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Guided waves in a multi-layered optical structure

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Abstract
Motivated by the study of the propagation of electromagnetic waves through a multi-layered optical medium, we prove the existence of two different kinds of homoclinic solutions to the origin in a Schrödinger equation with a nonlinear term. We use a Krasnoselskii fixed point theorem together with a compactness criterion due to Zima. The main results are illustrated with concrete examples of practical interest such as self-focusing nonlinearities of Kerr and non-Kerr type.

Mathematics Subject Classification: 34C37, 34C60

1. Introduction

The purpose of this paper is to study the existence and multiplicity of solutions $u \in H^1(\mathbb{R})$ for the scalar nonlinear differential equation

$$-\ddot{u}(x) + a(x)u(x) = b(x)f(u(x)), \tag{1}$$

where $a, b \in L^\infty(\mathbb{R})$ are non-negative almost everywhere and f is a given continuous function. Under some hypotheses, this equation models the propagation of electromagnetic waves through a medium consisting of layers of dielectric material (we refer to [15] for a detailed explanation of the physical background). If the medium is stratified in planes of homogeneous composition perpendicular to the x-axis and we look for solutions of the Maxwell's equations with the special ansatz $E(x, y, z, t) = u(x)\cos(kz - \omega t)e_2$ (which corresponds to a monochromatic electric field propagating in the direction z and transverse to the direction of propagation), we are led to the study of solutions of the differential equation

$$-\ddot{u}(x) + k^2u(x) = \frac{\omega^2}{c^2}\varepsilon\left(x, \frac{1}{2}u(x)^2\right)u(x), \tag{2}$$

where c is the speed of light in vacuum, $2\pi/k$ is the wave length and ε is the dielectric function. Such solutions must verify the so-called guidance conditions,

$$\lim_{|x|\to+\infty} u(x) = \lim_{|x|y\to+\infty} \dot{u}(x) = 0 \tag{3}$$

and

$$\int_{\mathbb{R}} u^2(x)\mathrm{d}x + \int_{\mathbb{R}} \dot{u}^2(x)\mathrm{d}x < +\infty, \tag{4}$$

that mathematically amount to looking for non-trivial solutions such that $u \in H^1(\mathbb{R})$ (see [15, proposition 2.1]). In the context of nonlinear optics, such solutions are known as guided waves or bright solitons [10].

This problem has been studied by several authors under adequate restrictions on the dielectric function. The most common choice for ε found in the related literature is

$$\varepsilon(x, s) = A(x) + B(x)s,$$

with $A, B \in L^\infty(\mathbb{R})$, which is known as a Kerr nonlinearity. For instance, [2] considered a dielectric function of the form

$$\varepsilon(x, s) = \begin{cases} q^2 + p^2 & \text{if } |x| < d, \\ q^2 + s & \text{if } |x| > d, \end{cases} \tag{5}$$

with $q, p \in \mathbb{R}$, and obtained the existence of asymmetric solutions bifurcating from a branch of symmetric ones. This property is known as a symmetry-breaking phenomenon. This choice of ε corresponds to a medium consisting of three layers and d gives the thickness of the internal one. Later, symmetry-breaking behaviour was observed in the multi-layered case [12] but only in the perturbative case (that is, a small parameter is introduced in the first term of the equation). Similar results are stated and proved in [3, 18]. Finally, [4] addresses the global case (without perturbative parameter) in a symmetric medium stratified in three layers, obtaining a positive asymmetric bound state. It is interesting to note that all these papers study the propagation in a medium of Kerr type. However, at high intensity regimes some kind of saturation is physically expected (i.e. an asymptotic finite value) for ε. Specific examples of materials with saturation are discussed in the literature and several models of non-Kerr-like dielectric responses have been proposed [10, 15].

