ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics Letters ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Applied Mathematics Letters

www.elsevier.com/locate/aml

KAM dynamics and stabilization of a particle sliding over a periodically driven curve

Daniel Núñez[a], Pedro J. Torres[b,*]

[a] *Departamento de Matemática y Computación, Universidad del Zulia, Maracaibo, 4001, Venezuela*
[b] *Departamento de Matemática Aplicada, Universidad de Granada, Av. Fuentenueva sn, 18071 Granada, Spain*

Received 23 February 2006; received in revised form 8 May 2006; accepted 11 May 2006

Abstract

The dynamics of a bead sliding without friction along a periodically pulsating wire is under consideration. If the arc length of the wire is taken as the relevant coordinate, the motion of the bead is described by a periodic newtonian equation. Sufficient conditions are derived assuring that a given equilibrium is of twist type, a property that implies its nonlinear stability as well as a KAM scenario around it. Special attention is paid to the stabilization of unstable equilibria, in parallel with the stabilization of the inverted pendulum.

Keywords: KAM dynamics; Lyapunov stability; Twist condition

1. Introduction

A smooth function $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with a T-periodic dependence of the first variable can be viewed as a *pulsating curve* in the plane. The aim of this work is to study the dynamics of a particle sliding without friction over such a periodically driven curve. If the curve is given by $\alpha(t,s) = (X(t,s), Y(t,s)) \in C^{2,4}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}, \mathbb{R}^2)$, we assume the following standing properties:

(C1) $\alpha_s(t,s) \cdot \alpha_s(t,s) = 1$ for all $(t,s) \in \mathbb{R}^2$, that is, s is the arc length of the curve $s \to \alpha(t,s)$ for each fixed $t \in \mathbb{R}$.
(C2) The curve α is symmetric with respect to the axis OY, that is, $X(t,\cdot)$ is an odd function and $Y(t,\cdot)$ is an even function for every fixed t.

Here, $\cdot$ stands for the usual scalar product. By the symmetry condition, the curve has a critical point that is conserved on the axis OY. Then, a particle placed at this point never leaves this position and describes the periodic trajectory $\alpha(t,0)$. Our aim is to derive geometrical conditions for the stability of this trajectory. As a particular case, we may think of the celebrated problem of the stabilization of the upper equilibrium of the pendulum by moving its suspension point [1,2,5,6].

* Corresponding author.
E-mail address: ptorres@ugr.es (P.J. Torres).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.05.023

Please cite this article in press as: D. Núñez, P.J. Torres, KAM dynamics and stabilization of a particle sliding over a periodically driven curve, Appl. Math. Lett. (2006), doi:10.1016/j.aml.2006.05.023

By fixing a suitable system of measures, the gravitational constant can be taken as 1, so the potential energy is $U(t,s) = Y(t,s)$. By using the condition (C1), the lagrangian can be written as

$$L(t,s,\dot{s}) = \frac{1}{2}\dot{s}^2 + (\alpha_t(t,s)\cdot\alpha_s(t,s))\dot{s} + \frac{1}{2}\alpha_t(t,s)\cdot\alpha_t(t,s) - Y(t,s).$$

Therefore, Lagrange's equation of motion becomes

$$s'' + (\alpha_{tt}(t,s) + e_2)\cdot\alpha_s(t,s) = 0, \tag{1}$$

with $e_2 = (0,1)$. This is a newtonian equation with a periodic dependence of time.

Our intention is to fix conditions over the curve $\alpha(t,s)$ such that $s(t) = 0$ is an equilibrium of twist type of Eq. (1). Roughly speaking, a periodic solution is said to be of twist type if the first coefficient of the Birkhoff Normal Form associated with the Poincaré map is different from zero. The celebrated Moser's twist theorem [12] implies that a solution of twist type is Lyapunov stable. Also, the Poincaré–Birkhoff theorem provides the existence of infinitely many subharmonics with minimal periods tending to infinity, whereas KAM theory (see for instance [9]) implies the existence of quasiperiodic solutions (invariant tori) and Smale horseshoes. In conclusion, around a solution of twist type the typical KAM scenario arises. There are a variety of twist criteria available in the literature, among which [10, 7] will be useful for us. The classical way to detect KAM dynamics is to prove that the system is close to an integrable one. In contrast, we will obtain non-perturbative results, that is, the pulsating curve is not near a stationary one.

