EXISTENCE OF PERIODIC SOLUTIONS FOR FUNCTIONAL EQUATIONS WITH PERIODIC DELAY

Daniel Franco[1], Eduardo Liz[2] and Pedro J. Torres[3]

[1] Departamento de Matemática Aplicada
Universidad Nacional de Educación a Distancia
Apartado de Correos 60149. Madrid 28080, Spain

[2] Departamento de Matemática Aplicada II
E.T.S.I. Telecomunicación, Universidad de Vigo
Campus Marcosende, 36280 Vigo, Spain

[3] Departamento de Matemática Aplicada
Facultad de Ciencias, Universidad de Granada
18071 Granada, Spain

Abstract. We deal with the existence of positive periodic solutions for functional differential equations with periodic delay which appear in population models. Our technique is based on a fixed point theorem on conical shells. We improve recent results in the literature.

1. Introduction and main result

The study of the existence of periodic solutions in delay differential equations was traditionally motivated by the observance of periodic phenomena in population models. For autonomous equations, one can find many results about periodic solutions in the literature, by using different methods (see, for example, [2, 10]). However, sometimes it is more realistic to consider that the parameters involved in the model (including the delay parameter) are periodic rather than constant (see, e.g., [4]). Recently, Krasnoselskii–type theorems have become an effective tool in proving existence of periodic solutions in delay differential equations with periodic coefficients [1, 8, 14, 15]. In this paper, we improve some results from the above cited references by using a new fixed point theorem for an integral operator defined in a Banach space. To prove this abstract result, we follow the ideas introduced by Lan in [11] (see also [12] and references therein). Our approach allows us to deal easily with different models recently investigated in the literature. First, let us consider the equation

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$$u'(t) = -a(t)u(t) + \lambda h(t) f(u(t - \tau(t)))\,, \tag{1}$$

where $\lambda > 0$, $a, h \in C(\mathbb{R}, [0, \infty))$ are ω–periodic functions with $\int_0^\omega a(s)\,ds > 0$, $\int_0^\omega h(s)\,ds > 0$, $\tau \in C(\mathbb{R}, \mathbb{R})$ is ω–periodic, and $f \in C([0, \infty), [0, \infty))$ is positive for $u > 0$.

Equation (1) with constant coefficients was widely studied due to its applications in many fields including population dynamics, neurophysiology, metabolic regulation, and agricultural commodity markets (see, e.g., [7, p. 78]).

Cheng and Zhang [1] have obtained some existence results for Equation (1) depending on the limits

$$f_0 = \lim_{x \to 0} \frac{f(x)}{x} \quad , \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}, \tag{2}$$

where $f_0, f_\infty \in [0, \infty]$. In addition, similar results were stated in [1] for equation

$$u'(t) = a(t)u(t) - \lambda h(t) f(u(t - \tau(t)))\,, \tag{3}$$

under the same assumptions made for (1).

A more detailed study can be found in the recent paper by Wang [14] , where the following slightly more general form of (3) was investigated:

$$u'(t) = a(t)g(u(t))u(t) - \lambda h(t) f(u(t - \tau(t)))\,, \tag{4}$$

where $g \in C([0, \infty), (0, \infty))$, and there exist constants $l, L > 0$ such that $g(u) \in [l, L)$ for all $u \geq 0$.

In Section 2, we show how our method allows to improve the range of values of the parameter λ (depending on the functions a, g, h and the limits f_0, f_∞ defined in (2)) for which there exists at least a positive periodic solution for equations (1) and (4), respectively. The proofs of these new existence results are easily derived from an abstract result, stated below as Theorem 1, which gives sufficient conditions to guarantee the existence of a positive fixed point for an integral operator from a Banach space into itself. This abstract result is based on a well known fixed point theorem for compact maps on conical shells, and, for convenience of the reader, its proof is presented in Section 3.

Before stating the above mentioned Theorem 1, we need to introduce some notation and definitions.

