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Non-trivial periodic solutions of a non-linear Hill's equation with positively homogeneous term

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Abstract

The existence of non-trivial periodic solutions of a general family of second order differential equations whose main model is a Hill's equation with a cubic nonlinear term arising in different physical applications is proved.

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1. Introduction and main result

The main motivation of this short note is the search for non-trivial T-periodic solutions of differential equations like

$$x'' + a(t)x = x^3, \tag{1}$$

where a is a T-periodic coefficient. This equation constitutes our main model and arises, for instance, in the study of localized solutions of certain one-dimensional nonlinear Schrödinder equations with a periodic potential in the framework of Bose–Einstein condensates [2] or as one-mode approximations of nonlinear wave equations which are relevant in different problems about fluid mechanics, theory of elasticity and optical modes. More concretely, it is useful in the modulation of an electromagnetic wave propagating in a stratified medium of Kerr type whose dielectric permittivity is periodic (see [1,3] and their references).

The presence of the trivial solution in equations like (1) supposes an additional difficulty in their study. This is because most of the classical methods in the qualitative theory of differential

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equations such as upper and lower solutions, degree theory or different fixed point theorems (Schauder is the most popular one) usually trap the trivial solution, which of course is not interesting for us. This problem was overcome in [7] by using a fixed point theorem for absolutely continuous operators defined on cones of a Banach space due to M.A. Krasnosel'skii. The main restriction in [7] was the assumption of some condition over the norm of a leading to the positivity of the Green's function of the linear part. Here, we will relax this condition for Eq. (1) and improve some of the results presented in [7].

Let us consider in general the nonlinear differential equation

$$x'' + a(t)x = g(t, x) \tag{2}$$

where $a \in L^{\infty}(\mathbb{R})$ is a T-periodic function such that $a(t) \geq 0$ for a.e. $t \in [0, T]$, and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, T-periodic in the first variable and with continuous derivative in the second one. Two main hypotheses are assumed.

(H1) The Hill's equation $x'' + a(t)x = 0$ is non-resonant, that is, it does not have non-trivial T-periodic solutions.

(H2) The function $g(t, x)$ is positively homogeneous of order $r > 1$ in the second variable, that is,

$$g(t, \lambda x) = \lambda^r g(t, x),$$

for a.e. $t \in [0, T]$ and for all $\lambda > 0, x \in \mathbb{R}$.

Condition (H2) implies in particular that $g(t, 0) = 0$ for a.e. t. In order to avoid trivialities, it is supposed that g is not identically zero. The following result is our main theorem.

Theorem 1. *Under the hypotheses* (H1)–(H2)*, if*

$$\lim_{x \to +\infty} g(t, x) = +\infty \qquad \textit{unif. for a.e. } t, \tag{3}$$

then Eq. (2) *has a non-trivial T-periodic solution.*

This results applies directly to the model equation (1). The proof will use a combination of a classical perturbation result with the upper and lower solution method. The rest of the paper is organized as follows. Section 2 introduces the basic tools which are necessary for the proof. Section 3 contains the proof itself. Finally, Section 4 presents some remarks and comments.

2. Preliminary results

The first lemma below is a particular case of a well-known result that can be found in many classical texts (see for instance [5, Theorem 3.7] or [6, Corollary 1.11]). In the following, $\|.\|_\infty$ denotes the supremum norm.

Lemma 1. *Let us assume that* (H1) *holds and that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a general Caratheodory function, T-periodic in the first variable and with continuous derivative in the second one. Then, there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$, the perturbed equation*

$$x'' + a(t)x = \varepsilon f(t, x)$$

admits a unique T-periodic solution $x(t, \varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \|x(t, \varepsilon)\|_\infty = 0$$

and this solution depends continuously on ε.

This result will be used in combination with the method of upper and lower solutions. We include the definition and a very popular result by completeness. A T-periodic function $\alpha \in W^{2,1}(\mathbb{R})$ is a *strict lower solution* of the equation $x'' = f(t, x)$ if

$$\alpha'' > f(t, \alpha) \qquad \text{for a.e. } t.$$

A T-periodic function $\beta \in W^{2,1}(\mathbb{R})$ is a *strict upper solution* if this inequality is reversed. The following basic result can be found for instance in [4].

Proposition 1. *Let α, β a couple of lower and upper solutions respectively of equation $x'' = f(t, x)$, being f a Caratheodory function T-periodic in the first variable and with continuous derivative in the second one. If $\alpha(t) < \beta(t)$ for all t, then there exists a T-periodic solution x of equation $x'' = f(t, x)$ such that $\alpha(t) < x(t) < \beta(t)$ for all t.*

3. Proof of Theorem 1

The proof is divided into several steps:

- **Step 1**: Let us consider the perturbed differential equation

$$\alpha'' + a(t)\alpha = \varepsilon(g(t, \alpha) + 1).$$

Then, there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$ there exists a T-periodic solution $\alpha(t, \varepsilon)$ such that $\max_t \alpha(t, \varepsilon) > 0$.