In our opinion, the stabilization of maxima is especially interesting, since they are unstable if the curve is stationary. In this context, the paradigm is the stabilization of the inverted pendulum, mentioned before. In general, the stabilization of unstable equilibria has drawn the attention of many mathematicians and constitutes one of the main problems in control theory. We show some examples along these lines, paying some attention to the stabilization by high frequency vibrations, a phenomenon analogous to the well-known stabilizing effect for an inverted pendulum of rapidly moving its pivot [6].

The rest of the work is organized as follows. The main result is stated and proved in Section 2. The required conditions are interpreted in terms of the second variation of the curvature in the equilibrium. Section 3 presents some illustrative examples and applications, with special attention given to the analysis of the pulsating ellipse as our main motivation. Finally, Section 4 presents the final conclusions of this work.

2. Main result

From the introduction, the differential equation (1) under study is $s'' + f(t,s) = 0$, where

$$\begin{aligned} f(t,s) &= (\alpha_{tt}(t,s) + e_2)\cdot\alpha_s(t,s) \\ &= X_{tt}(t,s)X_s(t,s) + Y_s(t,s)(Y_{tt}(t,s)+1). \end{aligned}$$

Note that $Y_s(t,s)$, $X_{tt}(t,s)$ are odd functions in the second variable and hence they vanish in $s=0$. In consequence, $f(t,0) = 0$ for all t. In other words, $s = 0$ is an equilibrium solution of Eq. (1). We are interested in the twist character of this equilibrium. With this in mind, let us write the Taylor expansion of $f(t,s)$ around $s=0$ up to the third degree,

$$f(t,s) = a(t)s + b(t)s^2 + c(t)s^3 + \cdots, \tag{2}$$

where $a(t) = f_s(t,0)$, $b(t) = \frac{1}{2}f_{ss}(t,0)$, $c(t) = \frac{1}{6}f_{sss}(t,0)$.

Lemma 1. *The coefficients in (2) are given by*

$$\begin{aligned} a(t) &= Y_{ss}(t,0)(Y_{tt}(t,0)+1) \\ b(t) &= 0 \\ c(t) &= \frac{1}{6}\{Y_{ss}(t,0)Y_{sstt}(t,0) + Y_{ssss}(t,0)(Y_{tt}(t,0)+1) - 2Y_{sst}(t,0)^2\}. \end{aligned} \tag{3}$$

Proof. The key point is a suitable usage of condition (C1), which can be written as

$$X_s(t,s)^2 + Y_s(t,s)^2 = 1. \tag{4}$$

Please cite this article in press as: D. Núñez, P.J. Torres, KAM dynamics and stabilization of a particle sliding over a periodically driven curve, Appl. Math. Lett. (2006), doi:10.1016/j.aml.2006.05.023

Note that $Y_s(t,0)=0$, so $X_s(t,0)=1$. By deriving (4) with respect to t, we get $X_{ts}(t,0)=0$. Hence, $X_{tts}(t,0)=0$. Simple calculus gives

$$\begin{aligned} f_s(t,s) &= \alpha_{tts}(t,s)\cdot\alpha_s(t,s) + (\alpha_{tt}(t,s)+e_2)\cdot\alpha_{ss}(t,s) \\ &= X_{tts}(t,s)X_s(t,s) + X_{tt}(t,s)X_{ss}(t,s) + Y_{tts}(t,s)Y_s(t,s) + Y_{ss}(t,s)(Y_{tt}(t,s)+1), \end{aligned}$$

and therefore

$$a(t) = f_s(t,0) = Y_{ss}(t,0)(Y_{tt}(t,0)+1).$$

Higher coefficients can be computed in the same way. ■

Now, we are in an appropriate situation for applying the wide variety of criteria based on the third approximation available in the literature. In the rest of the section, a, b, c are supposed to take the values given in (3).

Theorem 1. *Let us assume that*

(i) *The linear equation $\ddot{s} + a(t)s = 0$ is stable.*
(ii) $Y_{ss}(t,0)Y_{sstt}(t,0) + Y_{ssss}(t,0)(Y_{tt}(t,0)+1) > (<) 2Y_{sst}(t,0)^2$ *for all t.*

Then, $s = 0$ is a solution of twist type of (1).