Our purpose is to complete the mentioned bibliography with a new approach to the problem. Until now, the methods of proof have relied either on the explicit integrability of the problem or on a variational approach. We propose the use of a fixed point theorem due to Krasnoselskii for completely continuous operators defined in cones of a Banach space together with a suitable study of the Green's function for the linear part of the problem. This method has been employed successfully in some scalar problems on the half-line [20] with sublinear nonlinearities and more generally in a variety of integral equations in infinite intervals of the Volterra type (see for instance [1, 7, 13] and references therein). Using this technique, we are able to prove the existence of guided waves when the nonlinear contribution of the dielectric response has finite support. In particular, under symmetry conditions odd bound states are found, that is, solutions such that $u(-x) = -u(x)$ for any $x \in \mathbb{R}$. In the framework of optical solitons, changing-sign solutions are meaningful although related references are difficult to find [16].

The rest of the paper is organized as follows. In section 2 some preliminary results are collected. Section 3 contains the main result about the existence of a positive guided wave. Section 4 proves that, if in addition to the hypotheses of the main result of section 3 we assume that the coefficients a, b are even, 0 is outside the support of b and f is odd, then there exists a second guided wave which is odd. Finally, section 5 contains some illustrative examples of applicability to multi-layered optical media under the main assumption that the nonlinear contribution of the dielectric response is confined to the internal layers.

We use the notation $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$. We write $a \succ 0$ (respectively, $a \succeq 0$) if $a \in L^\infty(\mathbb{R})$ is positive (respectively, non-negative) almost everywhere. For a given

$a \in L^\infty(I\!R)$, the essential infimum is denoted as a_*. The support of a given function a is denoted by $\text{Supp}(a)$. The limit value of a given function u in $+\infty$ (or $-\infty$) is written simply as $u(+\infty)$ (or $u(-\infty)$). Finally, $\|.\|$ denotes the norm of the supremum.

2. Auxiliary results

We begin with some basic properties of the homogeneous boundary value problem

$$\begin{aligned} &-\ddot{u} + a(x)u = 0, \\ &u(\alpha) = 0, u(+\infty) = 0. \end{aligned} \tag{6}$$

with $a \succ 0$. The properties of this problem are analogous in the cases $\alpha = -\infty$ and $\alpha = 0$, so both of them are studied jointly. From now on, we denote by J the whole real line $I\!R$ or the half-line $I\!R^+$ without distinction.

Due to the sign of the coefficient, the equation $-\ddot{u} + a(x)u = 0$ presents a typical exponential dichotomy. The origin is hyperbolic and the stable manifold has codimension one. The associated Green's function is given by

$$G(x,s) = \begin{cases} u_1(x)u_2(s), & \alpha < x \leqslant s < +\infty \\ u_1(s)u_2(x), & \alpha < s \leqslant x < +\infty, \end{cases} \tag{7}$$

where u_1, u_2 are solutions such that $u_1(\alpha) = 0$, $u_2(+\infty) = 0$. It is easy to verify that a solution of $-\ddot{u}+a(x)u = 0$ vanishes at most once in the interval $[\alpha, +\infty]$. Hence, u_1, u_2 can be chosen as positive functions in J. Moreover, u_1 is strictly increasing and u_2 is strictly decreasing in J.

The following result generalizes some known properties of the Green's function (see [20]) for the special case of a constant coefficient $a(t)$. First, some preliminaries are needed. Note that u_1, u_2 intersect in a unique point x_0. Then, we define

$$p(x) = \begin{cases} \dfrac{1}{u_2(x)}, & x \leqslant x_0, \\ \dfrac{1}{u_1(x)}, & x > x_0. \end{cases} \tag{8}$$

Proposition 2.1. *The following properties for the Green's function defined by* (7) *hold.*

$(P1)$ $G(x,s) > 0$ *for every* $(x,s) \in J \times J$.
$(P2)$ $G(x,s) \leqslant G(s,s)$ *for every* $(x,s) \in J \times J$.
$(P3)$ *Given a non-empty compact subset* $P \subset J$, *we define*

$$m_1(P) = \min\{u_1(\inf P), u_2(\sup P)\}. \tag{9}$$

Then,

$$G(x,s) \geqslant m_1(P)p(s)G(s,s) \text{ for all } (x,s) \in P \times J.$$

$(P4)$ $G(s,s)p(s) \geqslant G(x,s)p(x)$ *for every* $(x,s) \in J \times J$.