The existence is a direct consequence of Lemma 1. Besides, $\lim_{\varepsilon \to 0} \|\alpha(t, \varepsilon)\|_\infty = 0$, so by using the continuity of g in the second variable there exists ε_1 such that if $0 < \varepsilon < \varepsilon_1$ then $g(t, \alpha(t, \varepsilon)) + 1 > 0$ for a.e. t. Integrating the equation over a period,

$$\int_0^T a(t)\alpha(t, \varepsilon)\mathrm{d}t = \varepsilon \int_0^T (g(t, \alpha(t, \varepsilon)) + 1)\mathrm{d}t > 0,$$

and now an elementary application of the mean value theorem provides a value ξ such that $\alpha(\xi, \varepsilon) > 0$, and therefore $\max_t \alpha(t, \varepsilon) > 0$.

- **Step 2**: The differential equation

$$y'' + a(t)y = \varepsilon g(t, y) \tag{4}$$

has a non-trivial T-periodic solution $y(t, \varepsilon)$ for all $0 < \varepsilon < \varepsilon_1$.

It is clear that the function $\alpha(t, \varepsilon)$ obtained in Step 1 is a lower solution of (4). Let us find an ordered upper solution. First, let us prove that (3) implies that $g(t, 1)$ is uniformly bounded below by some positive constant. In fact, (3) implies that for all $C > 0$ there exists $\delta > 0$ (not depending on t) such that $g(t, x) \geq C$ for all $x \geq \delta$ and for a.e. t. Then, by using (H2) we get $g(t, 1) = \delta^{-r} g(t, \delta) \geq \delta^{-r} C > 0$ for a.e. t. After this consideration, it is clear that the constant

$$M = \max_t \left\{ \|\alpha(t, \varepsilon)\|_\infty, \left\| \frac{a(t)}{\varepsilon g(t, 1)} \right\|_\infty^{\frac{1}{r-1}} \right\}.$$

is well-defined. Then, it is easy to verify that any constant $\beta > M$ is a well-ordered upper solution of Eq. (4). Hence, by Proposition 1 there exists a T-periodic solution $y(t, \varepsilon)$ such that

$$\alpha(t, \varepsilon) < y(t, \varepsilon) < \beta$$

for all t. Moreover, $y(t, \varepsilon)$ is non-trivial since $\alpha(t, \varepsilon)$ is positive somewhere.

- **Step 3**: Conclusion. Let us define $x(t) = \lambda y(t, \varepsilon)$, where λ is a positive constant to be fixed later. Due to condition (H2), it verifies

$$x'' + a(t)x = \varepsilon\lambda^{1-r} g(t, x).$$

Hence, by taking $\lambda = \varepsilon^{\frac{1}{1-r}}$, $x(t)$ is a non-trivial T-periodic solution of Eq. (2).

4. Further remarks

Our main result applies directly to the model equation (1), generalizing the results of [7] for this equation. More concretely, the conditions required in [7] imply in particular that the periodic problem for equation $x'' + a(t)x = 0$ has a positive Green's function. In fact, with this condition the Hill's equation is in the first stability region. With the result presented in this note, this assumption is relaxed to the mere existence of the Green's function. For example, in the Mathieu equation $a(t) = a + b\cos t$ only a numerable number of curves in the parameter plane $a - b$ is excluded, and it is indifferent if the linear part of (1) is stable or unstable.

It is interesting to remark that the cubic term in (1) can be multiplied by a periodic positive coefficient $\xi(t)$, which in the physical model [1] represents a nonlinear susceptibility of the medium.

Finally, let us note that, eventually, our method could be useful in more general situations such as equations depending nonlinearly on the derivative, Hamiltonian or more general systems of differential equations. These extensions should be developed elsewhere.

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References

[1] G.L. Alfimov, V.V. Konotop, On the existence of gap solitons, Physica D 146 (2000) 307–327.
[2] G.L. Alfimov, V.V. Konotop, M. Salerno, Matter solitons in Bose–Einstein condensates with optical lattices, Europhys. Lett. 58 (2002) 7–13.
[3] J. Coste, J. Peyraud, Stationary waves in a nonlinear periodic medium: Strong resonances and localized structures. II. The continuous model, Phys. Rev. B 39 (1989) 13096.
[4] C. De Coster, P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results, in: F. Zanolin (Ed.), Nonlinear Analysis and Boundary Value Problems for Ordinary Differential Equations, in: CISM-ICMS, vol. 371, Springer-Verlag, New York, 1996, pp. 1–78.
[5] A. Halanay, Differential Equations, Academic Press, 1965.
[6] N. Rouche, J. Mawhin, Équations différentielles ordinaries, Tome II: Stabilit'e et solutions périodiques, Masson et Cie, 1973.
[7] P. Torres, Existence of one-signed periodic solutions of some second order differential equations via a Krasnoselskii fixed point theorem, J. Differential Equations 190 (2003) 643–662.