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L^1 criteria for stability of periodic solutions of a newtonian equation

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Abstract

We develop new criteria for the stability of a periodic solution of a given newtonian equation based on the L^1-norm of the coefficients of the third order approximation, by proving that the twist coefficient is different form zero.

1. *Introduction*

In this paper we develop L^1-criteria for the Lyapunov stability of the trivial solution of the equation

$$\ddot{x} + a(t)x + b(t)x^2 + c(t)x^3 = 0, \tag{1·1}$$

where $a, b, c \in L^1(\mathbb{R} \setminus 2\pi\mathbb{Z})$ are measurable 2π-periodic coefficients. The main motivation to perform this study is to obtain criteria of L^1 type for the stability of a 2π-periodic solution $u(t)$ of the scalar newtonian equation

$$\ddot{x} + f(t, x) = 0, \tag{1·2}$$

where $f(t, x)$ is a Caratheodory function which is 2π-periodic in the first variable and has continuous derivatives in x up to order 4.

Let us consider the variational equation of (1·2) at $x = u(t)$

$$\ddot{x} + f_x(t, u(t))x = 0. \tag{1·3}$$

If (1·3) has Floquet multipliers λ_1, λ_2 such that $|\lambda_i| = 1, \lambda_i \neq \pm 1, i = 1, 2$, then $u(t)$ is said *elliptic* or *linearly stable*. However, the Lyapunov stability of $u(t)$ depends essentially on the nonlinear terms of the Taylor expansion of (1·2) around $u(t)$. Following the works of Ortega [**5**, **6**, **7**], it is clear that in most cases the stability of $u(t)$ can be determined from the third order approximation (1·1) of equation (1·2),

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where the coefficients are given by

$$a(t) = f_x(t, u(t)), \quad b(t) = \frac{1}{2} f_{xx}(t, u(t)), \quad c(t) = \frac{1}{6} f_{xxx}(t, u(t)). \tag{1·4}$$

More precisely, Ortega succeed in computing an explicit formula for the twist coefficient $\beta = \beta(a, b, c)$ of (1·1), later reformulated in [**3**] (see (4·1), (4·2)). This coefficient corresponds to the first nonlinear term in the Birkhoff normal form of the Poincaré map. If β is different from zero, we say that u is of *twist type*, and Moser Twist Theorem [**9**] implies that such a solution is Lyapunov stable. Generically, a solution of twist type has also a complicated dynamics in its behaviour, arising from the typical self-similar KAM scenario (including subharmonics with periods tending to infinity and quasiperiodic solutions). In general, the stability of the zero solution of (1·2) can be presented by the condition such that the twist coefficient β is non-zero. To this end, estimates of the rotation number of the linearized equation

$$\ddot{x} + a(t)\, x = 0 \tag{1·5}$$

and the L^4 norm of $r(t)$, the positive periodic solution of the Ermakov–Pinney equation

$$\ddot{r} + a(t)\, r = \frac{1}{r^3},$$

play a key role, as is shown in [**3**]. In the cited paper, such estimates are obtained provided that uniform bounds of $a(t)$ are known. Results along these lines, with stability criteria basic on uniform bounds of $a(t)$, can be found in [**3**–**7**, **11**, **13**]. However, in a purely Caratheodory context this assumption looks rather unnatural since it is possible that such uniform bounds do not exist. In this sense, [**7**] includes the following result as a particular case.

[**7**, corollary 3·3]. *If there exists a number θ^*, $\pi/2 < \theta^* < 2\pi/3$ such that*
(i) $a(t) > 0$ *for all t, and* $2\pi \int_0^{2\pi} a^+(t)\, dt \leqslant 4(\theta^*/\pi)^2$,
(ii) $b(t) \leqslant 0$ *for all t, or* $b(t) \geqslant 0$ *for all t,*
(iii) $c(t) < 0$ *for all t,*
then $x = 0$ is a solution of twist type of (1·1).

In this paper, we are interested in L^1-conditions like (i). This concrete assumption implies that the rotation number is in the first or second region of stability. Other related result asserts that if $b \equiv 0$, then $x = 0$ is twist type if the linearized equation is elliptic and (iii) holds [**6**]. On the other hand, results in [**3**] allow more flexible L^1-conditions on the nonlinear coefficients b and c, but a is required to be bounded with more restrictive assumptions over such bounds. Our aim is to combine these two situations and get new stability criteria without imposing uniform bounds in the coefficients.