Proof.

$(P1)$ Trivial because u_1, u_2 are positive.
$(P2)$ Trivial by the monotonicity of u_1, u_2.
$(P3)$ We prove it for the case $x \leqslant s$, the remaining possibility is analogous. By the increasing character of u_1,

$$G(x,s) \geqslant u_1(\inf P)u_2(s) \geqslant m_1(P)\frac{G(s,s)}{u_1(s)} \geqslant m_1(P)p(s)G(s,s).$$

($P4$) Again, we only prove the case $x \leqslant s$. Looking at the definition, one realizes that p is increasing for $x < x_0$ and decreasing for $x > x_0$. We consider three cases: if $x \leqslant s \leqslant x_0$ then $p(s) \geqslant p(x)$, which combined with ($P2$) gives the result. If $x_0 \leqslant x \leqslant s$, the equality holds. Finally, if $x \leqslant x_0 \leqslant s$, then $G(s,s)p(s) = u_1(s) \geqslant u_2(s) = G(x,s)p(x)$. ■

Remark 2.1. *The property* ($P4$) *is not used in the proofs and has been included by the author in the belief that it could be useful in the future developments of this line of research.*

Finally, we state a well-known fixed point theorem due to Krasnoselskii for a completely continuous operator defined on a Banach space [11, p.148]. Let us recall that a given operator is completely continuous if the image of a bounded set is relatively compact. Given a Banach space $\mathcal{B}$, we say that $\mathcal{P} \subset \mathcal{B}$ is a cone if it is closed, non-empty, $\mathcal{P} \neq \{0\}$ and whenever $x, y \in \mathcal{P}$ and $\lambda, \mu \in I\!R$ with $\lambda \geqslant 0$, $\mu \geqslant 0$ then $\lambda x + \mu y \in \mathcal{P}$.

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset B$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $\mathcal{A} : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that one of the following conditions is satisfied.*

1. $\|\mathcal{A}u\| \leqslant \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geqslant \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_2$.
2. $\|\mathcal{A}u\| \geqslant \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leqslant \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, $\mathcal{A}$ has at least one fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

This result has been extensively employed in the study of boundary value problems with separated boundary conditions (see for instance [5, 6, 8, 9] and their references) and more recently for the periodic problem [14, 17]. However, for problems defined in non-compact intervals such as ours, there is the difficulty that the Ascoli–Arzela theorem is not valid for proving the complete continuity of the operator. In [19], the following compactness criterion is proved. The space of bounded continuous functions $u : I\!R \to I\!R$ is denoted by $BC(I\!R)$.

Proposition 2.2. *Let $\Omega \subset BC(I\!R)$. Suppose that the functions $u \in \Omega$ are equicontinuous in each compact interval of $I\!R$ and uniformly bounded in the sense of the norm*

$$\|u\|_\xi = \sup_{x \in I\!R} \xi(|x|)|u(x)|,$$

where $\xi : I\!R_+ \to I\!R^+$ is a continuous function such that

$$\lim_{x \to +\infty} \xi(x) = +\infty. \tag{10}$$

Then, Ω is relatively compact.

3. Existence of positive homoclinic orbits to the origin

In order to avoid trivialities, from now on it is assumed that Supp(b) is a non-empty compact set. At this point, we introduce the additional assumption that the essential infimum of a is positive, $a_* > 0$. Then the following lemma holds.

Lemma 3.1. *If $a_* > 0$, and $h \in L^1(I\!R)$, the unique solution of the linear boundary value problem,*

$$-\ddot{u} + a(x)u = h(x),$$
$$u(-\infty) = 0, u(+\infty) = 0,$$

belongs to $H^1(I\!R)$.

