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Weak singularities may help periodic solutions to exist

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Abstract

In a periodically forced semilinear differential equation with a singular nonlinearity, a weak force condition enables the achievement of new existence criteria through a basic application of Schauder's fixed point theorem. The originality of the arguments relies in that, contrary to the customary situation in the available references, a weak singularity facilitates the arguments of the proofs.

Keywords: Periodic solution; Weak singularity; Schauder's fixed point theorem

1. Introduction

In the latter years, the periodic problem for the semilinear singular equation

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \tag{1}$$

with $a, b, c \in L^1[0, T]$ and $\lambda > 0$, has deserved the attention of many specialists in differential equations. Although surely incomplete, we will try to give a brief account of the present state of this problem.

The interest in this type of equations began with the paper of Lazer and Solimini [12]. They proved that for $a(t) \equiv 0$, $b(t) \equiv 1$ and $\lambda \geqslant 1$ (called *strong force condition* in a terminology first introduced by Gordon [8,9]), a necessary and sufficient condition for existence of a positive periodic solution is that the mean value of c is negative, $\bar{c} < 0$. Moreover, if $0 < \lambda < 1$ (weak

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force condition) they found examples of functions c with negative mean values and such that periodic solutions do not exist. Since then, the strong force condition became standard in the related works, see, for instance, [3,5,6,10,11,13–16,18,24,27], the recent review [17] and their bibliographies. With a strong singularity, the energy near the origin becomes infinity and this fact is helpful for obtaining either the a priori bounds needed for a classical application of the degree theory, either the fast rotation needed in recent versions of the Poincaré–Birkhoff theorem. In this paper, our objective is to show how a weak singularity can play an important role if a different classical tool is chosen in the proof, namely, the Schauder's fixed point theorem.

If compared with the literature available for strong singularities, the study of the existence of periodic solutions under the presence of a weak singularity is much more recent and the number of references is considerably smaller. The first existence result appears in [16]. From now on, let us denote by c^* and c_* the essential supremum and infimum of a given function $c \in L^1[0,T]$, if they exist. In the mentioned paper, it is proved that for $0 < k^2 < \mu_1 := (\frac{\pi}{T})^2$ and $\lambda, b > 0$, the equation

$$x'' + k^2 x = \frac{b}{x^\lambda} + c(t) \tag{2}$$

has a T-periodic solution if

$$c_* > -\left(\frac{\pi^2 - T^2 k^2}{T^2 \lambda b}\right)^{\frac{\lambda}{\lambda+1}} (\lambda+1) b. \tag{3}$$

For the critical case $k^2 = \mu_1$, the condition is just $c_* > 0$. At least for strong singularities, this condition is optimal because in [3] it is proved that the equation

$$x'' + \mu_1 x = \frac{b}{x^3} + \epsilon \sin\left(\frac{2\pi}{T} t\right)$$

has no T-periodic solutions for $\epsilon > 0$ sufficiently small.

Excluding this critical value, that is for $0 < k^2 \leqslant \mu_1$, the main result of [21] provides the alternative condition

$$c_* < 0, \qquad c^* \leqslant \frac{c_*}{\cos^\lambda(\frac{kT}{2})} + \frac{k}{T} \sin kT \left(\frac{b}{|c_*|}\right)^{\frac{1}{\lambda}}. \tag{4}$$

This condition has been slightly improved in [7]. Both conditions (3) and (4) are of independent interest: (3) is a uniform lower bound that cover the critical value μ_1 but do not cover natural examples like c nearly constant, whereas (4) is a condition over the "oscillation" of the forcing term but requires a bounded c and do not cover the critical case. The first result concerning Eq. (2) with a genuine unbounded $c \in L^1[0,T]$ was proved in [2, Theorem 5 and Example 2]. To explain it, let us define the function

$$\gamma(t) = \int_t^{t+T} c(s) \sin\left(\pi \frac{s-t}{T}\right) ds,$$

and let be $k^2 = \mu_1$, then $\gamma_* > 0$ implies the existence of a periodic solution of (2). Although not explicitly observed in [2], it is interesting to point out that $\gamma(t)$ is the unique T-periodic solution of the linear equation $x'' + \mu_1 x = c(t)$. It turns out that our results extend and improve this existence criterion.