2. *Main results*

Let us consider $a \in L^1(\mathbb{R} \setminus 2\pi\mathbb{Z})$ such that its mean value $\overline{a} = (1/2\pi) \int_0^{2\pi} a(t)\, dt$ is positive. Define positive constants σ, δ, σ_1 and σ_2 by

$$\sigma^2 = \overline{a}, \qquad \delta = \int_0^{2\pi} |a(t) - \sigma^2|\, dt \tag{2·1}$$

and

$$\sigma_1 = \sigma - \frac{\delta}{4\pi\sigma}, \qquad \sigma_2 = \sigma + \frac{\delta}{4\pi\sigma}. \tag{2·2}$$

In addition, $p^+(t) = \max\{p(t), 0\}$, $p^-(t) = \max\{-p(t), 0\}$ is the positive and negative part respectively of a given function $p(t)$. Our first result is the following.

THEOREM 2·1. *Assume that σ_1, σ_2 defined by (2·2) satisfy $\sigma_1, \sigma_2 \in M_0 = (0, 1/4)$. Suppose that $b, c \in L^1(\mathbb{R} \setminus 2\pi\mathbb{Z})$ are such that either*

$$\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^2 \|c_-\|_1 - \|c_+\|_1 > \frac{7}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2} \|b_+\|_1 \|b_-\|_1, \tag{2·3}$$

or

$$\|c_+\|_1 - \left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^2 \|c_-\|_1$$
$$> \frac{\sqrt{2}}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{5/2}\left[\|b_+\|_1^2 + \|b_-\|_1^2 + \frac{1}{2\sqrt{2}}(3\cot\pi\sigma_1 + \cot 3\pi\sigma_1)\|b\|_1^2\right]. \tag{2·4}$$

Then the trivial solution $x = 0$ of (1·1) is of twist type.

To our knowledge, this is the first result available in the literature not imposing some kind of uniform bounds on the coefficients a, b, c. As we will see, the condition $\sigma_1, \sigma_2 \in M_0 = (0, 1/4)$ implies that the rotation number is in the first region of stability.

Our next aim is to get results dealing with higher regions of stability. For that, the set

$$\Omega_0 = \{\omega \in (0, \infty) : \ \omega \neq p/q \ \text{ for all } p, q \in \mathbb{N} \ \text{ with } \ 1 \leqslant q \leqslant 4\} \tag{2·5}$$

will play an important role in our results. Let us begin with a definition.

Definition 2·2. A function $a \in L^1(\mathbb{R} \setminus 2\pi\mathbb{Z})$ is said to be *admissible* if

$$[\sigma_1, \sigma_2] \subset \Omega_0.$$

As we will see in the next section, if a is admissible, then the linear part of (1·1) does not have resonances up to order 4.

For convenience, we will define several functions which are involved in our next results. Let

$$N(\sigma, \sigma_1, \sigma_2) = \frac{\sqrt{2\pi}}{\sigma}\max\left\{\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^{1/2}, \left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2}\right\},$$

$$K_1(\theta) = \frac{\sqrt{2}}{8} + \max\left\{-\frac{3}{16}\cot\left(\frac{\theta}{2}\right), 0\right\} + \max\left\{-\frac{1}{16}\cot\left(\frac{3\theta}{2}\right), 0\right\},$$

$$\tilde{K}_1(\theta) = \frac{\sqrt{2}}{8} + \max\left\{\frac{3}{16}\cot\left(\frac{\theta}{2}\right), 0\right\} + \max\left\{\frac{1}{16}\cot\left(\frac{3\theta}{2}\right), 0\right\},$$

$$K_2(\theta) = \begin{cases} |2 + 3\cos(\theta)|/(8|\sin(3\theta/2)|), & \text{if } \theta \in (0, 2\pi/3) \cup (4\pi/3, 2\pi), \\ |\cos(\theta)|\sqrt{-2\cos(\theta)}/(8|\sin(3\theta/2)|), & \text{if } \theta \in (2\pi/3, 4\pi/3), \end{cases}$$

and $K_2(\theta)$ is extended by 2π-periodicity. Then, $K_1, \tilde{K}_1, K_2$ are well defined in the domain $\Theta_0 = \{\theta > 0 \, : \, \theta \neq 2n\pi/3 \text{ for all } n \in \mathbb{N}\}$. Define $K, \tilde{K} \colon \Theta_0 \to \mathbb{R}$ by

$$K(\theta) = \min\{K_1(\theta),\ K_2(\theta)\}, \qquad \tilde{K}(\theta) = \min\{\tilde{K}_1(\theta),\ K_2(\theta)\}.$$