Proof. Multiplying the equation by u and integrating by parts over the whole real line we get

$$\int_{I\!R} \dot{u}^2(x)\mathrm{d}x + a_* \int_{I\!R} u^2(x)\mathrm{d}x = \int_{I\!R} h(x)u(x)\mathrm{d}x < \|u\| \int_{I\!R} |h(x)|\mathrm{d}x < +\infty,$$

so clearly $u \in H^1(I\!R)$. ■

The problem of finding a solution $u \in H^1(I\!R)$ of equation (1) is equivalent to finding a fixed point for the operator $T : H^1(I\!R) \to H^1(I\!R)$ defined as

$$Tu := \int_{I\!R} G(x,s)b(s)f(u(s))\mathrm{d}s = \int_{\mathrm{Supp}(b)} G(x,s)b(s)f(u(s))\mathrm{d}s,$$

where $G(x, s)$ is the Green's function defined by (7) with $\alpha = -\infty$. Note that the image of T is in fact contained in $H^1(I\!R)$ as a direct consequence of the previous lemma.

With the idea of an application of theorem 2.1, we define the cone

$$\mathcal{P} = \left\{ u \in H^1(I\!R) \ : \ u(x) \geqslant 0 \text{ for all } x, \min_{x \in \mathrm{Supp}\ (b)} u(x) \geqslant m_1 p_0 \|u\| \right\}. \qquad (11)$$

Here, $p_0 = \inf_{\mathrm{Supp}(b)} p(x)$, where $p(x)$ is defined by (8) and the constant $m_1 \equiv m_1(\mathrm{Supp}\ (b))$ is defined by (9). Note that the compactness of Supp (b) implies that $p_0 > 0$. Besides, from $(P3)$ it is easy to see that $m_1 p_0 < 1$, and hence this cone is non-empty.

The following is the main result in this section.

Theorem 3.1. *Let us assume the following hypotheses.*

(i) $a_* > 0$, $b \succeq 0$.
(ii) *Supp*(b) *is a non-empty compact set.*
(iii) $f(s) \geqslant 0$ *for every* $s \geqslant 0$.
(iv) *There exists* $r > 0$ *such that*

$$f(u) \max_{x \in \mathrm{Supp}\ (b)} \int_{\mathrm{Supp}\ (b)} G(x,s)b(s)\mathrm{d}s \leqslant r$$

for every $u \in [0, r]$.
(v) *There exists* $R > r > 0$ *such that*

$$f(u) \min_{x \in \mathrm{Supp}\ (b)} \int_{\mathrm{Supp}\ (b)} G(x,s)b(s)\mathrm{d}s \geqslant \frac{R}{m_1 p_0}$$

for every $u \in [R, (1/m_1 p_0)R]$.

Then, there exists a positive solution $u \in H^1(I\!R)$ *of equation (1) such that* $r \leqslant \|u\| \leqslant (1/m_1 p_0)R$.

Remark 3.1. The hypotheses (iv) and (v) are very technical and in general could be difficult to verify; however, they can be replaced by the following simple (stronger) condition

$$(iv)' \qquad \lim_{s \to 0^+} \frac{f(s)}{s} = 0, \qquad \lim_{s \to +\infty} \frac{f(s)}{s} = +\infty.$$

However, (iv) and (v) will be useful in the study of materials with saturation, as is shown in section 5. Note that $v = \int_{\mathrm{Supp}\ (b)} G(x,s)b(s)\mathrm{d}s$ is the unique solution in $H^1(I\!R)$ of the linear equation $-\ddot{v} + a(x)v = b(x)$. This fact could be useful in order to find specific bounds for particular situations of practical interest.

The proof of theorem 3.1 consists of an application of theorem 2.1 which will be prepared through a number of preliminary lemmas.

Lemma 3.2. $T(\mathcal{P}) \subset \mathcal{P}$.