Up to our knowledge, the most recent result concerning Eq. (2) has been proved in [22]. Let us denote $\mu_n = (\frac{n\pi}{T})^2$ for any $n \in \mathbb{N}$ and take $k^2 \neq \mu_{2n}$, then for any $\tilde{c} \in L^1[0,T]$ there exists $C_0 > 0$ such that Eq. (2) with $c(t) = \bar{c} + \tilde{c}(t)$ possesses a unique positive T-periodic solution for any $\bar{c} > C_0$. The proof is tricky because there is a rescaling such that the equation is transformed in a perturbation of the linear part, which is nonresonant. This rescaling is not possible with more general nonlinearities.

Finally, let us mention that the results in [2,16] are only applicable to the general equation with variable coefficients in a very limited sense. In particular, such results do not cover the relevant particular case $c(t) \equiv 0$, the generalized Brillouin equation, which have generated a wide interest (see [1,4,19,20,25,26,28] and the references therein).

The paper is organized as follows. In Section 2 it is proved a result for the equation with a general singularity (strong or weak) which complement and/or generalize the results in [2,16,21]. Sections 3 and 4 are devoted to show that the additional assumption of a weak singularity enables the obtention of new criteria for the existence of periodic solutions. The importance of such results relies in that, for the first time in this topic, we have existence results which are valid for a weak singularity whereas the validity of such results under a strong force assumption remains as an open problem.

Let us fix some notation to be used in the following: 'for a.e.' means 'for almost every.' Given $a \in L^1(0,T)$, we write $a \succ 0$ if $a \geqslant 0$ for a.e. $t \in [0,T]$ and it is positive in a set of positive measure. The set of positive real numbers is denoted by $\mathbb{R}^+$. We write $f \in \mathrm{Car}([0,T] \times \mathbb{R}^+, \mathbb{R})$ if $f : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ is a L^1-Carathéodory function, that is, it is continuous in the second variable and for every $0 < r < s$ there exists $h_{r,s} \in L^1(0,T)$ such that $|f(t,x)| \leqslant h_{r,s}(t)$ for all $x \in [r,s]$ and a.e. $t \in [0,T]$. The usual L^p-norm is denoted by $\|.\|_p$. For a given p, let us denote by $p^* = \frac{p}{p-1}$ if $1 \leqslant p < \infty$ and $p^* = 1$ if $p = +\infty$. Finally, we denote the set of continuous T-periodic functions as C_T.

2. The general equation

Let us consider the general equation

$$x'' + a(t)x = f(t,x) + c(t), \tag{5}$$

with $a, c \in L^1[0,T]$ and $f \in \mathrm{Car}([0,T] \times \mathbb{R}^+, \mathbb{R})$. From now on, we assume the standing hypothesis

(H1) The Hill's equation $x'' + a(t)x = 0$ is nonresonant and the corresponding Green's function $G(t,s)$ is nonnegative for every $(t,s) \in [0,T] \times [0,T]$.

Under this assumption, let us define the function

$$\gamma(t) = \int_0^T G(t,s)c(s)\,ds,$$

which is just the unique T-periodic solution of the linear equation $x'' + a(t)x = c(t)$.

Before going to the main theorem of this section, it is pertinent to make some remarks concerning assumption (H1). When $a(t) \equiv k^2$, this condition is equivalent to $0 < k^2 \leqslant \mu_1$. For a nonconstant $a(t)$, there is a L^p-criterion proved in [21] (the proof relies on an anti-maximum principle from [23]) which is given in the following lemma for the sake of completeness.

Lemma 1. [21] *Let us define*

$$K(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}}\left(\frac{2}{2+q}\right)^{1-2/q}\left(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})}\right)^2, & \text{if } 1 \leqslant q < \infty, \\ \frac{4}{T}, & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function. Assume that $a(t) \succ 0$ and $a \in L^p(0,T)$ for some $1 \leqslant p \leqslant \infty$. If

$$\|a\|_p \leqslant K(2p^*),$$

then $G(t,s) \geqslant 0$ for all $(t,s) \in [0,T] \times [0,T]$.

The following one is the main result of this section.

Theorem 1. *Let us assume that there exist $b \succ 0$ and $\lambda > 0$ such that*

$$0 \leqslant f(t,x) \leqslant \frac{b(t)}{x^\lambda}, \quad \text{for all } x > 0, \text{ for a.e. } t.$$

If $\gamma_* > 0$, then there exists a positive T-periodic solution of (5).

