i(t, s), \qquad \sigma_i = m_i / M_i. \tag{2.3}$$

Obviously, $M_i > m_i > 0$ and $0 < \sigma_i < 1$.

We define the function $\gamma : \mathbb{R} \to \mathbb{R}^N$ by

$$\gamma_i(t) = \int_0^T G_i(t, s) e_i(s)\, ds, \quad i = 1, 2, \dots, N,$$

which is the unique T-periodic solution of

$$\ddot{x} + a(t)x = e(t).$$

Throughout this paper, we use the following notations

$$\gamma_* = \min_{i,t} \gamma_i(t), \qquad \gamma^* = \max_{i,t} \gamma_i(t).$$

3. Existence result (I)

In this section, we state and prove the first existence result. The proof is based on the following nonlinear alternative of Leray–Schauder, which can be found in [18].

Lemma 3.1. *Assume Ω is a relatively compact subset of a convex set K in a normed space X. Let $T : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds*:

(I) *T has at least one fixed point in $\overline{\Omega}$.*
(II) *There exist $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda T x$.*

In applications below, we take $X = \mathbb{C}[0, T] \times \cdots \times \mathbb{C}[0, T]$ (N copies) and denote by $\|\cdot\|$ the supremum norm of $\mathbb{C}[0, T]$. Define the operator $T : X \to X$ by $Tx = (T_1x, T_2x, \dots, T_Nx)^T$, where

$$(T_i x)(t) = \int_0^T G_i(t, s) f_i\big(s, x(s) + \gamma(s)\big)\, ds, \quad i = 1, 2, \dots, N. \tag{3.1}$$

It is easy to see that finding a fixed point for the operator T is equivalent to finding a T-periodic solution of system (3.2) below.

Theorem 3.1. *Suppose that $a(t)$ satisfies* (A). *Furthermore, we assume that*

(H_1) *for each constant $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that each component f_i of f satisfies $f_i(t,x) \geqslant \phi_L(t)$ for all $t \in [0,T]$ and $x \in [-L, L]$;*

(H_2) *for each component f_i of f, there exist continuous, non-negative functions $g_i(x)$, $h_i(x)$ and $k_i(t)$, such that*

$$0 \leqslant f_i(t,x) \leqslant k_i(t)\bigl\{g_i(x) + h_i(x)\bigr\} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}_+^N \setminus \{0\},$$

and $g_i(x) > 0$ is non-increasing and $h_i(x)/g_i(x)$ is non-decreasing in x;

(H_3) *there exists a positive number $r > 0$ such that*

$$\frac{r}{g_i(\gamma_*, \ldots, \gamma_*, \sigma_i r + \gamma_*, \gamma_*, \ldots, \gamma_*)\{1 + \frac{h_i(r+\gamma^*, \ldots, r+\gamma^*)}{g_i(r+\gamma^*, \ldots, r+\gamma^*)}\}} > K_i^*$$

for all $i = 1, 2, \ldots, N$, here $K_i(t) = \int_0^T G_i(t,s)k_i(s)\,ds$.

If $\gamma_* \geqslant 0$, then (1.1) *has at least one positive T-periodic solution x with $x(t) > \gamma(t)$ for all t and $0 < |x - \gamma| < r$.*

Proof. We first show that

$$\ddot{x} + a(t)x = f\bigl(t, x(t) + \gamma(t)\bigr) \tag{3.2}$$

has a positive T-periodic solution x satisfying $x(t) + \gamma(t) > 0$ for $t \in [0,T]$ and $0 < |x| < r$. If this is true, it is easy to see that $u(t) = x(t) + \gamma(t)$ will be a positive T-periodic solution of (1.1) with $0 < |u - \gamma| < r$ since

$$\ddot{u} + a(t)u = \ddot{x} + \ddot{\gamma} + a(t)x + a(t)\gamma = f(t, x+\gamma) + e(t) = f(t,u) + e(t).$$

Since (H_3) holds, we can choose $n_0 \in \{1, 2, \ldots\}$ such that $\frac{1}{n_0} < \sigma r + \gamma_*$ and

$$K_i^* g_i(\gamma_*, \ldots, \gamma_*, \sigma_i r + \gamma_*, \gamma_*, \ldots, \gamma_*)\left\{1 + \frac{h_i(r+\gamma^*, \ldots, r+\gamma^*)}{g_i(r+\gamma^*, \ldots, r+\gamma^*)}\right\} + \frac{1}{n_0} < r$$

for all $i = 1, 2, \ldots, N$. Here $\sigma = \min\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$.

