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Existence of closed solutions for a polynomial first order differential equation

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Abstract

We find new criteria for the existence of closed solutions in a first order polynomial differential equation which contains the Abel equation as a particular case. Such results are applied to the problem of the existence of limit cycles in planar polynomial vector fields.

Keywords: Abel equation; Closed solution; Limit cycle; Planar polynomial vector field

This short note is motivated by some results by Andersen and Sandqvist [1] concerning the existence of closed solutions of the Abel equation. Let us consider the differential equation

$$\frac{dx}{dt} = \sum_{k=0}^{n} a_k(t)x^k, \tag{1}$$

where $a_0, \dots, a_n \in C([0, 2\pi])$, $n \geqslant 2$. Our purpose is to obtain new criteria for the existence of closed solutions, that is, solutions such that $x(0) = x(2\pi)$.

The interest of this equation relies on its relation with the existence of limit cycles of polynomial planar systems and the 16th Hilbert problem. Under some conditions, a polynomial planar system can be written in polar coordinates in the form (1). Then, an isolated closed solution of Eq. (1) corresponds to a limit cycle of the planar system. The case $n = 3$ (Abel equation) is particularly important. The number of related references is huge and we can cite [1–7,9] only

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to mention some of them (see also the references therein). The second part of the 16th Hilbert problem refers to the study of the maximum number of limit cycles of a given polynomial planar system. In contrast, our results provide in some special cases a lower bound for this maximum number.

The paper is structured through three sections and Appendix A. In Section 1, the equation without independent term is considered. This case has the particularity that the methods of proof must exclude the trivial solution. Section 2 is devoted to the complete equation. Finally, Section 3 contains some illustrative examples about how to apply the previous results in the study of planar polynomial systems. We have collected in Appendix A some useful tools for the proofs, mainly a classical perturbation result and the method of upper and lower solution for first order equations.

1. The case $a_0 \equiv 0$

In this section, we focus our attention on the case $a_0(t) = 0$ for all t, that is, in the equation

$$\frac{dx}{dt} = \sum_{k=1}^{n} a_k(t)x^k. \tag{2}$$

Then, the trivial solution $x \equiv 0$ is present and the objective is to obtain no-trivial closed solutions. In the rest of the paper, $\dot{x}$ denotes dx/dt.

If a sign condition over the leading coefficient is assumed (say $a_n(t) > 0$ for all t), then it is standard to find existence conditions for closed solutions. The most usual argument consists in finding fixed points of the Poincaré map π by using Bolzano Theorem or similar. The interplay between the stability in zero and infinity can be exploited as well.

On the other hand, the case when $a_n(t) \geqslant 0$ for all $t \in [0, 2\pi]$ is by far more delicate. In [1] this case is studied for the Abel equation $n = 3$ (see Example 6). The result in [1] imposes that a_3 must have a finite number of zeros in $[0, 2\pi]$ and that the function $a_2(t)/a_3(t)$ must be bounded at the right side of the zeros of a_3. This assumption is rather unnatural and in particular implies that a_2 must vanish in the same set as a_3. By using upper and lower solutions, we are able to derive a different type of condition.

Theorem 1. *Let us assume that* $\int_0^{2\pi} a_1(t)\,dt < 0$. *Let us assume that there exists some* $j = 2, \dots, n$ *such that* $a_k(t) \geqslant 0$ *for all* $k = j, \dots, n$ *and* $t \in [0, 2\pi]$ *and* $\sum_{k=j}^{n} a_k(t) > 0$ *for all* $t \in [0, 2\pi]$. *Then, there exists a positive isolated closed solution of Eq.* (2).

Remark 1. In some sense, our condition goes in the opposite direction to that of [1], in the sense that in our case the coefficients should not vanish at the same point. The simple case $\dot{x} = -x + (\cos^2 t)x^2 + (\sin^2 t)x^3$ is covered by out result but not by [1]. Other difference is that in our case the leading coefficient can have infinitely many zeros, like for instance $\dot{x} = -x + (\cos^2 t)x^2 + t\sin^2(1/t)x^3$.

Proof. First, let us note that the conditions over the coefficients imply that $n \geqslant 3$. Let us perform the change $w(t) = 1/x(t)$. The resulting equation is

$$\dot{w} + a_1(t)w = -a_2(t) - \sum_{k=3}^{n} a_k(t)w^{-k+2}. \tag{3}$$

Note that for the case $n = 3$ this is an Abel equation of the second kind. Such equation has a singularity in 0, but anyway a positive closed solution of (3) is equivalent to a positive closed solution of Eq. (2).