THEOREM 2·3. *Assume that $a(t) \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ is admissible. Then, there exists a constant $\mu = \mu(\sigma, \sigma_1, \sigma_2) > 0$ such that $x = 0$ as a periodic solution of* (1·1) *is of twist type provided that $b(t)$ and $c(t)$ satisfy*

$$\max_{t \in \mathbb{R}} c(t) < -\mu \|b\|_4^2. \tag{2·6}$$

In fact, μ can be defined as

$$\mu(\sigma, \sigma_1, \sigma_2) := \frac{8}{3} K(2\pi\sigma_2) N(\sigma, \sigma_1, \sigma_2).$$

Evidently, this first result works only if $c(t)$ is negative for all t. However, the opposite sign is also interesting since it arises in several examples of superlinear [**8**] and singular equations [**10**, **11**]. The following result considers this situation.

THEOREM 2·4. *Assume that $a(t) \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ is admissible. Then, there exists a constant $\tilde{\mu} = \tilde{\mu}(\sigma, \sigma_1, \sigma_2) > 0$ such that $x = 0$ as a periodic solution of* (1·1) *is of twist type provided that $b(t)$ and $c(t)$ satisfy*

$$\min_{t \in \mathbb{R}} c(t) > \tilde{\mu} \|b\|_4^2. \tag{2·7}$$

In fact, $\tilde{\mu}$ can be defined as

$$\tilde{\mu}(\sigma, \sigma_1, \sigma_2) := \frac{8}{3} \tilde{K}(2\pi\sigma_2) N(\sigma, \sigma_1, \sigma_2).$$

3. *Estimation of the rotation number*

In this section we consider the Hill's equation

$$\ddot{x} + a(t)x = 0 \tag{3·1}$$

with $a \in L^1(\mathbb{R} \setminus 2\pi\mathbb{Z})$. In the following, we assume that the mean value $\overline{a} = (1/2\pi) \int_0^{2\pi} a(t)\, dt$ of a is positive. Let us consider the constants σ, δ defined by (2·1). Note that δ is a measure of the variation of a with respect to its mean value. Let $x = r\cos\psi, \dot{x} = -r\sin\psi$; then $\psi(t)$ satisfies the equation

$$\dot{\psi} = \cos^2\psi + a(t)\sin^2\psi. \tag{3·2}$$

Since (3·2) is periodic with respect to t and ψ, the limit

$$\rho = \rho(a) = \lim_{t \to \infty} \psi(t)/t$$

exists and is independent to the choice of the solution $\psi(t)$ [**1**]. This limit is called *rotation number* of (3·1). The relationship between the rotation number and the Floquet multipliers of (3·1) is given in the following Lemma [**2**].

LEMMA 3·1. *Equation* (3·1) *is elliptic if and only if $\rho = \rho(a) \notin \frac{1}{2}\mathbb{Z}^+$. In this case, the Floquet multipliers are given by $\lambda_{1,2} = e^{\pm i\theta}$, with θ given by*

$$\theta = 2\pi\rho.$$

An elliptic equation (3·1) is called 4-*elementary* if its Floquet multipliers $\lambda = e^{\pm i\theta}$ are not roots of unity up to order 4, that is, $\lambda^n \neq 1$ for $1 \leqslant q \leqslant 4$. In terms of the rotation number, this is equivalent to

$$\rho \in \mathbf{\Omega}_0,$$

where $\mathbf{\Omega}_0$ was defined in (2·5). Note that $\mathbf{\Omega}_0$ consists of a countable number of open intervals.