Let $N_0 = \{n_0, n_0+1, \ldots\}$. Fix $n \in N_0$. Consider the family of systems

$$\ddot{x} + a(t)x = \lambda f^n\bigl(t, x(t) + \gamma(t)\bigr) + \frac{a(t)}{n}, \tag{3.3}$$

where $\lambda \in [0,1]$, and for each $i = 1, 2, \ldots, N$,

$$f_i^n(t,x) = \begin{cases} f_i(t,x) & \text{if } x_i \geqslant \frac{1}{n}, \\ f_i(t, x_1, \ldots, x_{i-1}, \frac{1}{n}, x_{i+1}, \ldots, x_N) & \text{if } x_i \leqslant \frac{1}{n}. \end{cases}$$

Solving of (3.3) is equivalent to the following fixed point problem

$$x_i(t)=\lambda\int_0^T G_i(t,s)f_i^n\big(s,x(s)+\gamma(s)\big)\,ds+\frac{1}{n}=\lambda\big(T_i^n x\big)(t)+\frac{1}{n} \tag{3.4}$$

for each $i=1,2,\ldots,N$.

We claim that any fixed point x of (3.4) for any $\lambda\in[0,1]$ must satisfy $|x|\neq r$. Otherwise, assume that x is a fixed point of (3.4) for some $\lambda\in[0,1]$ such that $|x|=r$. Without loss of generality, we assume that $|x_j|=r$ for some $j=1,2,\ldots,N$. Thus we have

$$\begin{aligned}
x_j(t)-\frac{1}{n}&=\lambda\int_0^T G_j(t,s)f_j^n\big(s,x(s)+\gamma(s)\big)\,ds\\
&\geqslant \lambda m_j\int_0^T f_j^n\big(s,x(s)+\gamma(s)\big)\,ds\\
&=\sigma_j M_j\lambda\int_0^T f_j^n\big(s,x(s)+\gamma(s)\big)\,ds\\
&\geqslant \sigma_j\max_t\left\{\lambda\int_0^T G_j(t,s)f_j^n\big(s,x(s)+\gamma(s)\big)\,ds\right\}\\
&=\sigma_j\left\|x_j-\frac{1}{n}\right\|.
\end{aligned}$$

Hence, for all t, we have

$$x_j(t)\geqslant\sigma_j\left\|x_j-\frac{1}{n}\right\|+\frac{1}{n}\geqslant\sigma_j\left(\|x_j\|-\frac{1}{n}\right)+\frac{1}{n}\geqslant\sigma_j r.$$

Therefore,

$$x_j(t)+\gamma_j(t)\geqslant\sigma_j r+\gamma_*>\frac{1}{n}$$

since $\frac{1}{n}\leqslant\frac{1}{n_0}<\sigma r+\gamma_*$.

Thus we have from condition (H_2), for all t,

$$x_j(t)=\lambda\int_0^T G_j(t,s)f_j^n\big(s,x(s)+\gamma(s)\big)\,ds+\frac{1}{n}$$

$$= \lambda \int_0^T G_j(t,s) f_j\big(s, x(s)+\gamma(s)\big)\,ds + \frac{1}{n}$$

$$\leqslant \int_0^T G_j(t,s) f_j\big(s, x(s)+\gamma(s)\big)\,ds + \frac{1}{n}$$

$$\leqslant \int_0^T G_j(t,s) k_j(s) g_j\big(x(s)+\gamma(s)\big)\left\{1 + \frac{h_j(x(s)+\gamma(s))}{g_j(x(s)+\gamma(s))}\right\} ds + \frac{1}{n}$$

$$\leqslant g_j(\gamma_*, \ldots, \gamma_*, \sigma_j r + \gamma_*, \gamma_*, \ldots, \gamma_*)\left\{1 + \frac{h_j(r+\gamma^*, \ldots, r+\gamma^*)}{g_j(r+\gamma^*, \ldots, r+\gamma^*)}\right\} K_j^* + \frac{1}{n_0}$$

since $x_i(t) \geqslant \frac{1}{n}$ for all $i \in \{1, \ldots, N\} \setminus \{j\}$ and $\gamma_* \geqslant 0$.