Let $E_\omega = \{u \in C(\mathbb{R}, \mathbb{R}) : u(t) = u(t + \omega)\}$ be the vectorial space of the ω-periodic continuous functions, which is a Banach space with the norm

$$\|u\| = \sup_{t \in [0, \omega]} |u(t)|.$$

We define the integral operator $S\colon E_\omega \to E_\omega$ by

$$[Su](t) = \int_t^{t+\omega} k(t, s) F(s, u(s - \tau(s)))\,ds, \quad t \in \mathbb{R}, \tag{5}$$

where $\tau\colon \mathbb{R} \to \mathbb{R}$ is a continuous ω-periodic function, $F\colon \mathbb{R}^2 \to [0,\infty)$ is continuous and ω-periodic in the first variable, and $k\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying

$$k(t+\omega, s+\omega) = k(t,s) \ \text{ for all } \ (t,s) \in \mathbb{R}^2. \tag{6}$$

Now, we are in a position to enunciate the announced fixed point theorem for S.

Theorem 1. *Assume that*

(D1) *there exist a constant $c \in (0,1]$ and a continuous function $\Phi\colon [0,\omega] \to [0,\infty)$ such that*

$$c\Phi(s) \le k(t,s) \le \Phi(s) \quad \textit{for all } (t,s) \in [0,\omega] \times [0,\omega];$$

(D2) *there exist a constant $\alpha > 0$ and a function $\phi\colon \mathbb{R} \to (0,\infty)$, continuous and ω-periodic, such that*

$$F(t,v) \ge c\alpha\phi(t) \quad \textit{for all } (t,v) \in [0,\omega] \times [c\alpha, \alpha],$$

and

$$\inf_{t\in[0,\omega]} \int_0^\omega k(t,s)\phi(s)\,ds \ge 1;$$

(D3) *there exist a constant $\beta > 0$ and a function $\psi\colon \mathbb{R} \to (0,\infty)$, continuous and ω–periodic, such that*

$$F(t,v) \le \beta\psi(t) \quad \textit{for all } (t,v) \in [0,\omega] \times [c\beta, \beta],$$

and

$$\sup_{t\in[0,\omega]} \int_0^\omega k(t,s)\psi(s)\,ds \le 1.$$

Then, the following results hold:

(a) *if $\beta < c\alpha$, then the operator S has a positive fixed point u which satisfies*

$$\beta \le \|u\| \le \alpha \ \ \textit{and} \ \ c\beta \le \min_{t\in[0,\omega]} u(t) \le c\alpha\,;$$

(b) *if $\alpha < \beta$, then the operator S has a positive fixed point u which satisfies*

$$c\alpha \le \|u\| \le \beta \ \ \textit{and} \ \ c\alpha \le \min_{t\in[0,\omega]} u(t).$$

Remark 1. From the proof of Theorem 1 (see Section 3), it is clear that its conclusions remain valid if the kernel k in (5) depends on u, if we assume that $k = k(u)$ satisfies (6), (D1), (D2) and (D3) with the same constants c, α, β and functions Φ, ϕ, ψ for all $u \in E_\omega$. This remark will be useful to apply Theorem 1 to equation (4).

2. Population models with periodic delays

This section is devoted to obtain some applications of Theorem 1 to the existence of positive periodic solutions of equations (1) and (4).

Define $F(t,u) = \lambda h(t) f(u)$. Hence, (1) can be written as

$$u'(t) = -a(t)u(t) + F(t, u(t-\tau(t)))\,. \tag{7}$$

Next, (see [1] for details), u is a ω–periodic solution of (7) if and only if u is a fixed point of the integral operator $S\colon E_\omega \to E_\omega$ defined by

$$[Sx](t) = \int_t^{t+\omega} G(t,s)F(s, u(s-\tau(s)))\,ds\,,\ t \in \mathbb{R},$$

where

$$G(t,s) = \frac{e^{\int_t^s a(r)\,dr}}{\sigma - 1}\ ,\ \sigma = e^{\int_0^\omega a(r)\,dr} > 1.$$

Moreover,

$$\frac{1}{\sigma-1} \le G(t,s) \le \frac{\sigma}{\sigma-1}\,.$$

Hence, (D1) holds with $\Phi = \sigma/(\sigma-1)$, $c = \sigma^{-1} \in (0,1)$.