Lemma 3·2. *If a is admissible then equation* (3·1) *is* 4-*elementary.*

Proof. By the change of variables

$$x(t) = \rho(t)\cos\phi(t)/\sigma,\ \dot{x}(t) = -\sigma\rho(t)\sin\phi(t)/\sigma,$$

it is seen that $\phi(t)$ satisfies

$$\dot{\phi} = \sigma^2 \sin^2 \phi/\sigma + a(t)\cos^2\phi/\sigma. \tag{3·3}$$

Let us denote by $\phi(t;\phi_0)$ the solution of (3·3) with initial value $\phi(0;\phi_0) = \phi_0$. It is easy to prove that there is ϕ_0 such that

$$\rho = \frac{\phi(2\pi;\phi_0) - \phi_0}{2\,\pi\,\sigma}.$$

By using the inequality

$$\min\{\sigma^2, a(t)\} \leqslant \sigma^2\sin^2\phi/\sigma + a(t)\cos^2\phi/\sigma \leqslant \max\{\sigma^2, a(t)\},$$

after an integration in equation (3·3) we get

$$\int_0^{2\pi} \min\{\sigma^2, a(t)\}\,dt \leqslant \phi(2\pi;\phi_0) - \phi_0 \leqslant \int_0^{2\,\pi} \max\{\sigma^2, a(t)\}\,dt. \tag{3·4}$$

Let us define

$$I = \{t \in [0, 2\pi]\ |a(t) \leqslant \sigma^2\},\quad J = [0, 2\pi]\backslash I.$$

Then

$$\begin{aligned}\int_0^{2\pi} \max\{\sigma^2, a(t)\}\,dt &= \int_I \sigma^2\,dt + \int_J a(t)\,dt\\ &= 2\pi\sigma^2 + \int_J (a(t) - \sigma^2)\,dt,\\ \int_0^{2\pi} \min\{\sigma^2, a(t)\}\,dt &= \int_J \sigma^2\,dt + \int_I a(t)\,dt\\ &= 2\pi\sigma^2 - \int_I (\sigma^2 - a(t))\,dt.\end{aligned}$$

On the other hand, it is easy to verify that

$$\int_J (a(t) - \sigma^2)\,dt = \int_I (\sigma^2 - a(t))\,dt = \frac{\delta}{2}.$$

Therefore, (3·4) gives

$$2\pi\sigma^2 - \frac{\delta}{2} \leqslant \phi(2\pi;\phi_0) - \phi_0 \leqslant 2\pi\sigma^2 + \frac{\delta}{2}$$

for any ϕ_0. dividing by $2\pi\sigma$, we see that the rotation number satisfies

$$\rho \in [\sigma_1, \sigma_2] \subset \Omega_0$$

and consequently the proof is complete.

4. *Estimation of the twist coefficient*

Let $\Psi(t)$ be the (complex) solution of Hill's equation (3·1) with initial conditions $\Psi(0) = 1, \dot{\Psi}(0) = i$. When (3·1) is elliptic, $\Psi(t)$ is different from zero for all t, so it can be written in polar coordinates as $\Psi(t) = r(t)e^{i\varphi(t)}$, with r, φ satisfying the initial conditions

$$r(0) = 1, \quad \dot{r}(0) = 0, \quad \varphi(0) = 0, \quad \dot{\varphi}(0) = 1,$$

and $r(t) > 0$ for all t. Let us assume that Hill's equation (3·1) is 4-elementary. When (3·1) comes from the linearization of a nonlinear equation (1·2) around a given periodic solution $u(t)$, it is said that such a solution is *out of strong resonances*. In this situation, the twist character of $u(t)$ can be determined by proving that a given twist coefficient β is not zero. Such β depends on the coefficients of the third order approximation (1·1) and can be written explicitly, as it was shown by Ortega in [**5**, **7**]. After some changes of variables and reformulations [**3**], β can be written as

$$\beta = -\frac{3}{8}\int_0^{2\pi} c(t)r^4(t)\,dt + \int\int_{[0,2\pi]^2} b(t)b(s)r^3(t)r^3(s)\chi_1(|\varphi(t)-\varphi(s)|)\,dt\,ds$$
$$+\frac{3}{16}\cot\left(\frac{\theta}{2}\right)\left|\int_0^{2\pi} b(t)r^3(t)e^{-i\varphi(t)}dt\right|^2 + \frac{1}{16}\cot\left(\frac{3\theta}{2}\right)\left|\int_0^{2\pi} b(t)r^3(t)e^{3i\varphi(t)}dt\right|^2, \tag{4·1}$$