Therefore,

$$r = |x_j| \leqslant g_j(\gamma_*, \ldots, \gamma_*, \sigma_j r + \gamma_*, \gamma_*, \ldots, \gamma_*)\left\{1 + \frac{h_j(r+\gamma^*, \ldots, r+\gamma^*)}{g_j(r+\gamma^*, \ldots, r+\gamma^*)}\right\} K_j^* + \frac{1}{n_0}.$$

This is a contradiction to the choice of n_0 and the claim is proved.

From this claim, Lemma 3.1 guarantees that

$$x(t) = \big(T^n x\big)(t) + \frac{1}{n} \tag{3.5}$$

has a fixed point, denoted by x^n, in $B_r = \{x \in X\colon\ |x| < r\}$, i.e.,

$$\ddot{x} + a(t)x = f^n\big(t, x(t)+\gamma(t)\big) + \frac{a(t)}{n} \tag{3.6}$$

has a T-periodic solution x^n with $|x^n| < r$. Since $x_i^n(t) \geqslant \frac{1}{n} > 0$ for all $i = 1, \ldots, N$ and $t \in [0, T]$, x^n is actually a positive T-periodic solution of (3.6).

Next we claim that $x^n(t) + \gamma(t)$ have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$\min_{i,t}\big\{x_i^n(t) + \gamma_i(t)\big\} \geqslant \delta \tag{3.7}$$

for all $n \in N_0$. Since (H_1) holds, there exists a continuous function $\phi_{r+\gamma^*}(t) \succ 0$ such that each component f_i of f satisfies $f_i(t,x) \geqslant \phi_{r+\gamma^*}(t)$ for all t and $|x| \leqslant r + \gamma^*$. Let $x^{r+\gamma^*}(t)$ be the unique T-periodic solution to

$$\ddot{x} + a(t)x = \Phi(t)$$

with $\Phi(t) = (\phi_{r+\gamma^*}(t), \ldots, \phi_{r+\gamma^*}(t))^T$, then we have

$$x_i^{r+\gamma^*}(t) + \gamma_i(t) = \int_0^T G_i(t,s)\phi_{r+\gamma^*}(s)\,ds + \gamma_i(t) \geqslant \Phi_* + \gamma_* > 0$$

for each $i = 1, \ldots, N$, here

$$\Phi_* = \min_{i,t} \Phi_i(t), \qquad \Phi_i(t) = \int_0^T G_i(t,s)\phi_{r+\gamma^*}(s)\,ds.$$

Next we show that (3.7) holds for $\delta = \Phi_* + \gamma_* > 0$. To see this, for each $i = 1, \ldots, N$, since $x_i^n(t) + \gamma_i(t) \leqslant r + \gamma^*$ and $x_i^n(t) + \gamma_* \geqslant \frac{1}{n}$, we have

$$\begin{aligned} x_i^n(t) + \gamma_i(t) &= \int_0^T G_i(t,s) f_i^n\big(s, x^n(s) + \gamma(s)\big)\,ds + \gamma_i(t) + \frac{1}{n} \\ &\geqslant \int_0^T G_i(t,s)\phi_{r+\gamma_*}(s)\,ds + \gamma_i(t) \\ &= \int_0^T G_i(t,s)\phi_{r+\gamma_*}(s)\,ds + \gamma_i(t) \\ &\geqslant \Phi_* + \gamma_* = \delta. \end{aligned}$$

In order to pass the solutions x^n of the truncation systems (3.6) to that of the original system (3.2), we need the following fact

$$\big|\dot{x}^n\big| \leqslant H \tag{3.8}$$

for some constant $H > 0$ and for all $n \geqslant n_0$. To this end, by the periodic boundary conditions, $\dot{x}^n(t_0) = 0$ for some $t_0 \in [0, T]$. Integrating (3.6) from 0 to T, we obtain

$$\int_0^T a(t)x^n(t)\,dt = \int_0^T \left[f^n\big(t, x^n(t) + \gamma(t)\big) + \frac{a(t)}{n} \right] dt.$$