Denote

$$A = \max_{t\in[0,\omega]} \int_0^\omega G(t,s)h(s)\,ds \quad , \quad B = \min_{t\in[0,\omega]} \int_0^\omega G(t,s)h(s)\,ds\,.$$

We have the following consequence of Theorem 1, which improves Theorem 2.5 in [1]

Theorem 2. *Assume that $f_0, f_\infty \in (0,\infty)$, and*

$$\frac{1}{B\max\{f_0, f_\infty\}} < \lambda < \frac{1}{A\min\{f_0, f_\infty\}}. \tag{8}$$

Then equation (1) has a positive ω–periodic solution.

Proof: We assume that $f_0 < f_\infty$. The case $f_0 > f_\infty$ is similarly addressed by using part (b) in Theorem 1.

Let $\varepsilon > 0$ be such that

$$\frac{1}{B(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{A(f_0+\varepsilon)}. \tag{9}$$

By the definition of f_∞, there exists $\alpha > 0$ such that $f(u) \ge (f_\infty - \varepsilon)u$ for all $u \ge c\alpha$. Choosing $\phi(t) = \lambda h(t)(f_\infty - \varepsilon)$, hypothesis (D2) clearly holds. Analogously, there exists $\beta \in (0, c\alpha)$ such that $f(u) \le (f_0+\varepsilon)u$ for all $u \le \beta$. Hence, (D3) is satisfied with $\psi(t) = \lambda h(t)(f_0+\varepsilon)$. The result now follows from Theorem 1 (a). □

Now we consider equation (4). As it is shown in [14], $u \in E_\omega$ is a ω–periodic solution of (4) if and only if u is a fixed point of the integral operator $\tilde{S}\colon E_\omega \to E_\omega$ defined by

$$[\tilde{S}x](t) = \int_t^{t+\omega} G_u(t,s)F(s, u(s-\tau(s)))\,ds\,,\ t \in \mathbb{R},$$

where

$$G_u(t,s) = \frac{e^{-\int_t^s a(r)g(u(r))\,dr}}{1 - e^{-\int_0^\omega a(r)g(u(r))\,dr}}\,,$$

and $F(t,u) = \lambda h(t) f(u)$. Let $\sigma = e^{-\int_0^\omega a(r)\,dr} \in (0,1)$. Since

$$\frac{\sigma^L}{1-\sigma^L} \le G_u(t,s) \le \frac{1}{1-\sigma^l}\,,$$

for all $u \in E_\omega$, condition (D1) holds with $c = \sigma^L(1-\sigma^l)(1-\sigma^L)^{-1} \in (0,1)$.

Repeating the same arguments used in the proof of Theorem 2, and using Remark 1, we get the following result, which improves Theorem 1.3 in [14]

Theorem 3. *Assume that $f_0, f_\infty \in (0,\infty)$, and*

$$\frac{1-\sigma^L}{\sigma^L \int_0^\omega h(s)ds}\frac{1}{\max\{f_0, f_\infty\}} < \lambda < \frac{1-\sigma^l}{\int_0^\omega h(s)ds}\frac{1}{\min\{f_0, f_\infty\}}. \tag{10}$$

Then equation (4) has a positive ω–periodic solution.

We emphasize that Theorem 1 is very easy to apply; roughly speaking, it only requires an integral representation of the considered equation and some bounds for the kernel of the equivalent integral equation.

Example 1. Let us consider the following modification of the celebrated Nicholson's blowflies equation [6]:

$$u'(t) = -u(t) + \lambda(1+\sin(2\pi t))u(t-\tau(t))\left(\alpha + \beta e^{-\gamma u(t-\tau(t))}\right), \tag{11}$$

where $\alpha, \beta, \gamma, \lambda$ are positive constants, and τ is a 1-periodic delay.