or, equivalently,

$$\beta = -\frac{3}{8}\int_0^{2\pi} c(t)r^4(t)\,dt + \int\int_{[0,2\pi]^2} b(t)b(s)r^3(t)r^3(s)\chi_2(|\varphi(t)-\varphi(s)|)\,dt\,ds, \tag{4·2}$$

where the kernels χ_1 and χ_2 are given by

$$\chi_1(x) = \frac{3\sin x - 2\sin^3 x}{8}, \quad x \in [0,\theta],$$
$$\chi_2(x) = \frac{3}{16}\frac{\cos(x-\theta/2)}{\sin(\theta/2)} + \frac{1}{16}\frac{\cos 3(x-\theta/2)}{\sin(3\theta/2)}, \quad x \in [0,\theta].$$

More precisely, the original twist coefficient is the previous one up to multiplication by a constant, which is not important for us. At this point, it is necessary to emphasize that this formulation can be translated to the Caratheodory context without change. In order to estimate β, the key point is to control the range of $r(t)$. There is a nice relation, proved in [**3**], between $r(t)$, the Ermakov–Pinney equation and the Ricatti equation.

LEMMA 4·1. *Let us assume that Hill's equation* (3·1) *is elliptic. Then* $r(t)$ *is the unique positive periodic solution of the Ermakov–Pinney equation*

$$\ddot{r} + a(t)r = r^{-3}. \tag{4·3}$$

Moreover, $w(t) = -\frac{\dot r}{r} - \frac{i}{r^2}$ *is a periodic solution of the Riccati equation*

$$\dot w = w^2 + a(t). \tag{4·4}$$

Hence, to estimate the norm of the periodic solution $r(t)$ of the Ermakov–Pinney equation, we only need to fix a bound for the imaginary part of the periodic solution of the corresponding Riccati equation. For this purpose, we shall find an estimate for the critical values of $r(t)$. Let t_0 be critical point of $r(t)$, and let $r_0 = r(t_0)$ be the critical value; then $w(t_0) = -i/r_0^2$. It is known that the critical value $w(t_0)$ is exactly the fixed point of the Poincaré map of the Riccati equation (4·4). Hereinafter, we will assume without loss of generality that $t_0 = 0$. Let us denote by $w(t; z)$ the solution of (4·4) with initial value $w(0) = z$. Then $w(t; z)$ is well defined for all $t \in I\!R$, with values in the Riemman surface. It is known that the Poincaré map of the Riccati equation has the form of a Möbius transformation

$$T(z) = w(2\pi; z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbf{R}.$$

Then, from the above discussion, $z_0 = -i/r_0^2$ is a fixed point of T. On the other hand, since z_0 is imaginary, we have $a = d$ and $b/c < 0$. When $a = d = 0$, we have $w(2\pi; 0) = T(0) = \infty$, which is excluded in our discussion. Hence, we can assume that $a = d = 1$, and $r_0 = (-c/b)^{1/4}$. To estimate r_0, we only need to find some estimates for b and c. Note that $b = T(0)$ and $c = -1/T(\infty)$. In the following Proposition, we use $\langle a, b\rangle$ to denote $[a, b]$ if $a \leqslant b$ or $[b, a]$ if $a \geqslant b$.

PROPOSITION 4·2. *Let us assume that* $a(t)$ *is admissible in the sense of the previous section. Then, the positive* 2π*-periodic solution* $r(t)$ *of* (4·3) *satisfies*

$$r(t) \in \left\langle \sigma^{-1/2}\left(\frac{\tan(2\pi\sigma_1)}{\tan(2\pi\sigma_2)}\right)^{-1/4}, \sigma^{-1/2}\left(\frac{\tan(2\pi\sigma_2)}{\tan(2\pi\sigma_1)}\right)^{-1/4}\right\rangle$$

for all t.