The function

$$G(t,s) = \begin{cases} \frac{1}{1-\sigma} e^{-\int_s^t a_1(\tau)\,d\tau}, & \text{if } 0 \leqslant s < t \leqslant 2\pi, \\ \frac{\sigma}{1-\sigma} e^{-\int_s^t a_1(\tau)\,d\tau}, & \text{if } 0 \leqslant t \leqslant s \leqslant 2\pi, \end{cases} \tag{4}$$

with $\sigma = e^{-\int_0^{2\pi} a(\tau)\,d\tau}$, is the Green's function of the operator $L[x] = x' + a_1(t)x$ with periodic boundary conditions. This function is negative for all (t,s). Then, it is an easy task to verify that $\alpha(t) = -\mu \int_0^{2\pi} G(t,s)\,ds$ is a positive strict lower solution for μ positive and big enough.

Now, let us prove that $\beta(t) = \varepsilon > 0$ is a (constant) strict upper solution if ε is very small. To see this, it is sufficient to verify that

$$a_1(t)\varepsilon + a_2(t) + \sum_{k=3}^{n} a_k(t)\varepsilon^{-k+2} > 0$$

for all t. Multiplying the left-hand side by ε^{n-2} we get

$$\sum_{k=1}^{n} a_k(t)\varepsilon^{n-k} > \sum_{k=1}^{j-1} a_k(t)\varepsilon^{n-k} + \varepsilon^{n-j}\sum_{k=j}^{n} a_k(t) > 0,$$

if ε is small enough.

Besides, these upper and lower solutions can be chosen such that $\beta < \alpha$, so the proof is done by Lemma A.2 of Appendix A. □

The next objective is to derive conditions that do not require any kind of sign conditions in the higher coefficients.

Theorem 2. *Let us write $a_2(t) = \lambda p(t)$, with p a fixed continuous function. Let us assume that $\int_0^{2\pi} a_1(t)\,dt \neq 0$ and that the unique closed solution of the linear equation $x' + a_1(t)x = -p(t)$ is positive. Then, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists at least a positive isolated closed solution of Eq.* (2)*.*

Proof. We use again the change $w(t) = 1/x(t)$. Now Eq. (3) is

$$\dot{w} + a_1(t)w = -\lambda p(t) - \sum_{k=3}^{n} a_k(t)w^{-k+2}. \tag{5}$$

Let us consider the perturbed equation

$$\dot{y} + a_1(t)y = -p(t) - \sum_{k=3}^{n} \epsilon^{k-1} a_k(t)y^{-k+2}. \tag{6}$$

Lemma A.3 gives ϵ_0 such that (6) has an isolated closed solution for any $0 < \epsilon < \epsilon_0$. Besides, if ϵ_0 is taken small enough, such a solution (denoted by y_ϵ) is near to the unique closed solution y_0 of the unperturbed equation $\dot{y} = a_1(t)y + p(t)$, which is positive by assumption. Hence, y_ϵ is positive for ϵ small. Finally, it is straightforward to verify that $w(t) = \epsilon y_\epsilon$ is a (positive) closed solution of (5) with $\lambda = 1/\epsilon$. Hence, the result holds with $\lambda_0 = 1/\epsilon_0$. □

Remark 2. Note that if $\int_0^{2\pi} a_1(t)\,dt \neq 0$, the unique closed solution of $x' + a_1(t)x = -p(t)$ is $-\int_0^{2\pi} G(t,s)p(s)\,ds$ where $G(t,s)$ is given by (4). Hence, this assumption is explicit. For instance, if $\int_0^{2\pi} a_1(t)\,dt < 0$ it is sufficient (but not necessary) that $p(t)$ is positive for all t. On the other hand, let us remark that this result provides large closed solutions, in fact $x(t) = O(\lambda)$.

The same idea works for proving a similar result with a different decomposition of the coefficient.

Theorem 3. *Let us write $a_2(t) = \lambda + p(t)$. If $\int_0^{2\pi} a_1(t)\,dt < 0$, then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists at least a positive isolated closed solution of Eq.* (2).

Proof. After the same change as before, the adequate perturbed equation to be used is

$$\dot{y} + a_1(t)y = -1 - \epsilon p(t) - \sum_{k=3}^{n} \epsilon^{k-2} a_k(t) y^{-k+2}.$$

The rest of the proof is analogous. □

2. The complete case

In the general case, a sign condition over the leading coefficient a_n is sufficient for the existence of a closed solution by a trivial use of constant upper and lower solutions. More generally, it is not hard to derive conditions over the higher coefficients like that of Theorem 1. On the other hand, the following result only imposes conditions in a_0, a_1.