Proof. A T-periodic solution of (5) is just a fixed point of the completely continuous map $\mathcal{F}: C_T \to C_T$ defined as

$$\mathcal{F}[x](t) := \int_0^T G(t,s)\big[f\big(s,x(s)\big) + c(s)\big]\,ds = \int_0^T G(t,s) f\big(s,x(s)\big)\,ds + \gamma(t). \tag{6}$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that $\mathcal{F}$ maps the closed convex set defined as

$$K = \big\{x \in C_T\colon\ r \leqslant x(t) \leqslant R \text{ for all } t\big\} \tag{7}$$

into itself, where $R > r > 0$ are positive constants to be fixed properly. In fact, let us fix $r := \gamma_*$, which is positive by hypothesis. Given $x \in K$, by the nonnegative sign of G and f,

$$\mathcal{F}[x](t) \geqslant \gamma_* = r$$

for all t.

If we define the function $\beta(t) = \int_0^T G(t,s)b(s)\,ds$, then for every $x \in K$

$$\mathcal{F}[x](t) \leqslant \frac{\beta^*}{r^\lambda} + \gamma^* =: R.$$

In conclusion, $\mathcal{F}(K) \subset K$ if $r = \gamma_*$ and $R = \frac{\beta^*}{\gamma_*^\lambda} + \gamma^*$. Clearly, $R > r > 0$, so the proof is finished. □

Remark 1. In the particular case of Eq. (2), the information given by this result is exactly the same as in [2]. Note that the assumption $b \succ 0$ covers, in particular, the possibility of to "switch off" or deactivate the singularity for open intervals of time, a case which (up to our knowledge) has not been studied before.

3. The case $\gamma_* = 0$

The aim of this section is to show that the presence of a weak nonlinearity makes possible to find periodic solutions under different assumptions. Our first result is the following.

Theorem 2. *Let us assume* (H1) *and that there exist* $b, \hat{b} \succ 0$ *and* $0 < \lambda < 1$ *such that*

$$\text{(H2)} \qquad 0 \leqslant \frac{\hat{b}(t)}{x^\lambda} \leqslant f(t,x) \leqslant \frac{b(t)}{x^\lambda}, \quad \text{for all } x > 0, \text{ for a.e. } t.$$

If $\gamma_* = 0$, *then there exists a positive* T*-periodic solution of* (5).

Proof. We follow the same strategy and notation as in the proof of Theorem 1. Again, we need to fix $r < R$ such that $\mathcal{F}(K) \subset K$. If we define the functions $\beta(t) = \int_0^T G(t,s)b(s)\,ds$, $\hat{\beta}(t) = \int_0^T G(t,s)\hat{b}(s)\,ds$, some easy computations prove that it is sufficient to find $r < R$ such that

$$\frac{\hat{\beta}_*}{R^\lambda} \geqslant r, \qquad \frac{\beta^*}{r^\lambda} + \gamma^* \leqslant R.$$

Note that $\hat{\beta}_*, \beta_* > 0$ as a consequence of (H1). Taking $R = 1/r$, it is sufficient to find $R > 1$ such that

$$\hat{\beta}_* R^{1-\lambda} \geqslant 1, \qquad \beta^* R^\lambda + \gamma^* \leqslant R,$$

and these inequalities hold for R big enough because $\lambda < 1$. □

As a corollary for Eq. (2), for a weak singularity we improve the result in [2]. In the particular case $c(t) \equiv 0$, we can provide sharp bounds of the periodic solution. The knowledge of optimal bounds could be important for a future analysis of the stability properties, as in [19,20].