Proof. As already noted the question is to find an appropriate bound for $b = T(0)$ and $c = -1/T(\infty)$. By the above discussion, $b = w(2\pi; 0)$ where $w(t; 0)$ is the solution of Riccati equation (4·4). Let us perform the change of variable

$$w(t) = \sigma \tan \frac{\theta(t)}{\sigma}.$$

Then, $\theta(t)$ satisfies the equation

$$\dot\theta = \sigma^2 \sin^2 \frac{\theta}{\sigma} + a(t)\cos^2 \frac{\theta}{\sigma}. \tag{4·5}$$

Note that this equation is exactly the same as (3·3). By mimicking the proof of Lemma 3·2 and assuming that $\theta(0) = 0$, it follows that

$$2\pi\sigma^2 - \frac{\delta}{2} \leqslant \theta(2\pi) \leqslant 2\pi\sigma^2 + \frac{\delta}{2}.$$

Consequently, if

$$(n - 1/2)\pi < 2\pi\sigma \pm \frac{\delta}{2\sigma} < (n + 1/2)\pi \tag{4·6}$$

for some $n \in I\!N$, then

$$\sigma \tan\left(2\pi\sigma - \frac{\delta}{2\sigma}\right) \leqslant b \leqslant \sigma \tan\left(2\pi\sigma + \frac{\delta}{2\sigma}\right). \tag{4·7}$$

Similarly, for c, let $w^*(t^*) = 1/w(-t^*)$, then $-c = w^*(2\pi)$. Consider the alternative equation

$$\dot{w}^* = a(-t^*)w^{*2} + 1,$$

and let

$$w^*(t) = \sigma^{-1} \tan \sigma\theta^*(t).$$

Then

$$\dot{\theta}^* = \sigma^{-2}a(-t^*)\sin^2\sigma\theta^* + \cos^2\sigma\theta^*$$

and

$$\sigma^{-2}\left(2\pi\sigma^2 - \frac{\delta}{2}\right) \leqslant \theta^*(2\pi) \leqslant \sigma^{-2}\left(2\pi\sigma^2 + \frac{\delta}{2}\right).$$

Hence, if (4·6) is satisfied, then

$$\sigma^{-1} \tan\left(2\pi\sigma - \frac{\delta}{2\sigma}\right) \leqslant -c \leqslant \sigma^{-1} \tan\left(2\pi\sigma + \frac{\delta}{2\sigma}\right). \tag{4·8}$$

From (4·7), (4·8), if either

$$(n - 1/2)\pi < 2\pi\sigma \pm \frac{\delta\sigma}{2\sigma} < n\pi \tag{4·9}$$

or

$$n\pi < 2\pi\sigma \pm \frac{\delta\sigma}{2\sigma} < (n + 1/2)\pi \tag{4·10}$$

for some $n \in I\!N$, then

$$-b/c \in \left\langle \sigma^2 \frac{\tan(2\pi\sigma - \frac{\delta}{2\sigma})}{\tan(2\pi\sigma + \frac{\delta}{2\sigma})}, \sigma^2 \frac{\tan(2\pi\sigma + \frac{\delta}{2\sigma})}{\tan(2\pi\sigma - \frac{\delta}{2\sigma})} \right\rangle,$$

or, equivalently,

$$-b/c \in \left\langle \sigma^2 \frac{\tan(2\pi\sigma_1)}{\tan(2\pi\sigma_2)}, \sigma^2 \frac{\tan(2\pi\sigma_2)}{\tan(2\pi\sigma_1)} \right\rangle.$$

Note that one of the previous conditions (4·9) or (4·10) is automatically verified since $a(t)$ is admissible. Consequently, since a critical value of $r(t)$ verifies $r_0 = (-b/c)^{-1/4}$, the proof is complete.

Now, we have all the information necessary to prove our main results.