In terms of our Theorem 2, we have

$$f(u) = u\left(\alpha + \beta e^{-\gamma u}\right) \quad ; \quad h(t) = 1 + \sin(2\pi t)$$

$$f_0 = \lim_{x\to 0}\frac{f(x)}{x} = \alpha + \beta \quad ; \quad f_\infty = \lim_{x\to\infty}\frac{f(x)}{x} = \alpha.$$

Next, since $a(t) \equiv 1$, $G(t,s) = e^{s-t}(e-1)^{-1}$. Denoting by

$$g(t) = \int_0^1 G(t,s)h(s)\,ds = e^{-t}\left(1 - \frac{2\pi}{1+4\pi^2}\right),$$

we easily get

$$A = g(0) \approx 0.844777 \quad ; \quad B = g(1) \approx 0.310776.$$

Theorem 2 provides the existence of a positive 1-periodic solution of (11) for all values of λ in the interval

$$I = \left(\frac{1}{(\alpha+\beta)B}, \frac{1}{\alpha A}\right).$$

For example, if we choose $\alpha = 0.2$, $\beta = 1.2$, Theorem 2 ensures that (11) has a positive 1-periodic solution for all $\lambda \in (2.3, 5.9)$. We notice that Theorem 2.5 in [1] does not aply in this case, since the required inequalities do not hold for any value of λ.

Remark 2. It is easy to use Theorem 1 to obtain similar results when the limits f_0, f_∞ in (2) take values in $[0,\infty]$. In this way, the corresponding existence results in [1, 14] are also improved.

On the other hand, by using similar ideas to those in [3], the results in Section 2 can be extended to a system of integral operators

$$[S_i u_i](t) = \int_t^{t+\omega} k_i(t,s) F_i(s, u(s-\tau(s)))\, ds,\ t \in \mathbb{R}\,,\ i = 1,2,\dots,n, \qquad (12)$$

where k_i, F_i are in the conditions given for (5). In this way, we can apply our scheme to the nonautonomous n–dimensional system considered in [15]

$$u'(t) = -A(t)u(t) + \lambda H(t) F(u(t-\tau(t))),$$

where

$$A(t) = \operatorname{diag}\left[a_1(t), \dots, a_n(t)\right],$$
$$H(t) = \operatorname{diag}\left[h_1(t), \dots, h_n(t)\right],\quad F(u) = \left[f_1(u), \dots, f_n(u)\right]^t,$$

$\lambda, a_i(t), h_i(t)$ are in the conditions given for $\lambda, a(t), h(t)$ in equation (1), respectively, and $f_i : \mathbb{R}^n_+ \to [0,\infty)$ is continuous with $f_i(u) > 0$ for $\|u\| > 0$, $i = 1, \dots, n$.

3. Proof of Theorem 1

As it was indicated in the introduction, the proof of Theorem 1 is based on a fixed point theorem for compact maps on conical shells. We recall the statement of this result below, after introducing some definitions and notations.

Let $(E, \|\cdot\|)$ be a Banach space, we say that $K \subset E$ is a cone if it is closed, nonempty, $K \neq \{0\}$ and whenever $x, y \in K$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda \geq 0$, $\mu \geq 0$ then $\lambda x + \mu y \in K$. If D is a subset of E, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

As usual, we define a compact map $S\colon E \to E$ as a continuous map such that $\overline{S(E)}$ is a compact subset of E. A map S is said to be completely continuous if it is continuous and $\overline{S(C)}$ is a compact subset of E for each bounded subset $C \subset E$. We are now in a position to state the above mentioned fixed point theorem, whose proof is based on the properties of the fixed point index (see, e.g., [5]).

Theorem 4. *Assume Ω^1, Ω^2 are open bounded sets with $\Omega^1_K \neq \emptyset$, $\overline{\Omega^1}_K \subset \Omega^2_K$, and let $S\colon \overline{\Omega^2}_K \to K$ be a compact map such that either*

- *there exists $e \in K \setminus \{0\}$ such that $u \neq Su + \lambda e$ for all $u \in \partial_K \Omega^2$ and all $\lambda > 0$, and*
- *$\|Su\| \leq \|u\|$ for all $u \in \partial_K \Omega^1$.*

or

- *there exists $e \in K \setminus \{0\}$ such that $u \neq Su + \lambda e$ for all $u \in \partial_K \Omega^1$ and all $\lambda > 0$, and*
- *$\|Su\| \leq \|u\|$ for all $u \in \partial_K \Omega^2$.*

Then S has a fixed point in $\overline{\Omega^2}_K \setminus \Omega^1_K$.