Therefore, for each $i = 1, \ldots, N$,

$$\begin{aligned} \big\|\dot{x}_i^n\big\| &= \max_t \left| \int_{t_0}^t \ddot{x}_i^n(s)\,ds \right| \\ &= \max_t \left| \int_{t_0}^t \left[f_i^n\big(s, x_n(s) + \gamma(s)\big) + \frac{a_i(s)}{n} - a_i(s)x_i^n(s) \right] ds \right| \end{aligned}$$

$$\leqslant \int_0^T \left[f_i^n\big(s, x_n(s) + \gamma(s)\big) + \frac{a_i(s)}{n} \right] ds + \int_0^T a_i(s) x_i^n(s)\, ds$$

$$= 2 \int_0^T a_i(s) x_i^n(s)\, ds < 2r|a_i|_1 = H_i,$$

here $|a_i|_1 = \max_i |\int_0^T a_i(s)\, ds|$. Then (3.8) is satisfied for $H = \max_i\{H_i\}$.

The fact $|x^n| < r$ and (3.8) show that for each $i = 1, 2, \ldots, N$, $\{x_i^n\}_{n \in N_0}$ is a bounded and equi-continuous family on $[0, T]$. Now the Arzela–Ascoli Theorem guarantees that $\{x_i^n\}_{n \in N_0}$ has a subsequence, $\{x_i^{n_k}\}_{k \in \mathbb{N}}$, converging uniformly on $[0, T]$ to a function $x_i \in \mathbb{C}[0, T]$. Let $x = (x_1, \ldots, x_N)$, from the fact $|x_n| < r$ and (3.7), x satisfies $\delta \leqslant x_i(t) + \gamma_i(t) \leqslant r + \gamma^*$ for all t and $i = 1, \ldots, N$. Moreover, $x_i^{n_k}$ satisfies the integral equation

$$x_i^{n_k}(t) = \int_0^T G_i(t, s) f_i\big(s, x^{n_k}(s) + \gamma(s)\big)\, ds + \frac{1}{n_k}, \quad i = 1, \ldots, N.$$

Letting $k \to \infty$, we arrive at

$$x_i(t) = \int_0^T G_i(t, s) f_i\big(s, x(s) + \gamma(s)\big)\, ds, \quad i = 1, \ldots, N,$$

where the uniform continuity of $f_i(t, x)$ on $[0, T] \times [\delta, r + \gamma^*]$ is used. Therefore, x is a positive periodic solution of (3.2) and satisfies $0 < |x| \leqslant r$.

Finally it is not difficult to show that $|x| < r$, by noting that if $|x| = r$, the argument similar to the proof of the first claim will yield a contradiction. □

Corollary 3.1. *Suppose that $a_1(t)$, $a_2(t)$ satisfy* (A) *and $\alpha > 0$, $\beta \geqslant 0$, then for each $e_1(t), e_2(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* \geqslant 0$, we have*

(i) *if $\beta < 1$, then* (1.4) *has at least one positive T-periodic solution for each $\mu > 0$;*
(ii) *if $\beta \geqslant 1$, then* (1.4) *has at least one positive T-periodic solution for each $0 < \mu < \mu_1$, where μ_1 is some positive constant.*

Proof. We will apply Theorem 3.1. To this end, the assumption (H_1) is fulfilled by $\phi_L(t) = (\sqrt{2}L)^{-\alpha}$. If we take

$$g_1(x, y) = g_2(x, y) = \sqrt{\big(x^2 + y^2\big)^{-\alpha}}, \qquad h_1(x, y) = h_2(x, y) = \mu\sqrt{\big(x^2 + y^2\big)^{\beta}}$$

and $k_1(t) = k_2(t) = 1$, then (H_2) is satisfied. Let

$$\omega_1(t) = \int_0^T G_1(t, s)\, ds, \qquad \omega_2(t) = \int_0^T G_2(t, s)\, ds.$$

