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Existence and stability of periodic solutions for second-order semilinear differential equations with a singular nonlinearity

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It is proved that a periodically forced second-order equation with a singular nonlinearity in the origin with linear growth in infinity possesses a T-periodic stable solution for high values of the mean value of the forcing term. The method of proof combines a rescaling argument with the analysis of the first twist coefficient of the Birkhoff normal form for the Poincaré map.

1. Introduction

In this paper we are concerned with the existence and stability of positive T-periodic solutions of the semilinear singular equation

$$x'' + a(t)x = \frac{b(t)}{x^{\alpha}} + p(t), \tag{1.1}$$

with $a, b, p \in L^1[0, T]$ and $\alpha > 0$. Interest in scalar equations with singularities began in the early 1960s with the work of Forbat and Huaux [7, 10], where the singular nonlinearity models the restoring force caused by a compressed perfect gas (see [17, § 6] for a more complete list of references). Later, interest in this problem increased with publication of the paper by Lazer and Solimini [12]. This work is a benchmark for the problem and since its publication many researchers have focused their attention on the study of singular equations, in such a way that a complete bibliography is out of the scope of this work. We refer the reader to [2, 3, 5, 6, 8, 21, 24, 27, 29, 30] as a brief list of examples.

Our study is mainly motivated by [3], where it is proven that the equation

$$x'' + \mu x = \frac{1}{x^{\alpha}} + p(t) \tag{1.2}$$

possesses a T-periodic solution for all $\alpha \geqslant 1$ and $\mu \neq (k\pi/T)^2$ for all $k \in \mathbb{Z}$. Lazer and Solimini's result [12] corresponds to the first 'resonant case', $\mu = 0$, showing that the additional condition $\bar{p} < 0$ is necessary and sufficient for existence. From now on, let us set $\mu_k = (k\pi/T)^2$. Remark 2.1 in [3] states explicitly the open problem of finding additional conditions on p in order to ensure the existence of T-periodic solutions in the cases $\mu = \mu_k$ for $k \in \mathbb{Z}$. For the next critical value, μ_1, a first solution was given in [21], where it is proven that it is sufficient that

p is positive. It is remarkable that the result in [21] does not require the 'strong force' assumption $\alpha \geqslant 1$, in contrast to that in [3]. Meanwhile, for the specific case $\alpha = 3$, a more general condition is found in [1] by using the isochronous character of the potential. On the other hand, and contrasting with the number of papers studying the existence of periodic solutions in singular equations, the stability of such solutions has been 'terra incognita' for a long time, and only very recent works [25, 26, 29] can be found in the literature.

It is interesting to observe that the sequence of critical values μ_k is composed of the eigenvalues of the Dirichlet and periodic boundary-value problems (BVPs) of the linear part in alternating order. For instance, μ_1 is the first positive eigenvalue of the Dirichlet BVP, whereas μ_2 is the first positive eigenvalue of the periodic BVP, and so on. The frequent appearance of these eigenvalues in the related literature is relevant. There is an interesting discussion in [30] about the role of the eigenvalues for the Dirichlet BVP in the context of singular equations. Intuitively speaking, this is justified by arguing that high-energy periodic solutions of singular equations look like solutions of the Dirichlet problem. However, if additional conditions are assumed in order to get an adequate distance of the periodic solution from the singularity, it is intuitively reasonable to expect the attainment of new sufficient conditions at least in the cases $\mu = \mu_{2k+1}$ for $k \in \mathbb{Z}$. This is our main task. In the next section, a classical perturbation result and a simple rescaling argument lead to the existence and uniqueness of periodic solution of (1.1) for high values of the mean value of p. Section 3 is devoted to the study of the Lyapunov stability of such a solution for equation (1.2) by proving that the first twist coefficient of the Birkhoff normal form of the Poincaré map is not zero. Additionally, classical arguments in Kolmogorov–Arnold–Moser (KAM) theory provide the typical KAM scenario around the solution, including subharmonics, quasiperiodic solutions (invariant tori) and Smale horseshoes.