Proof of Theorem 2·1. Following the notation of [**3**], let $r_0 = \min\{r(t) : t \in [0, 2\pi]\}$, $r_\infty = \max\{r(t) : t \in [0, 2\pi]\}$. Then, taking into account that $\sigma_1, \sigma_2 \in (0, 1/4)$ and using Proposition 4·2, we get

$$\sigma^{-1/2}\left(\frac{\tan(2\pi\sigma_2)}{\tan(2\pi\sigma_1)}\right)^{-1/4} < r_0 < r_\infty < \sigma^{-1/2}\left(\frac{\tan(2\pi\sigma_1)}{\tan(2\pi\sigma_2)}\right)^{-1/4}. \tag{4·11}$$

If condition (2·3) holds, the proof is similar to [**3**, theorem 3·2]. By repeating the arguments of [**3**, section 3·6], we get from (4·2) that

$$\beta \geqslant \frac{3}{8} r_0^4 \left\| c_- \right\|_1 - \frac{3}{8} r_\infty^4 \left\| c_+ \right\|_1 - \frac{7}{8} r_\infty^6 \left\| b_+ \right\|_1 \left\| b_- \right\|_1 .$$

Now, by using (4·11) and (2·3) it is easy to verify that the right-hand side of this inequality is positive. In consequence $\beta > 0$ and the proof is complete.

On the other hand, if the alternative condition (2·4) holds, we use formula (4·1) to prove that the twist coefficient is negative. The first term is

$$-\frac{3}{8} \int_0^{2\pi} c(t) r^4(t)\, dt \leqslant -\frac{3}{8} r_0^4 \left\| c_+ \right\|_1 + \frac{3}{8} r_\infty^4 \left\| c_- \right\|_1 .$$

On the other hand,

$$\begin{aligned} &\int\!\!\int_{[0,2\pi]^2} b(t) b(s) r^3(t) r^3(s) \chi_1(|\varphi(t) - \varphi(s)|)\, dt\, ds \\ &\leqslant \frac{\sqrt{2}}{8} r_\infty^6 \int\!\!\int_{[0,2\pi]^2} b_+(t) b_+(s) + b_-(t) b_-(s)\, dt\, ds < \frac{\sqrt{2}}{8} r_\infty^6 \left(\left\| b_+ \right\|_1^2 + \left\| b_- \right\|_1^2 \right), \end{aligned} \qquad (4·12)$$

where it is used that $0 < \chi_1(x) \leqslant \sqrt{2}/8$ for all x. Finally, taking into account that $0 < 2\pi\sigma_1 < \theta = 2\pi\rho < 2\pi\sigma_2 < \pi/2$, we can estimate the last two terms of (4·1) as follows

$$\frac{3}{16} \cot\left(\frac{\theta}{2}\right) \left| \int_0^{2\pi} b(t) r^3(t) e^{-i\varphi(t)}\, dt \right|^2 \leqslant \frac{3}{16} \cot(\pi\sigma_1) r_\infty^6 \left\| b \right\|_1^2,$$

$$\frac{1}{16} \cot\left(\frac{3\theta}{2}\right) \left| \int_0^{2\pi} b(t) r^3(t) e^{-i\varphi(t)}\, dt \right|^2 \leqslant \frac{1}{16} \cot(3\pi\sigma_1) r_\infty^6 \left\| b \right\|_1^2.$$

By using the previous inequalities in formula (4·1) we get

$$\begin{aligned} \beta \leqslant &-\frac{3}{8} r_0^4 \left\| c_+ \right\|_1 + \frac{3}{8} r_\infty^4 \left\| c_- \right\|_1 + \frac{\sqrt{2}}{8} r_\infty^6 \left(\left\| b_+ \right\|_1^2 + \left\| b_- \right\|_1^2 \right) \\ &+ \frac{3}{16} \cot(\pi\sigma_1) r_\infty^6 \left\| b \right\|_1^2 + \frac{1}{16} \cot(3\pi\sigma_1) r_\infty^6 \left\| b \right\|_1^2 . \end{aligned}$$

By using (4·11) and (2·4) it is proved that the right-hand term of this inequality is negative. Consequently $\beta < 0$ and the proof is finished.

Proofs of Theorems 2·3 and 2·4. Note that by Proposition 4·2 it is easy to verify that

$$\left\| r \right\|_4^2 \leqslant N(\sigma, \sigma_1, \sigma_2)$$

where $N(\sigma, \sigma_1, \sigma_2)$ is defined in Section 2. Now, the proof of Theorem 2·3 is identical to the proof of theorem 3·1 of [**3**]. The twist coefficient β is then positive. On the other hand, the proof of Theorem 2·4 is similar, but now the estimates in formulae (4·1) and (4·2) are used to prove that the twist coefficient β is negative.

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