Proof. Evidently, the property $(P1)$ of the Green's function together with (ii) implies that $Tu(x) \geqslant 0$ for all x. Let us call x_m the point where $\min_{x\in \text{Supp }(b)} Tu(x)$ is attained. Then, for all $x \in I\!R$,

$$\begin{aligned} Tu(x_m) &= \int_{\text{Supp }(b)} G(x_m, s)b(s)f(u(s))\text{d}s \\ &\geqslant m_1 \int_{\text{Supp }(b)} p(s)G(s,s)b(s)f(u(s))\text{d}s \\ &\geqslant m_1 p_0 \int_{\text{Supp }(b)} G(x,s)b(s)f(u(s))\text{d}s = m_1 p_0 Tu(x), \end{aligned}$$

where we have used $(P2)$ and $(P3)$. This completes the proof. ■

Lemma 3.3. *$T : \mathcal{P} \to \mathcal{P}$ is continuous and completely continuous.*

Proof. The continuity is trivial. We have to prove that given $\Omega \subset \mathcal{P}$ a bounded set (call M a valid bound for this set), the image $T(\Omega)$ is relatively compact. By the Ascoli–Arzela theorem, it is standard to prove that the functions belonging to $T(\Omega)$ are equicontinuous in each compact interval of $I\!R$. With proposition 2.2 in mind, let us define $q(x)$

$$q(x) = \begin{cases} \dfrac{1}{u_1(x)}, & x \leqslant 0, \\ \dfrac{1}{u_2(x)}, & x > 0, \end{cases} \tag{12}$$

where u_1, u_2 are the same functions of the definition of the Green's function (7). This function is discontinuous at 0, but the basic properties of u_1, u_2 enable us to fix an even function $\xi : I\!R \to I\!R^+$ such that $\xi(x) = \xi(|x|) < q(x)$ for all $x \in I\!R$ and condition (10) holds. It only remains to prove that the functions of $T(\Omega)$ are uniformly bounded in the sense of the norm $\|\cdot\|_\xi$. Take $u \in \Omega$ and suppose that $x > 0$ as a first case. Then, calling $M^* = \max_{s\in[0,M]} f(s)$,

$$\begin{aligned} \xi(|x|)|Tu(x)| &\leqslant q(x)Tu(x) = \frac{1}{u_2(x)}Tu(x) \leqslant \frac{M^*}{u_2(x)} \int_{I\!R} b(s)G(x,s)\text{d}s \\ &\leqslant \frac{M^*}{u_2(x)} \left[\int_{-\infty}^{x} b(s)u_1(s)u_2(x)\text{d}s + \int_{x}^{+\infty} b(s)u_1(x)u_2(s)\text{d}s \right] \\ &\leqslant M^* \int_{-\infty}^{x} b(s)u_1(s)\text{d}s + M^* \int_{x}^{+\infty} b(s)u_1(x)\frac{u_2(s)}{u_2(x)}\text{d}s \\ &\leqslant M^* \int_{\text{Supp } b} b(s)u_1(s)\text{d}s, \end{aligned}$$

where we have used that u_2 is decreasing and consequently $(u_2(s))/(u_2(x)) \leqslant 1$ if $s > x$. Further, the last bound is finite because of the compactness of Supp (b) and does not depend on u. When $x \leqslant 0$, an analogous argument gives

$$\xi(|x|)|Tu(x)| \leqslant M^* \int_{\text{Supp }(b)} b(s)u_2(s)\text{d}s,$$

and the proof is completed. ■

Lemma 3.4. *If $a_* > 0$ and $h \in L^\infty(I\!R)$ has compact support, then any positive maximum of a solution of the linear equation*

$$-\ddot{v} + a(x)v = h(x)$$

is attained at Supp (h).

Proof. By contradiction, if $v(x_0) > 0$ is a maximum of v and $x_0 \notin$ Supp (h), then $-\ddot{v} + a(x)v = 0$ around x_0, so v is strictly convex in x_0, which is a contradiction. ■

Proof of theorem 3.1. We define the open sets Ω_1 and Ω_2 as the open balls in $H^1(I\!R)$ of radius r and $R/m_1 p_0$, respectively. We also define

$$M_r = \max_{s\in[0,r]} f(s), \qquad M_R = \min_{s\in[R,\frac{R}{m_1 p_0}]} f(s).$$