Now, in order to apply Theorem 4 in the proof of Theorem 1, we need to choose an adequate cone on E_ω and to check that hypotheses in Theorem 1 guarantee that the cone is invariant by S.

We will consider the cone

$$K = \left\{ u \in E_\omega : \min_{t\in[0,\omega]} u(t) \geq c\|u\| \right\}, \tag{13}$$

where $c \in (0, 1]$ was introduced in (D1). We have the following result:

Lemma 1. *Assume (D1) holds. Then S maps K into K and it is completely continuous.*

Proof: Using an standard reasoning (see for example [11]) one can show that S is a completely continuous operator. So, for simplicity, we omit the proof of that part. Moreover, the periodicity properties of the functions F, τ and k guarantee that S maps E_ω into E_ω.

Next, to show that S maps K into K, let $u \in K$ and $t \in \mathbb{R}$. We have from (D1) that

$$|Su(t)| = \int_t^{t+\omega} k(t,s)F(s, u(s-\tau(s)))\,ds \leq \int_t^{t+\omega} \Phi(s)F(s, u(s-\tau(s)))\,ds.$$

Therefore,

$$\|Su\| \leq \int_t^{t+\omega} \Phi(s)F(s, u(s-\tau(s)))\,ds. \tag{14}$$

On the other hand, using (D1) again, we have for each $u \in K$ and $t \in \mathbb{R}$ that

$$\begin{aligned} Su(t) &= \int_t^{t+\omega} k(t,s)F(s, u(s-\tau(s)))\,ds \\ &\geq c\int_t^{t+\omega} \Phi(s)F(s, u(s-\tau(s)))\,ds \geq c\|Su\|. \end{aligned}$$

Thus, $\min_{t\in[0,\omega]} Su(t) \geq c\|Su\|$, and therefore $S(K) \subset K$. □

The following pieces that we need in order to apply Theorem 4 are suitable open sets. For each $r > 0$ we write

$$A^r = \left\{ u \in E_\omega : \min_{t\in[0,\omega]} u(t) < cr \right\},$$

and

$$B^r = \{u \in E_\omega : \|u\| < r\}.$$

The sets defined above satisfy the following result, which is an adaptation of Lemma 2.5 in [11] to the Banach space E_ω. Since the proof is essentially what appears in [11], we omit it.

Lemma 2. *Sets A^r and B^r verify*

(a) *A^r_K and B^r_K are open relative to K.*
(b) $B^{cr}_K \subset A^r_K \subset B^r_K$.

(c) $u \in \partial_K A^r$ *if and only if* $u \in K$ *and* $\min_{t\in[0,\omega]} u(t) = cr$.

(d) *If* $u \in \partial_K A^r$, *then* $cr \leq u(t) \leq r$ *for each* $t \in [0, \omega]$.

It is clear that sets A^r are unbounded sets for each $r > 0$, so we can not use Theorem 4 with them. However we will be able to apply Theorem 4 taking into account that, for each $\delta > r$, the following relations hold:

$$A^r_K = (A^r \cap B^\delta)_K \qquad \text{and} \qquad \overline{A^r}_K = (\overline{A^r \cap B^\delta})_K.$$

The first equality follows immediately from Lemma 2 (b). For the second let $u \in \overline{A^r}_K$. Then from Lemma 2 (c) we have that $c\|u\| \leq \min_{t\in[0,\omega]} u(t) \leq cr < c\delta$ so $u \in (\overline{A^r} \cap B^\delta) \cap K$. Now, since A^r and B^δ are open sets we have $\overline{A^r} \cap B^\delta \subseteq \overline{A^r \cap B^\delta}$ and so

$$(\overline{A^r} \cap B^\delta)_K \subset (\overline{A^r \cap B^\delta})_K.$$

Thus $u \in (\overline{A^r \cap B^\delta})_K$, and therefore $\overline{A^r}_K \subseteq (\overline{A^r \cap B^\delta})_K$. The reverse inclusion is trivial.