Theorem 4. *Let us assume that $\int_0^{2\pi} a_1(t)\,dt \neq 0$ and let us write $a_0(t) = \lambda p(t)$. Then, there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ there exists at least one isolated closed solution of Eq.* (1).

Proof. Let us write the auxiliary equation

$$\dot{y} = p(t) + a_1(t)y + \sum_{k=2}^{n} a_k(t)\lambda^{k-1} y^k. \tag{7}$$

We are under the assumptions of Lemma A.3, so there exists $\lambda_0 > 0$ such that (7) has a closed solution y_λ for all $0 < \lambda < \lambda_0$. Then, $x(t) = \lambda y_\lambda$ is a closed solution of the original equation (1) with $a_0(t) = \lambda p(t)$. □

3. Some applications to polynomial planar systems

The main motivation to analyze first order polynomial differential equations is the study of the existence and multiplicity of limit cycles of polynomial vector fields in $\mathbb{R}^2$, an interesting and complicate branch in the qualitative theory of differential equations which in particular contains the second part of the 16th problem proposed by Hilbert in 1900.

Let us consider the planar system

$$\dot{x} = \sum_{k=1}^{n} P_k(x,y),$$

$$\dot{y} = \sum_{k=1}^{n} Q_k(x, y), \tag{8}$$

where P_k, Q_k are homogeneous polynomials of degree k. An isolated periodic orbit of system (8) is called a *limit cycle*.

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the system (8) is transformed to

$$\dot{r} = \sum_{k=1}^{n} f_k(\theta) r^k,$$
$$\dot{\theta} = \sum_{k=1}^{n} g_k(\theta) r^{k-1}, \tag{9}$$

where

$$f_k(\theta) = \cos\theta\, P_k(\cos\theta, \sin\theta) + \sin\theta\, Q_k(\cos\theta, \sin\theta),$$
$$g_k(\theta) = \cos\theta\, Q_k(\cos\theta, \sin\theta) - \sin\theta\, P_k(\cos\theta, \sin\theta). \tag{10}$$

In some particular cases, this system can be rewritten as a single equation like (1) with independent variable θ. Then, an isolated closed solution of (1) would correspond to a limit cycle of (8). The results obtained in the previous sections will be used in order to study some particular examples. The first example belongs to the class of *rigid systems* (see [5,6] and their references).

Proposition 1. *Let us consider the system*

$$\dot{x} = ax - cy + \sum_{k=1}^{n-1} x F_k(x, y),$$
$$\dot{y} = cx + ay + \sum_{k=1}^{n-1} y F_k(x, y), \tag{11}$$

where $a < 0 < c$ and F_k are homogeneous polynomials of degree k. Let us assume that there exists some $j = 2, \ldots, n-1$ such that $F_k(\cos\theta, \sin\theta) \geqslant 0$ for all $k = j, \ldots, n$ and $\theta \in [0, 2\pi]$, and moreover $\sum_{k=j}^{n-1} F_k(\cos\theta, \sin\theta) > 0$ for all $\theta \in [0, 2\pi]$. Then, the system (11) *has at least a limit cycle.*

Proof. The system in polar coordinates reads

$$\dot{r} = ar + \sum_{k=2}^{n} F_{k-1}(\cos\theta, \sin\theta) r^k,$$
$$\dot{\theta} = c. \tag{12}$$

By taking r as a function of θ we get the single differential equation

$$\frac{dr}{d\theta} = \frac{a}{c} r + \frac{1}{c} \sum_{k=2}^{n} F_{k-1}(\cos\theta, \sin\theta) r^k,$$

and the result follows from a direct application of Theorem 1. □

Our results are also applicable to different systems as it is illustrated by the following proposition.

Proposition 2. *Let us consider the system*

$$\dot{x} = \lambda x + P_{m+1}(x, y) + \sum_{k=2}^{n} x F_{mk}(x, y),$$
$$\dot{y} = \lambda y + Q_{m+1}(x, y) + \sum_{k=2}^{n} y F_{mk}(x, y), \tag{13}$$

where $m \in \mathbb{N}$ *and* F_{mk} *are homogeneous polynomials of degree* mk. *If* $g_{m+1}(\theta) > 0$ (< 0) *for all* $\theta \in [0, 2\pi]$ *and*

$$\int_0^{2\pi} \frac{f_{m+1}(\theta)}{g_{m+1}(\theta)}\, d\theta \neq 0,$$

then there exists $\lambda_0 > 0$ *such that* (13) *has at least a limit cycle for all* $0 < \lambda < \lambda_0$.