Now the existence condition (H_3) becomes

$$\mu < \frac{r[(\sigma_i r + \gamma_*)^2 + \gamma_*^2]^{\frac{\alpha}{2}} - \omega_i^*}{2^{\frac{\alpha+\beta}{2}}(r + \gamma^*)^{\alpha+\beta}}, \quad i = 1, 2,$$

for some $r > 0$. So (1.4) has at least one positive T-periodic solution for

$$0 < \mu < \mu_1 := \min_{i=1,2} \sup_{r>0} \frac{r[(\sigma_i r + \gamma_*)^2 + \gamma_*^2]^{\frac{\alpha}{2}} - \omega_i^*}{2^{\frac{\alpha+\beta}{2}}(r + \gamma^*)^{\alpha+\beta}}.$$

Note that $\mu_1 = \infty$ if $\beta < 1$ and $\mu_1 < \infty$ if $\beta \geqslant 1$. We have the desired results (i) and (ii). □

Remark 3.1. We emphasize that our results are applicable to the case of a strong singularity as well as the case of a weak singularity since we only need $\alpha > 0$, and that e does not need to be positive. Therefore the new results generalize and improve those in [10,11] and those in [15] even for the scalar cases.

Corollary 3.2. *Suppose that $a(t)$ satisfies* (A). *Assume further that there exist continuous functions $b, \hat{b} \succ 0$ and $\alpha > 0$, $0 \leqslant \beta < 1$ such that each component f_i of f satisfies*

$$\text{(F)} \quad 0 \leqslant \frac{\hat{b}(t)}{|x|^\alpha} \leqslant f_i(t,x) \leqslant \frac{b(t)}{|x|^\alpha} + b(t)|x|^\beta, \quad \textit{for all } t.$$

If $\gamma_* \geqslant 0$, then (1.1) *has at least one positive T-periodic solution.*

Proof. We will apply Theorem 3.1. To this end, we take

$$\phi_L(t) = \frac{\hat{b}(t)}{L^\alpha}, \qquad k_i(t) = b(t), \qquad g_i(x) = \frac{1}{|x|^\alpha}, \qquad h_i(x) = |x|^\beta.$$

Then (H_1) and (H_2) are satisfied and the existence condition (H_3) becomes

$$\frac{r(\sigma_i r + \gamma_*)^\alpha}{1 + (r + \gamma^*)^{\alpha+\beta}} > \beta_i^*, \quad i = 1, \ldots, N, \tag{3.9}$$

for some $r > 0$, here

$$\beta_i(t) = \int_0^T G_i(t,s)b(s)\,ds.$$

Since $\alpha > 0$, $0 \leqslant \beta < 1$ and $\gamma_* \geqslant 0$, we can choose $r > 0$ large enough such that (3.9) is satisfied. □

Remark 3.2. At the cost of a more involved notation, the assumption (H_1) in Theorem 3.1 can be generalized by considering the function ϕ_L is different for each component of f. Consequently, in Corollary 3.2 the functions $b, \hat{b} \succ 0$ can be different for each i. In this way, new cases are covered like the possibility of adding some weights in the singular part of the model system

$$\begin{cases} \ddot{x} + a_1(t)x = b_1(t)\sqrt{(x^2+y^2)^{-\alpha}} + \mu\sqrt{(x^2+y^2)^{\beta}} + e_1(t), \\ \ddot{y} + a_2(t)y = b_2(t)\sqrt{(x^2+y^2)^{-\alpha}} + \mu\sqrt{(x^2+y^2)^{\beta}} + e_2(t) \end{cases} \tag{3.10}$$

with $b_i \succ 0$.

Remark 3.3. In the proof of Theorem 3.1, the positiveness of Green's function $G(t,s)$ plays an important role, and then it is not applicable to the critical case, such as $k = \mu_1$ for the case $a(t) = k^2$. The validity of our results for the critical case remains still open to the authors. In Section 4, we will state a different existence result, which can deal with the critical case. However, it can only cover the case of a weak singularity.

Finally in this section, we consider the system with an attractive singularity (1.2). Recall that $a_1, a_2, \dots, a_N$ are the components of $a(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$. If $a_i(t) \succ 0$, it is well known that the linear equation

$$-x'' + a_i(t)x = e_i(t)$$

with periodic boundary conditions has a positive Green's function (see for instance [26]), in other words, the standing hypothesis (A) holds. Then, the problem of finding a T-periodic solution of system (1.2) is expressed as a fixed point problem for the same operator defined in (3.1). This means that all the results obtained in Section 3 are automatically valid for the system (1.2). For instance, the counterpart of Theorem 3.1 for the attractive case is as follows.