Proof. By continuity, condition (ii) means that the strict inequality holds in one sense or the other. Then, the result follows as a consequence of the main result in [10]. ■

In the previous result, the stability of the linear equation is understood in the Lyapunov sense. In practice, any stability criterion for Hill's equation together with condition (ii) provides a different result for (1). For instance, the classical Lyapunov criterion gives the following result.

Corollary 1. *Let us assume that*

(i) $\int_0^T a(t)\mathrm{d}t > 0$, $\int_0^T a(t)^+\mathrm{d}t \le 4/T$.
(ii) $Y_{ss}(t,0)Y_{sstt}(t,0) + Y_{ssss}(t,0)(Y_{tt}(t,0)+1) > (<) 2Y_{sst}(t,0)^2$ *for all t.*

Then, $s = 0$ is a solution of twist type of (1).

Here, $a^+(t) = \max\{a(t), 0\}$ denotes the positive part of a. Let us mention that in [13] there is proved a quite flexible criterion for Hill's equation based in the L^p norm generalizing the classical Lyapunov criterion that can be used to improve the previous corollary. Other stability criteria can be found in [3,8,11]. We will not insist on this direction. More important in our opinion is the geometrical interpretation of the coefficients (3). Note that as the curve is parametrized by the arc, the curvature is given by [4]

$$k(t,s) = (\alpha_{ss}(t,s)\cdot\alpha_{ss}(t,s))^{1/2} = \sqrt{X_{ss}(t,s)^2 + Y_{ss}(t,s)^2}.$$

By the symmetry conditions imposed, the curvature $k(t,s)$ is an even function in s. Besides, $k(t,0) = |Y_{ss}(t,0)|$. When one fixes the positive orientation in the plane, it is possible to define a signed curvature by watching the orientation type of the frame formed by the tangent and normal vectors at each point. The signed curvature is defined then by

$$K(t,s) = X_s(t,s)Y_{ss}(t,s) - X_{ss}(t,s)Y_s(t,s).$$

Notice that $k(t,s) = |K(t,s)|$. As $X_s(t,0) = \pm 1$ we can choose the parametrization over the curve such that $X_s(t,0) = 1$. In this way we obtain

$$K(t,0) = Y_{ss}(t,0), \qquad Y_{ssss}(t,0) = K_{ss}(t,0) - K(t,0)^3.$$

Now, it is possible to write the coefficients of the third order expansion as

$$\begin{aligned} a(t) &= K(t,0)(Y_{tt}(t,0)+1) \\ b(t) &= 0 \\ c(t) &= \frac{1}{6}\{K(t,0)K_{tt}(t,0) + (K_{ss}(t,0) - K(t,0)^3)(Y_{tt}(t,0)+1) - 2K_t(t,0)^2\}. \end{aligned} \tag{5}$$

Please cite this article in press as: D. Núñez, P.J. Torres, KAM dynamics and stabilization of a particle sliding over a periodically driven curve, Appl. Math. Lett. (2006), doi:10.1016/j.aml.2006.05.023

Note that these coefficients involve the curvature function at zero, its velocity and acceleration, the second variation of curvature with respect to the arc s and the vertical acceleration at the equilibrium. In consequence we can refer to this approach as like a “second variation of curvature’s approach”.

3. Stabilization of unstable equilibria

The stabilization of unstable equilibria is of special interest in branches like control theory. We will present two examples of stabilization of an equilibrium which is a maximum of the given curve by means of a suitable periodic pulsation.

3.1. Stabilizing the upper equilibrium in the pulsating ellipse

A particle sliding over a stationary ellipse (with semiaxes p, q constant) has an unstable (hyperbolic) equilibrium, which corresponds to the upper vertex. The question is whether it is possible to find time-dependent semiaxes $p(t), q(t) \in C^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ stabilizing this equilibrium. The arc length parametrization of the curve is

$$\alpha(t, s) = (p(t) \sin\theta(t, s), q(t) \cos\theta(t, s)), \tag{6}$$

where $\theta(t, s)$ is implicitly defined by the relation

$$s(t, \theta) = \int_0^\theta (p(t)^2 \cos^2 u + q(t)^2 \sin^2 u)^{1/2} \mathrm{d}u. \tag{7}$$

The upper equilibrium corresponds to the solution $s = 0$.