2. Existence of periodic solutions

Given $p \in L^1[0, T]$, its mean value is denoted by

$$\bar{p} = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t.$$

Let us denote by $\tilde{L}^1[0, T]$ the subspace of $L^1[0, T]$ composed by the functions of mean value zero. The main result of this section is the following.

Theorem 2.1. *Let us assume that*

(H1) *the Hill equation $x'' + a(t)x = 0$ is non-resonant, that is, it does not have non-trivial T-periodic solutions;*

(H2) *the Green function $G(t, s)$ of the equation $x'' + a(t)x = 0$ with periodic conditions verifies*

$$\int_0^T G(t, s)\,\mathrm{d}s > 0, \quad \text{for all } t.$$

Then, for any $\tilde{p} \in \tilde{L}^1[0,T]$ *there exists* $P_0 > 0$ *such that the equation*

$$x'' + a(t)x = \frac{b(t)}{x^\alpha} + \bar{p} + \tilde{p}(t), \tag{2.1}$$

possesses at least a positive T*-periodic solution for any* $\bar{p} > P_0$.

Before the proof, we give some remarks and direct consequences of this theorem.

REMARK 2.2. Note that

$$x_0(t) = \int_0^T G(t,s)\,\mathrm{d}s$$

is the unique T-periodic solution of $x'' + a(t)x = 1$, so hypothesis (H2) imposes its positiveness. If $a(t) = \mu > 0$ as in (1.2), then (H1) holds if and only if $\mu \neq \mu_{2k}$, $k \in \mathbb{Z}$. In this case, (H2) is trivially verified, since $x_0 = 1/\mu$. Hence, we are giving a partial solution to the open problem posed in [3] for those critical values that are eigenvalues of the Dirichlet BVP.

REMARK 2.3. For a positive non-constant coefficient $a(t)$, it is more difficult to find effective criteria ensuring (H1) and (H2) hold. Of course, a maximum principle (that is, Green's function exists and is positive) is a sufficient condition for the verification of (H1) and (H2). For instance, a maximum principle with L^p-conditions in $a(t)$ is proved in [27, 28]. More concretely, $\|a\|_p$ must be less than or equal to a given best Sobolev constant. This maximum principe generalizes the classical L^∞-norm condition (which would read $0 < a(t) < \mu_1$ for all t) and makes possible the attainment of unbounded coefficients a (so a is crossing all the critical values μ_k) for which the thesis of theorem 2.1 holds.

REMARK 2.4. It is interesting to remark that no sign condition over $b(t)$ is required, so we are covering singularities whose effect changes in time from attractive to repulsive type. This type of nonlinearity has scarcely been considered in the related literature, in spite of its use in physical applications like some electromagnetic trapping mechanisms for cold neutral atoms similar to the Paul trap (see [11, 14]). It is also remarkable that any kind of 'strong force' condition is not necessary in the singularity.

The proof of theorem 2.1 relies on the following result.

LEMMA 2.5. *Let us consider the perturbed equation*

$$y'' + a(t)y = \varepsilon f(t, y, \varepsilon) + 1, \tag{2.2}$$

where $f : [0,T]\times\mathbb{R}^+\times\mathbb{R} \to \mathbb{R}$ *is a Carathéodory function with continuous derivatives with respect to* y *and* ε*. Under conditions (H1) and (H2), there exists* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$*, equation (2.2) has a* T*-periodic solution* y_ε*. Moreover,*

$$\lim_{\varepsilon\to 0^+} y_\varepsilon(t) = \int_0^T G(t,s)\,\mathrm{d}s \equiv y_0 \quad \textit{uniformly in } t. \tag{2.3}$$

This perturbation result is a particular case of a well-known theorem that can be found in many classical texts (see, for example, [9, theorem 3.7] or [22, corollary 1.11]).

With the help of this lemma, it is easy to prove theorem 2.1.

Proof. Take $f(t, y, \varepsilon) = \varepsilon^{1+\alpha} b(t)/x^\alpha + \varepsilon \tilde{p}(t)$. There then exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, there exists a positive T-periodic solution y_ε of equation (2.2). Now, it is not hard to verify that $x_\varepsilon(t) = y_\varepsilon(t)/\varepsilon$ is a T-periodic solution of equation (2.1) with $\bar{p} = 1/\varepsilon$. By defining $P_0 = 1/\varepsilon_0$, the proof is complete. □

Note that the uniqueness is trivial for lemma 2.5 since two different solutions of (2.1) for a given $\bar{p} > P_0$ would be equivalent to two different solutions of the perturbed equation (2.2), contradicting the lemma.