Theorem 3. *Let us assume* (H1) *and* (H2). *If* $c(t) \equiv 0$, *then there exists a positive* T*-periodic solution of* (5) *such that*

$$\left(\frac{\hat{\beta}_*}{\beta^{*\lambda}}\right)^{\frac{1}{1-\lambda^2}} \leqslant x(t) \leqslant \left(\frac{\beta^*}{\hat{\beta}_*^\lambda}\right)^{\frac{1}{1-\lambda^2}}.$$

Proof. Again, to prove that $\mathcal{F}(K) \subset K$ we need $r < R$ such that

$$\frac{\hat{\beta}_*}{R^\lambda} \geqslant r, \qquad \frac{\beta^*}{r^\lambda} \leqslant R.$$

Taking the equalities, some simple computations lead to

$$r = \left(\frac{\hat{\beta}_*}{\beta^{*\lambda}}\right)^{\frac{1}{1-\lambda^2}}, \qquad R = \left(\frac{\beta^*}{\hat{\beta}_*^\lambda}\right)^{\frac{1}{1-\lambda^2}},$$

and the proof is finished. □

Remark 2. It is interesting to note that these bounds are sharp in the sense that for $a(t) \equiv b(t) \equiv \hat{b}(t)$, it results that $\hat{\beta}_* = \beta^* = 1$, so Theorem 3 provides the exact solution $x(t) = 1$.

4. The case $\gamma^* \leqslant 0$

In the related literature, the cases where c is negative are technically more difficult and additional conditions are necessary to avoid a collision with the singularity as occurs in the classical counterexample of [12]. In this case, $\gamma(t)$ is also negative and the results in [2] do not apply. We will fill partially this gap with the following result.

Theorem 4. *Let us assume* (H1) *and* (H2). *If* $\gamma^* \leqslant 0$ *and*

$$\gamma_* \geqslant \left[\frac{\hat{\beta}_*}{\beta^{*\lambda}}\lambda^2\right]^{\frac{1}{1-\lambda^2}}\left(1 - \frac{1}{\lambda^2}\right) \tag{8}$$

then there exists a positive T*-periodic solution of* (5).

Proof. In this case, to prove that $\mathcal{F}(K) \subset K$ it is sufficient to find $r < R$ such that

$$\frac{\hat{\beta}_*}{R^\lambda} + \gamma_* \geqslant r, \qquad \frac{\beta^*}{r^\lambda} \leqslant R. \tag{9}$$

If we fix $R = \frac{\beta^*}{r^\lambda}$, then the first inequality holds if r verifies

$$\frac{\hat{\beta}_*}{\beta^{*\lambda}} r^{\lambda^2} + \gamma_* \geqslant r,$$

or equivalently,

$$\gamma_* \geqslant f(r) := r - \frac{\hat{\beta}_*}{\beta^{*\lambda}} r^{\lambda^2}.$$

The function $f(r)$ possesses a minimum in $r_0 := [\frac{\hat{\beta}_*}{\beta^{*\lambda}}\lambda^2]^{\frac{1}{1-\lambda^2}}$. Taking $r = r_0$, then the first inequality in (9) holds if $\gamma_* \geqslant f(r_0)$, which is just condition (8). The second inequality holds directly by the choice of R, and it would remain to prove that $R = \frac{\beta^*}{r_0^\lambda} > r_0$. This is easily verified through elementary computations. □

Remark 3. Of course, the lower bound $M_\lambda = [\frac{\hat{\beta}_*}{\beta^{*\lambda}}\lambda^2]^{\frac{1}{1-\lambda^2}}(1 - \frac{1}{\lambda^2})$ is negative because of the weak force condition $0 < \lambda < 1$. Besides, it can be shown that

$$\lim_{\lambda \to 0^+} M_\lambda = -\hat{\beta}_*.$$

As a corollary, we obtain an alternative criterion to those obtained in [16,21] for Eq. (2).

Corollary 1. *Let us consider Eq.* (2) *with* $0 < \lambda < 1$ *and* $0 < k^2 \leqslant \mu_1 := (\frac{\pi}{T})^2$. *Then, there exists a positive T-periodic solution if $c(t) < 0$ for a.e. t and*

$$c_* \geqslant \left[bk^{2\lambda}\lambda^{\frac{2\lambda^2}{1-\lambda}}\right]^{\frac{1}{1+\lambda}}\left(\lambda^2 - 1\right). \tag{10}$$

Proof. In this case, $\hat{\beta}_* = \beta^* = \frac{b}{k^2}$ and $\gamma_* \geqslant c_* \min_t \int_0^T G(t,s)\,ds = \frac{c_*}{k^2}$. Now, the corollary follows after some algebra with condition (8). □

If compared with condition (3), the main advantage of our assumption is that it provides an effective bound for a negative coefficient c even in the critical case $k^2 = \mu_1$.

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