Theorem 3.2. *Suppose that $a_i(t) \succ 0$ for $i = 1, \dots, N$ and that assumptions* (H$_1$)–(H$_3$) *hold. If $\gamma_* \geqslant 0$, then* (1.2) *has at least one positive T-periodic solution x with $x(t) > \gamma(t)$ for all t and $0 < |x - \gamma| < r$.*

4. Existence result (II)

In this section, we establish the second existence result for (1.1) by using Schauder's fixed point theorem.

Theorem 4.1. *Suppose that $a(t)$ satisfies* (B) *and $f(t,x)$ satisfies* (H$_1$)–(H$_2$). *Furthermore, assume that*

(G$_1$) *there exists a positive constant $R > 0$ such that $R > \Phi_*$, $\Phi_* + \gamma_* > 0$ and, for each $i = 1, \dots, N$,*

$$R \geqslant g_i(\Phi_* + \gamma_*, \dots, \Phi_* + \gamma_*)\left\{1 + \frac{h_i(R+\gamma^*, \dots, R+\gamma^*)}{g_i(R+\gamma^*, \dots, R+\gamma^*)}\right\}K_i^*,$$

here $\Phi_* = \min_{i,t} \Phi_i(t)$, $\Phi_i(t) = \int_0^T G_i(t,s)\phi_{R+\gamma^*}(s)\,ds$.

Then (1.1) *has at least one positive T-periodic solution.*

Proof. A T-periodic solution of (1.1) is just a fixed point of the map $T: X \to X$ defined by (3.1). Note that T is a completely continuous map.

Let R be the positive constant satisfying (G_1) and $r = \Phi_* > 0$. Then we have $R > r > 0$. Now we define the set

$$\Omega = \left\{x \in X:\ r \leqslant x_i(t) \leqslant R \text{ for all } t,\ i = 1, \ldots, N\right\}. \tag{4.1}$$

Obviously, Ω is a closed convex set. Next we prove $T(\Omega) \subset \Omega$.

In fact, for each $x \in \Omega$ and for each $i = 1, \ldots, N$, using that $G_i(t,s) \geqslant 0$ and condition (H_1),

$$(T_i x)(t) \geqslant \int_0^T G_i(t,s)\phi_{R+\gamma^*}(s)\,ds \geqslant \Phi_* = r > 0.$$

On the other hand, by conditions (H_2) and (G_1), we have

$$\begin{aligned}(T_i x)(t) &\leqslant \int_0^T G_i(t,s)k_i(s)g_i\big(x(s)+\gamma(s)\big)\left\{1 + \frac{h_i(x(s)+\gamma(s))}{g_i(x(s)+\gamma(s))}\right\} ds \\ &\leqslant g_i(\Phi_* + \gamma_*, \ldots, \Phi_* + \gamma_*)\left\{1 + \frac{h_i(R+\gamma^*, \ldots, R+\gamma^*)}{g_i(R+\gamma^*, \ldots, R+\gamma^*)}\right\} K_i^* \leqslant R.\end{aligned}$$

In conclusion, $T(\Omega) \subset \Omega$. By a direct application of Schauder's fixed point theorem, the proof is finished. □

As an application of Theorem 4.1, we consider the case $\gamma_* = 0$. The following corollary is a direct result of Theorem 4.1.

Corollary 4.1. *Suppose that $a(t)$ satisfies* (B) *and $f(t,x)$ satisfies conditions* (H_1) *and* (H_2). *Furthermore, assume that*

(G_1^*) *there exists a positive constant $R > 0$ such that $R > \Phi_*$ and for each $i = 1, \ldots, N$,*

$$R \geqslant g_i(\Phi_*, \ldots, \Phi_*)\left\{1 + \frac{h_i(R+\gamma^*, \ldots, R+\gamma^*)}{g_i(R+\gamma^*, \ldots, R+\gamma^*)}\right\} K_i^*.$$

If $\gamma_* = 0$, then (1.1) *has at least one positive T-periodic solution.*

Corollary 4.2. *Suppose that $a_1(t)$, $a_2(t)$ satisfy* (B) *and $0 < \alpha < 1$, $\beta \geqslant 0$, then for each $e_1(t), e_2(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* = 0$, we have*