Remark 2.6. Under the assumption (H1), let us suppose that the unique T-periodic solution of $x'' + a(t)x = p(t)$ is positive. Then, the same proof technique provides the existence of a unique T-periodic solution of the equation

$$x'' + a(t)x = \frac{b(t)}{x^\alpha} + sp(t),$$

for any s greater than a given constant S_0. This type of forcing term was considered in [4] for a non-singular equation.

3. Stability of periodic solutions

In this section, the solution given by theorem 2.1 for a fixed $\tilde{p} \in \tilde{L}^1[0, T]$ is denoted by $\gamma(t, \bar{p})$. A consequence of the method of proof together with (2.3) is that

$$\lim_{\bar{p} \to +\infty} \gamma(t, \bar{p}) = +\infty \quad \text{uniformly in } t. \tag{3.1}$$

In fact, $\gamma(t, \bar{p}) = O(\bar{p})$. This property will be crucial in the study of stability.

In a general conservative Newtonian equation $x'' + g(t, x) = 0$ with a T-periodic dependence on t, the first-order approximation is not sufficient for us to decide about the Lyapunov stability of a given T-periodic solution $\gamma(t)$. However, it is known that if the solution is not resonant up to order four (also called 4-elementary [20]), the third-order approximation determines in most of the cases the Lyapunov stability of the periodic solution. Although the link between the third-order approximation and the stability of solutions goes back at least to the time of Moser, a series of papers by Ortega [18–20] were a breakthrough from the point of view of applications due to the finding of an explicit formula for the first coefficient of the Birkhoff normal form of the Poincaré map, also called the twist coefficient. In the following, we will summarize some basic aspects of this relation.

Let us consider the third-order expansion

$$x'' + c_1(t)x + c_2(t)x^2 + c_3(t)x^3 + \cdots = 0,$$

where the coefficients

$$c_1(t) = g_x(t, \gamma(t)), \quad c_2(t) = \tfrac{1}{2} g_{xx}(t, \gamma(t)), \quad c_3(t) = \tfrac{1}{6} g_{xxx}(t, \gamma(t)) \tag{3.2}$$

correspond to the Taylor expansion up to degree three of the function $g(t, \gamma(t) + x)$ around $x = 0$. A Hill equation $x'' + a(t)x = 0$ is said to be *4-elementary* if it is stable and its Floquet multipliers verify that $\lambda^p \neq 1$, $1 \leqslant p \leqslant 4$. The T-periodic solution γ is called *4-elementary* if the linearized equation $x'' + c_1(t)x = 0$ is 4-elementary.

Then, the rotation number θ is defined by the relation $\lambda = \mathrm{e}^{\pm \mathrm{i}\theta}$. Such a solution is said to be *of twist type* if the twist coefficient

$$\beta = -\frac{3}{8}\int_0^T c_3(t)r^4(t)\,\mathrm{d}t + \iint_{[0,T]^2} c_2(t)c_2(s)r^3(t)r^3(s)\chi(|\varphi(t)-\varphi(s)|)\,\mathrm{d}t\,\mathrm{d}s \quad (3.3)$$

is non-zero, where $\Psi(t) = r(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$ is the (complex) solution of the linearized equation $x'' + c_1(t)x = 0$ with initial conditions $\Psi(0) = 1$, $\Psi'(0) = \mathrm{i}$ and the kernel χ is given by

$$\chi(x) = \frac{3}{16}\frac{\cos(x-\theta/2)}{\sin(\theta/2)} + \frac{1}{16}\frac{\cos 3(x-\theta/2)}{\sin 3\pi/2}, \quad x \in [0,\theta].$$

This formulation is a compact form, obtained in [31] (see also [15,16]), of the original Ortega's formula [20]. As a consequence of Moser's invariant curve theorem [23], a solution of twist type is Lyapunov stable. Additionally, from basic facts in KAM theory and the Poincaré–Birkhoff's fixed point theorem it follows that around a solution of twist type the typical KAM dynamics arises.