Lemma 2. *For the pulsating ellipse (6), the coefficients (3) are given by*

$$\begin{aligned} a(t) &= K(t)(1 + q''(t)) \\ b(t) &= 0 \\ c(t) &= \frac{1}{6}\left\{K(t)K''(t) - \frac{1 + q''(t)}{p(t)^6} q(t)(3p(t)^2 - 4q(t)^2) - 2K'(t)^2\right\} \end{aligned} \tag{8}$$

where $K(t) = -\frac{q(t)}{p(t)^2}$.

Note that the curvature $K(t)$ in $s = 0$ is negative. We have the following result.

Corollary 2. *Let us assume that* $p(t) \equiv P$ *is constant. For any* $q(t) \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ *verifying*

(i) $\|q'\|_2^2 > \int_0^T q(t)\mathrm{d}t$,
(ii) $q''(t) > -3/2$ *for all* t,

there exists a positive constant P_* *such that if* $P > P_*$, *then* $s = 0$ *is a solution of twist type.*

Proof. Taking $p(t) \equiv P$ constant, the coefficient $a(t)$ is

$$a(t) = -\frac{q(t)}{P^2}(1 + q''(t)).$$

We will prove that the linear equation $s'' + a(t)s = 0$ is stable by using Lyapunov’s criterion. First, note that

$$\int_0^T a(t)\mathrm{d}t = \frac{1}{P^2}\left(\|q'\|_2^2 - \int_0^T q(t)\mathrm{d}t\right) > 0,$$

by condition (i). Besides,

$$\int_0^T a(t)^+\mathrm{d}t \le \frac{1}{P^2}\|(qq'')^+\|_1 < 4/T$$

if $P > P_1 \equiv \frac{1}{2}(T\|(qq'')^+\|_1)^{1/2}$.

Please cite this article in press as: D. Núñez, P.J. Torres, KAM dynamics and stabilization of a particle sliding over a periodically driven curve, Appl. Math. Lett. (2006), doi:10.1016/j.aml.2006.05.023

On the other hand, the coefficient $c(t)$ reads

$$c(t) = \frac{1}{6P^4}\left[-2q(t)q''(t) - 3q(t) + \frac{4q(t)^3}{P^2}(1 + q''(t)) - 2q'(t)^2\right].$$

Now, condition (ii) implies that $c(t)$ is negative for P large enough, concretely if

$$P > P_2 \equiv \max_{t\in[0,T]}\left[\frac{4q(t)^3|1 + q''(t)|}{q(t)(2q''(t) + 3) + 2q'(t)^2}\right]^{1/2}.$$

Hence, taking $P_* = \max\{P_1, P_2\}$ the conditions of Corollary 1 are satisfied and the result is obtained. ■

It is important to remark that P_* is explicitly computable. Although it is not very easy to find explicit examples, numerical computations show that $q(t) = (1.23\sin(\sin t) + 10^{-4})^2$ holds with the assumptions of Corollary 2 and in this particular case $P_* \simeq 263\,352$.

3.2. An example of zero curvature in the equilibrium: A quartic curve

As a further example, let us consider a particle sliding over the curve

$$\alpha(t, s) = (\theta(t, s), -p(t)\theta(t, s)^4) \tag{9}$$

where $p(t) \in C^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ and the arc length parameter s is given by

$$s(t, \theta) = \int_0^\theta (1 + 16p(t)^2u^6)^{1/2}\mathrm{d}u. \tag{10}$$

In this case, $X(t, s) = \theta(t, s)$, $Y(t, s) = -p(t)\theta(t, s)^4$. Then,

$$Y_{ss}(t, s) = p(t)[12\theta(t, s)^2\theta_s(t, s)^2 + 4\theta(t, s)^3\theta_{ss}(t, s)].$$

Taking into account that $\theta(t, 0) = 0$, by using the formula in (3), the result is that the first order approximation coefficient is $a(t) = 0$ and therefore Theorem 1 does not apply. After some tedious but elementary computations we get that the third order coefficient is $c(t) = -4p(t)$, so it is negative and therefore the equilibrium is unstable for any $p(t) \in C^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ as a direct application of Theorem 2 in [7]. In the next result, we will see that the addition of a vertical oscillation is suitable for stabilizing this equilibrium. In the following, the mean value of a periodic function is denoted by $\overline{p} = \frac{1}{T}\int_0^T p(t)\mathrm{d}t$.