Remark 3. The formulas (10) provide an equivalence between f_k, g_k and P_k, Q_k, so it is very easy to find explicit examples of P_{m+1}, Q_{m+1} verifying the required conditions.

Proof. In this case, the system in polar coordinates reads

$$\dot{r} = \lambda r + f_{m+1}(\theta) r^{m+1} + \sum_{k=2}^{n} F_{mk}(\cos\theta, \sin\theta) r^{mk+1},$$
$$\dot{\theta} = g_{m+1}(\theta) r^m. \tag{14}$$

By taking r as a function of θ we get the single differential equation

$$\frac{dr}{d\theta} = \frac{1}{g_{m+1}(\theta) r^m} \left[\lambda r + f_{m+1}(\theta) r^{m+1} + \sum_{k=2}^{n} F_{mk}(\cos\theta, \sin\theta) r^{mk+1} \right].$$

Finally, the change $R = r^m$ leads to

$$\frac{dR}{d\theta} = \frac{m}{g_{m+1}(\theta)} \left[\lambda + f_{m+1}(\theta) R + \sum_{k=2}^{n} F_{mk}(\cos\theta, \sin\theta) R^k \right],$$

which is a differential equation of type (1). Now, the result follows from a direct application of Theorem 4. □

Finally, we consider a system without linear part.

Proposition 3. *Let us consider the system*

$$\dot{x} = P_{m+1}(x, y) + \lambda x F_{2m}(x, y) + \sum_{k=3}^{n} x F_{mk}(x, y),$$
$$\dot{y} = Q_{m+1}(x, y) + \lambda y F_{2m}(x, y) + \sum_{k=3}^{n} y F_{mk}(x, y), \tag{15}$$

where $m \in \mathbb{N}$ and F_{mk} are homogeneous polynomials of degree mk. Let us assume the following conditions:

(1) $g_{m+1}(\theta) > 0$ (< 0) *for all* $\theta \in [0, 2\pi]$,
(2) $\int_0^{2\pi} a_1(\theta)\,d\theta \neq 0$, *being* $a_1(\theta) = \frac{m f_{m+1}(\theta)}{g_{m+1}(\theta)}$,
(3) *the function*

$$M(\theta) = -\int_0^{2\pi} G(\theta, s) \frac{F_{2m}(\cos s, \sin s)}{g_{m+1}(s)}\,ds$$

is positive for all $\theta \in [0, 2\pi]$, *where* $G(\theta, s)$ *is the Green's function of the operator* $L[x] = \dot{x} + a_1(\theta)x$ *with periodic conditions given by* (4).

Then, there exists $\lambda_0 > 0$ *such that* (15) *has at least a limit cycle for all* $\lambda > \lambda_0$.

Proof. By using the same change $R = r^m$ as in the previous proposition, we arrive to the equation

$$\frac{dR}{d\theta} = \frac{m}{g_{m+1}(\theta)}\left[f_{m+1}(\theta)R + \lambda F_{2m}(\cos\theta, \sin\theta) + \sum_{k=2}^{n} F_{mk}(\cos\theta, \sin\theta)R^k \right],$$

and the result is direct from Theorem 2. □

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Appendix A

For the sake of completeness we include here some known results used in the proofs. Let us consider a first order differential equation

$$\dot{x} = f(t, x) \tag{A.1}$$

with $f : [0, 2\pi] \times I \to \mathbb{R}$, being I a given open interval of $\mathbb{R}$.

Lemma A.1. *If f is continuous in t and analytic in x, one of the following alternatives hold*:

(i) *Every solution is closed.*
(ii) *Every possible closed solution is isolated.*

Proof. This is a trivial consequence of the analyticity of the Poincaré map. □

Definition A.1. A function $\alpha \in C^1([0, 2\pi])$ is called a *strict lower solution* of the problem

$$x' = f(t, x), \qquad x(0) = x(2\pi)$$

if $\dot{\alpha} < f(t, \alpha(t))$ for all $t \in [0, 2\pi]$ and $\alpha(0) \leqslant \alpha(2\pi)$. A function $\beta \in C^1([0, 2\pi])$ is called a *strict upper solution* if the previous inequalities are reversed.