(i) *if $\alpha + \beta < 1 - \alpha^2$, then* (1.4) *has at least one positive periodic solution for each $\mu \geqslant 0$;*
(ii) *if $\alpha + \beta \geqslant 1 - \alpha^2$, then* (1.4) *has at least one positive T-periodic solution for each $0 \leqslant \mu < \mu_2$, where μ_2 is some positive constant.*

Proof. We apply Corollary 4.1 and follow the same notation as in the proof of Corollary 3.1. Then (H_1) and (H_2) are satisfied and the existence condition (G_1^*) becomes

$$\mu < \frac{2^{\frac{\alpha}{2}} R\Phi_*^{\alpha} - \omega_i^*}{2^{\frac{\alpha+\beta}{2}} \omega_i^* (R+\gamma^*)^{\alpha+\beta}}, \quad i = 1, 2, \tag{4.2}$$

for some $R > 0$ with $R > \Phi_*$. Note that

$$\Phi_* = 2^{-\frac{\alpha}{2}} (R+\gamma^*)^{-\alpha} \omega_*,$$

here $\omega_* = \min_{i=1,2}\{\omega_{i*}\}$. Therefore, (4.2) becomes

$$\mu < \frac{2^{\frac{\alpha-\alpha^2}{2}} R(R+\gamma^*)^{-\alpha^2} \omega_*^{\alpha} - \omega_i^*}{2^{\frac{\alpha+\beta}{2}} \omega_i^* (R+\gamma^*)^{\alpha+\beta}}, \quad i = 1, 2,$$

for some $R > 0$.

So (1.4) has at least one positive T-periodic solution for

$$0 < \mu < \mu_2 = \min_{i=1,2} \sup_{R>0} \frac{2^{\frac{\alpha-\alpha^2}{2}} R(R+\gamma^*)^{-\alpha^2} \omega_*^{\alpha} - \omega_i^*}{2^{\frac{\alpha+\beta}{2}} \omega_i^* (R+\gamma^*)^{\alpha+\beta}}.$$

Note that $\mu_2 = \infty$ if $\alpha + \beta < 1 - \alpha^2$ and $\mu_2 < \infty$ if $\alpha + \beta \geqslant 1 - \alpha^2$. We have the desired results (i) and (ii). □

The next results explore the case when $\gamma_* > 0$.

Theorem 4.2. *Suppose that $a(t)$ satisfies* (B) *and $f(t,x)$ satisfies condition* (H_2). *Furthermore, assume that*

(G_2) *there exists $R > \gamma^*$ such that, for all $i = 1, \dots, N$,*

$$g_i(\gamma_*, \dots, \gamma_*)\left\{1 + \frac{h_i(R+\gamma^*, \dots, R+\gamma^*)}{g_i(R+\gamma^*, \dots, R+\gamma^*)}\right\} K_i^* \leqslant R.$$

If $\gamma_* > 0$, then (1.4) *has at least one positive T-periodic solution.*

Proof. We follow the same strategy and notation as in the proof of Theorem 4.1. Let R be the positive constant satisfying (G_2) and $r = \gamma_*$, then $R > r > 0$ since $R > \gamma^*$. Next we prove $T(\Omega) \subset \Omega$.

For each $x \in \Omega$ and for each $i = 1, \dots, N$, by the non-negative sign of $G_i(t,s)$ and $f_i(t,x)$, we have

$$(T_i x)(t) = \int_0^T G_i(t,s) f_i\bigl(s, x(s)\bigr)\, ds + \gamma_i(t) \geqslant \gamma_* = r > 0.$$

On the other hand, by (H_2) and (G_2), we have

$$(T_i x)(t) \leqslant \int_0^T G_i(t,s)k_i(s)g_i\big(x(s)+\gamma(s)\big)\left\{1+\frac{h_i(x(s)+\gamma(s))}{g_i(x(s)+\gamma(s))}\right\}ds$$
$$\leqslant g_i(\gamma_*,\dots,\gamma_*)\left\{1+\frac{h_i(R+\gamma^*,\dots,R+\gamma^*)}{g_i(R+\gamma^*,\dots,R+\gamma^*)}\right\}K_i^* \leqslant R.$$

In conclusion, $T(\Omega) \subset \Omega$ and the proof is finished by Schauder's fixed point theorem. □