After these preliminaries, let us state the main result of this section.

THEOREM 3.1. *Let us assume that the Hill equation $x'' + a(t)x = 0$ is 4-elementary and $b(t) > 0$ for almost everywhere t. Then, for any $\tilde{p} \in \tilde{L}^1[0,T]$ there exists $P_1 > P_0$ (the constant given in theorem 2.1) such that, for any $\bar{p} > P_1$, the positive T-periodic solution $\gamma(t,\bar{p})$ of equation (2.1) is of twist type.*

Proof. Let us fix

$$g(t,x) = a(t)x - \frac{b(t)}{x^\alpha} + p(t).$$

Let us recall that $\gamma(t,\bar{p}) = O(\bar{p})$. Hence, the first coefficient of the third-order expansion is

$$c_1(t) = a(t) + \frac{\alpha b(t)}{\gamma(t,\bar{p})^{\alpha+1}} = a(t) + O\biggl(\frac{1}{\bar{p}^{\alpha+1}}\biggr).$$

As a consequence of the continuity of the Floquet multipliers with respect to the coefficients, if $x'' + a(t)x = 0$ is 4-elementary, then $x'' + c_1(t)x = 0$ is also 4-elementary for high values of $\bar{p}$. Therefore, it remains only to compute the sign of β given by (3.3).

In the following, it will be useful to specify the dependence of the coefficients with respect to $\bar{p}$ by writing $c_i(t,\bar{p})$. Let $\Psi(t) = r(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$ be the solution of $x'' + a(t)x = 0$ with initial conditions $\Psi(0) = 1$, $\Psi'(0) = \mathrm{i}$. Let us note that $r(t)$ is a positive periodic function (in fact, it is the unique periodic solution of the associated Ermakov–Pinney equation; see [15] for more details). If $\Psi(t,\bar{p}) = r(t,\bar{p})\mathrm{e}^{\mathrm{i}\varphi(t,\bar{p})}$ is the solution of $x'' + c_1(t)x = 0$ with the same initial conditions, it is then easy to deduce from $c_1(t,\bar{p}) = a(t) + O(1/\bar{p}^{\alpha+1})$ the existence of uniform bounds M_1 and M_2 (not depending on $\bar{p}$) such that

$$M_1 < r(t,\bar{p}) < M_2$$

for all t and high values of $\bar{p}$.

On the other hand, there exists $K > 0$ such that

$$c_2(t,\bar{p}) = -\frac{\alpha(\alpha+1)b(t)}{\gamma(t,\bar{p})^{\alpha+2}} = O\left(\frac{1}{\bar{p}^{\alpha+2}}\right),$$

$$c_3(t,\bar{p}) = \alpha(\alpha+1)(\alpha+2)\frac{b(t)}{\gamma(t,\bar{p})^{\alpha+3}} > \frac{K}{\bar{p}^{\alpha+3}}.$$

Now, it is possible to estimate β in (3.3), obtaining

$$\beta \leqslant -\frac{3}{8}\frac{KT\bar{b}}{\bar{p}^{\alpha+3}} + O\left(\frac{1}{\bar{p}^{2\alpha+4}}\right).$$

As $\alpha + 3 < 2\alpha + 4$ for all $\alpha > 0$, the twist coefficient β is negative for high values of $\bar{p}$, completing the proof. □

Remark 3.2. Of course, the condition that $x'' + a(t)x = 0$ is 4-elementary is stronger than (H1) and (H2) in theorem 2.1. If $a(t) = \mu$, as in (1.2), this condition means that

$$\mu \neq \left(\frac{k\pi}{mT}\right)^2 \quad \text{for all } n, m \in \mathbb{Z} \text{ with } 1 \leqslant m \leqslant 4,$$

which in particular excludes the critical values μ_k. On the other hand, as far as we know this is the first stability result involving weak singularities.

Remark 3.3. For a non-constant $a(t)$, the concept of *admissible function* introduced in [13] implies that the corresponding Hill equation is 4-elementary (see [13, lemms 3.2]).

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