Proposition 1. *Let us consider the curve*

$$\alpha(t, s) = (\theta(t, s), -p(t)\theta(t, s)^4 + p(t)), \tag{11}$$

where s is the arc length parameter and $p(t) \in C^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ verifies that $\|p'\|_2^2 > T\overline{p}$. Then, the equilibrium $s = 0$ is of twist type as a solution of (1).

Proof. By computations similar to those carried out before, one obtains that

$$a(t) = 0, \qquad b(t) = 0, \qquad c(t) = -4p(t)(p''(t) + 1).$$

Hence, $\int_0^T c(t)\mathrm{d}t = 4[\|p'\|_2^2 - T\overline{p}] > 0$, and the twist character of the equilibrium follows from [7]. ■

As a corollary, we obtain the stabilization through high frequency oscillations, in analogy with the well-known phenomenon of the stabilization of the inverted pendulum with vertical oscillations of its suspension point.

Corollary 3. *Let us consider the curve*

$$\alpha(t, s) = (\theta(t, s), -p(\omega t)\theta(t, s)^4 + p(\omega t)),$$

where $p(t) \in C^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^+)$ and s is the arc length parameter. If

$$\omega^2 > \frac{2\pi\overline{p}}{\|p'\|_2^2},$$

then the equilibrium $s = 0$ is of twist type as a solution of (1).

Please cite this article in press as: D. Núñez, P.J. Torres, KAM dynamics and stabilization of a particle sliding over a periodically driven curve, Appl. Math. Lett. (2006), doi:10.1016/j.aml.2006.05.023

4. Final conclusions

We have provided some stability criteria in the concrete physical model of a particle moving without friction over a pulsating symmetric wire. In this way it is shown that there is a close relation between curvature and stability. This general framework is illustrated by the stabilization of unstable equilibria in a pulsating ellipse and a variable quartic.

Several interesting problems remain open for future research, for example the study of pulsating curves with changing-sign curvature in a given equilibrium, the stabilization of an arbitrary maximum by vertical oscillations and the analysis of non-symmetric pulsating curves.

Acknowledgements

One of the key ideas of this work is taking the arc length as the relevant parameter for the curve. We are indebted to R. Ortega for this suggestion. Pedro J. Torres was supported by D.G.I. BFM2002-01308, Ministerio Ciencia y Tecnología, Spain.

References

[1] B.S. Bardin, A.P. Markeyev, The stability of the equilibrium of a pendulum for vertical oscillations of the point of suspension, J. Appl. Math. Mech. 59 (6) (1995) 879–886.
[2] H.W. Broer, I. Hoveijn, M. van Noort, The inverted pendulum: A singularity theory approach, J. Differential Equations 157 (1) (1999) 120–149.
[3] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Springer-Verlag, New York, Berlin, 1971.
[4] M.P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1976.
[5] M. Levi, Stability of the inverted pendulum—a topological explanation, SIAM Rev. 30 (4) (1988) 639–644.
[6] M. Levi, W. Weckesser, Stabilization of the inverted linearized pendulum by high frequency vibrations, SIAM Rev. 37 (2) (1995) 219–223.
[7] B. Liu, The stability of the equilibrium of a conservative system, J. Math. Anal. Appl. 202 (1996) 133–149.
[8] W. Magnus, S. Winkler, Hill's Equation, Dover, New York, 1979.
[9] J.N. Mather, Existence of quasi-periodic orbits for twist homeomorphisms of the annulus, Topology 21 (1982) 457–467.
[10] R. Ortega, The stability of the equilibrium of a non-linear Hill's equation, SIAM J. Math. Anal. 25 (1994) 1393–1401.
[11] V.M. Starzinskii, Survey of works on conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients, in: Amer. Math. Soc. Transl. Ser. 2, vol. 1, Providence, RI, 1955.
[12] C.L. Siegel, J.K. Moser, Lectures on Celestial Mechanics, Springer-Verlag, New York, Berlin, 1971.
[13] M. Zhang, W. Li, A Lyapunov-type stability criterion using L^α norms, Proc. Amer. Math. Soc. 130 (11) (2002) 3325–3333.

Please cite this article in press as: D. Núñez, P.J. Torres, KAM dynamics and stabilization of a particle sliding over a periodically driven curve, Appl. Math. Lett. (2006), doi:10.1016/j.aml.2006.05.023