Corollary 4.3. *Suppose that $a_1(t)$, $a_2(t)$ satisfy* (B) *and $\alpha, \beta \geqslant 0$, then for each $e_1(t), e_2(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* > 0$, we have*

(i) *if $\alpha+\beta<1$, then* (1.4) *has at least one positive T-periodic solution for each $\mu \geqslant 0$*;
(ii) *if $\alpha+\beta \geqslant 1$, then* (1.4) *has at least one positive T-periodic solution for each $0 \leqslant \mu < \mu_3$, where μ_3 is some positive constant.*

Proof. We apply Theorem 4.2 and follow the same notation as in the proof of Corollary 3.1. Then (H_1) and (H_2) are satisfied and the existence condition (G_2) becomes

$$\mu < \frac{2^{\frac{\alpha}{2}}R\gamma^{*\alpha}-\omega_i^*}{2^{\frac{\alpha+\beta}{2}}\omega_i^*(R+\gamma^*)^{\alpha+\beta}}, \quad i=1,2,$$

for some $R>0$. So (1.4) has at least one positive T-periodic solution for

$$0<\mu<\mu_3=\min_{i=1,2}\sup_{R>0}\frac{2^{\frac{\alpha}{2}}R\gamma^{*\alpha}-\omega_i^*}{2^{\frac{\alpha+\beta}{2}}\omega_i^*(R+\gamma^*)^{\alpha+\beta}}.$$

Note that $\mu_3=\infty$ if $\alpha+\beta<1$ and $\mu_3<\infty$ if $\alpha+\beta \geqslant 1$. We have the desired results (i) and (ii). □

Corollary 4.4. *Suppose that $a(t)$ satisfies* (B) *and $f(t,x)$ satisfies* (F) *with $\beta=0$. Then we have*

(i) *if $\alpha>0$ and $\gamma_*>0$, then* (1.1) *has at least one positive T-periodic solution*;
(ii) *if $0<\alpha<1$ and $\gamma_*=0$, then* (1.1) *has at least one positive T-periodic solution.*

Proof. We will apply Theorem 4.2 and Corollary 4.1. To this end, we take

$$\phi_L(t)=\frac{\hat{b}(t)}{L^\alpha}, \qquad k_i(t)=b(t), \qquad g_i(x)=\frac{1}{|x|^\alpha}, \qquad h_i(x)=0.$$

Then (H_1) and (H_2) are satisfied.

If $\gamma_*>0$, then the existence condition (G_2) becomes

$$R\gamma_*^\alpha \geqslant \beta_i^*, \quad i=1,\dots,N, \tag{4.3}$$

for some $R>0$. This is clear since $\alpha>0$ and $\gamma_*>0$, and thus we have the desired result (i).

If $\gamma_* = 0$, then the existence condition (G_1^*) becomes

$$R \geqslant \Phi_*^{-\alpha} \beta_i^*, \qquad \Phi_* = \frac{\omega_*}{(R+\gamma^*)^\alpha}. \tag{4.4}$$

Note that (4.4) is equivalent to

$$R \geqslant \frac{(R+\gamma^*)^{\alpha^2} \beta_i^*}{\omega_*^\alpha}, \quad i = 1, \dots, N. \tag{4.5}$$

Since $0 < \alpha < 1$, we can choose $R > 0$ large enough such that (4.5) is satisfied. So we have the desired result (ii). □

Remark 4.1. The validity of (ii) in Corollary 4.4 under strong force conditions remains still open to us. Such an open problem has been partially solved by Corollary 3.2. However, we do not solve it completely because we need the positivity of $G(t,s)$ in Corollary 3.2, and therefore it is not applicable to the critical case. The validity for the critical case remains open to the authors.

Remark 4.2. By employing Theorem 4.1 directly, we can deal with the case $\gamma_* < 0$, which is not covered in Section 3. In particular, we can get the same result as Theorem 3.1 in [11]. Here we omit it.

Remark 4.3. By using the same techniques in this paper, we can deal with (1.2) with the potential such as

$$V(t,x) = a(t)\frac{|x|^2}{2} - g\left(t, \frac{|x|^2}{2}\right)$$

with a T-periodic dependence on t and such that g presents a singularity of repulsive type.

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