Now we are in position to prove Theorem 1.

Proof of Theorem 1: Lemma 1 ensures that the restrictions $S\colon \overline{A^\alpha}_K \to K$ and $S\colon \overline{B^\beta}_K \to K$ are well defined compact maps for each $\alpha, \beta \in (0, \infty)$.

Next, we claim that:

(I) There exists $e \in K\setminus\{0\}$ such that $u \neq Su+\lambda e$ for $u \in \partial_K A^\alpha$ and $\lambda > 0$.

(II) $\|Su\| \leq \|u\|$ for all $u \in \partial_K B^\beta$.

We start with (I). Let $e(t) = 1$ for $t \in \mathbb{R}$. Then $e \in K \setminus \{0\}$. Next, suppose that there exists $u \in \partial_K A^\alpha$ and $\lambda > 0$ such that $u = Su + \lambda e$. Then from Lemma 2 (d) we have $c\alpha \leq u(t) \leq \alpha$ for $t \in [0, w]$. From (D2) we have, for each $t \in [0, w]$,

$$\begin{aligned} u(t) &= Su(t) + \lambda = \int_t^{t+\omega} k(t,s)F(s,u(s-\tau(s)))\,ds + \lambda \\ &\geq c\alpha \int_t^{t+\omega} k(t,s)\phi(s)\,ds + \lambda \geq c\alpha + \lambda. \end{aligned} \tag{15}$$

Hence, $\min_{t\in[0,\omega]} u(t) \geq c\alpha + \lambda > c\alpha$, contradicting the statement of Lemma 2 (c). This contradiction proves part (I) of our claim.

Next, let us consider part (II). If $u \in \partial_K B^\beta$ then $\|u\| = \beta$ and from (D3) we obtain, for each $t \in \mathbb{R}$,

$$\begin{aligned} |Su(t)| &\leq \int_t^{t+\omega} k(t,s)F(s,u(s-\tau(s)))\,ds \\ &\leq \beta \int_t^{t+\omega} k(t,s)\psi(s)\,ds \leq \beta \sup_{t\in[0,\omega]} \int_0^{\omega} k(t,s)\psi(s)\,ds < \beta. \end{aligned}$$

Hence, $\|Su\| \leq \|u\|$ for each $u \in \partial_K B^\beta$, and so (II) holds.

Now suppose that $\beta < c\alpha$. Then one has from Lemma 2 that $\overline{B^\beta}_K \subset B^{c\alpha}_K \subset A^\alpha_K$, and therefore it follows from Theorem 4 that S has at least a fixed point $u \in \overline{A^\alpha}_K \setminus B^\beta_K$. Hence $c\beta \le \min_{t\in[0,\omega]} u(t) \le c\alpha$ and $\|u\| \ge \beta$ hold. On the other hand, $c\|u\| \le \min_{t\in[0,\omega]} u(t) \le c\alpha$, and therefore $\|u\| \le \alpha$.

Finally, if $\alpha < \beta$ one has $\overline{A^\alpha}_K \subset B^\beta_K$, and then Theorem 4 guarantees the existence of at least one fixed point $u \in \overline{B^\beta}_K \setminus A^\alpha_K$ of S. Hence we obtain the inequalities

$$c\alpha \le \|u\| \le \beta \quad \text{and} \quad c\alpha \le \min_{t\in[0,\omega]} u(t).$$

□

Remark 3. The conclusions of Theorem 1 remain valid under more general hypotheses on f and k. For example, we can assume that f is a Carathéodory function with an explicit periodic dependence of time, $f(t,x)$. In such a case, the limits in (2) are required to be uniform. Also, k can be assumed not continuous, but only measurable and satisfying the limit relation

$$\lim_{t\to\nu} \int_0^\omega |k(t,s) - k(\nu,s)|\, ds = 0,$$

for every $\nu \in [0,\omega]$. The reader interested in these sharper conditions can consult [3, 12, 13].

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E-mail address: `dfranco@ind.uned.es, eliz@dma.uvigo.es, ptorres@ugr.es`