Lemma A.2. *Let us assume that f is continuous in t and analytic in x. If there exists a couple of strict lower and upper solutions α, β such that $\alpha(t) < \beta(t)$ (respectively $\alpha(t) > \beta(t)$) for all t, then Eq.* (A.1) *has at least one isolated closed solution x such that $\alpha(t) < x(t) < \beta(t)$ (respectively $\alpha(t) > x(t) > \beta(t)$) for all t.*

Proof. We can fix the inequality $\alpha(t) < \beta(t)$, since the reasonings for the contrary inequality are analogous. It is a very known result ([8,10] are classical references, see also [11] for a complete review) that under these conditions there exists minimal and maximal closed solutions $x_1(t), x_2(t)$ (possibly the same) such that $\alpha(t) < x_1(t) \leqslant x_2(t) < \beta(t)$ for all $t \in [0, 2\pi]$. Any other closed solution in the interval $[\alpha, \beta]$ should belong to $[x_1, x_2]$. In particular, this implies that not all the solutions are closed (take an initial condition $\alpha(0) < x_0 < x_1(0)$, by continuity the solution remains in $[\alpha, \beta]$ and it is not closed because then it should belong to $[x_1, x_2]$. Then, the analytic character of f implies that the closed solutions are isolated. □

The finding of adequate upper and lower solutions is sometimes very tricky. In this paper, we have used the following perturbation result.

Lemma A.3. *Let us consider the differential equation*

$$\dot{x} + a(t)x = b(t) + \epsilon c(t, x, \epsilon) \tag{A.2}$$

where $a, b : [0, 2\pi] \to \mathbb{R}$ are continuous functions and $c : [0, 2\pi] \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function with continuous derivatives in x, ϵ (I is a given open interval of $\mathbb{R}$). Let us assume that $\int_0^{2\pi} a(t)\,dt \neq 0$ and the unique solution x_0 of the unperturbed equation $\dot{x} + a(t)x = b(t)$ is such that $x_0(t) \in I$ for all $t \in [0, 2\pi]$. Then, there exist $\epsilon_0, \rho > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists a unique closed solution x_ϵ of Eq. (A.2) *in the ball $B[x_0, \rho]$ in $C^0([0, 2\pi])$. Moreover, x_ϵ depends continuously on ϵ and*

$$\lim_{\epsilon \to 0} x_\epsilon = x_0$$

uniformly in t.

Proof. It is known that $\int_0^{2\pi} a(t)\,dt \neq 0$ is the non-resonance condition of the linear part. Then, this result is a particular case of a classical result, see for instance [12, Corollary 1.11]. □

References

[1] K.M. Andersen, A. Sandqvist, Existence of closed solutions of an equation $\dot{x} = f(t, x)$ where $f'_x(t, x)$ is weakly convex on concave in x, J. Math. Anal. Appl. 229 (1999) 480–500.

[2] J. Devlin, N.G. Lloyd, J.M. Pearson, Cubic systems and Abel equations, J. Differential Equations 147 (1998) 435–454.

[3] A. Gasull, A. Guillamon, Limit cycles for generalized Abel equations, preprint.

[4] A. Gasull, J. Llibre, Limit cycles for a class of Abel equations, SIAM J. Math. Anal. 21 (1990) 1235–1244.

[5] A. Gasull, R. Prohens, J. Torregrosa, Limit cycles for rigid cubic systems, J. Math. Anal. Appl. 303 (2005) 391–404.

[6] A. Gasull, J. Torregrosa, Exact number of limit cycles for a family of rigid systems, Proc. Amer. Math. Soc. 133 (2005) 751–758.

[7] J. Giné, J. Llibre, Darboux integrability and limit cycles for a class of polynomial differential systems, Differential Equations with Symbolic Computation, Birkhäuser, in press.

[8] V. Lakshmikantham, S. Leela, Existence and monotone method for periodic solutions of first-order differential equations, J. Math. Anal. Appl. 91 (1983) 237–243.

[9] I. Malta, N.C. Saldanha, C. Tomei, Morin singularities and global geometry in a class of ordinary differential operators, Topol. Methods Nonlinear Anal. 10 (1997) 137–169.
[10] M.N. Nkashama, A generalized upper and lower solutions method and multiplicity results for nonlinear first-order ordinary differential equations, J. Math. Anal. Appl. 140 (1989) 381–395.
[11] F. Obersnel, P. Omari, Old and new results for first order periodic ODEs without uniqueness: A comprehensive study by lower and upper solutions, Adv. Nonlinear Stud. 4 (2004) 323–376.
[12] N. Rouche, J. Mawhin, Équations différentielles ordinaries, Tome II: Stabilité et solutions périodiques, Masson et Cie, 1973.