Let us take $u \in \mathcal{P} \cap \partial\Omega_1$. Note that $v = Tu$ is the unique positive solution in $H^1(I\!R)$ of the linear equation

$$-v'' + a(x)v = b(x)f(u(x)).$$

Therefore, using lemma 3.4 and hypothesis (iv),

$$\|Tu\| = \max_{x\in \text{ Supp }(b)} Tu(x) \leqslant M_r \max_{x\in \text{ Supp }(b)} \int_{\text{Supp }(b)} G(x,s)b(s)\mathrm{d}s \leqslant r.$$

On the other hand, let us take $u \in \mathcal{P} \cap \partial\Omega_2$. By the defining property of the cone, $R \leqslant u(x) \leqslant R/m_1 p_0$ for every $x \in$ Supp (b). Then

$$\|Tu\| = \max_{x\in \text{ Supp }(b)} Tu(x) = \max_{x\in \text{ Supp }(b)} \int_{\text{Supp }(b)} G(x,s)b(s)f(u(s))\mathrm{d}s$$
$$\geqslant M_R \min_{x\in \text{ Supp }(b)} \int_{\text{Supp }(b)} G(x,s)b(s)\mathrm{d}s \geqslant \frac{R}{m_1 p_0}$$

by hypothesis (v). The proof is closed by a direct application of theorem 2.1. ■

4. Odd homoclinic orbits to the origin in the equation with symmetric coefficients

In the previous section we have obtained positive solutions of the problem under consideration, which is the case considered in the wholeness of related papers known to the author. In this section, we prove the existence of a new kind of solution when the coefficients are even, which physically means that the layered structure of the optical medium is symmetric. In the following result, $p_0 = \inf_{\text{Supp }(b)\cap I\!R^+} p(x)$ and $m_1 \equiv m_1(\text{Supp }(b) \cap I\!R^+)$.

Theorem 4.1. *Under the conditions of theorem 3.1, if moreover a, b are even functions, $0 \notin$ Supp (b) and f is odd, then there exists an odd non-trivial solution $u \in H^1(I\!R)$ of equation (1) such that $r \leqslant \|u\| \leqslant 1/m_1 p_0 R$.*

Sketch of the proof. The proof mimics the steps of the proof of theorem 3.1, working now with the Green's function for the problem on the half-line $I\!R_+$. The operator $T : H^1(I\!R_+) \to H^1(I\!R_+)$ is defined as

$$Tu := \int_{I\!R_+} G(x,s)b(s)f(u(s))\mathrm{d}s.$$

The adequate cone is

$$\mathcal{P} = \left\{ u \in H^1(I\!R_+) \,:\, u(0) = 0,\ u(x) \geqslant 0 \text{ for all } x \in I\!R_+, \min_{x\in \text{ Supp } (b)\cap I\!R_+} u(x) \geqslant m_1 p_0 \|u\| \right\}.$$

Then, $T(\mathcal{P}) \subset \mathcal{P}$ and T is a continuous and completely continuous operator, since proposition 2.2 can be applied to functions u defined only in $I\!R_+$ and such that $u(0) = 0$ by simply extending as the zero constant function on the negative axis. Then, the proof of lemma 3.3 is valid by taking

$$\xi(|x|) := \frac{1}{u_2(|x|)}.$$

Finally, the sets Ω_1 and Ω_2 are defined again as the open balls of radius r and $R/m_1 p_0$, respectively. Everything works in the same way so the repetitive details are omitted. In conclusion, we obtain a positive non-trivial solution $u \in H_1(I\!R_+)$ such that $u(0) = 0$ and the odd extension gives the desired solution.

Note that the assumption $0 \notin$ Supp (b) is necessary in order to have $m_1 \neq 0$, which is a key point in the proper definition of the cone as well as in the choice of the set Ω_2. We will analyse briefly the physical implications of this assumption in section 5.

Remark 4.1. *By revising the proofs, one realizes that theorems 3.1 and 4.1 are still true for the more general equation*

$$-\ddot{u}(x) + a(x)u(x) = b(x) f(x, u(x)), \tag{13}$$

where f is a given L^∞-Caratheodory function (that is, $f(\cdot, s) \in L^\infty(I\!R)$ for every s and $f(x, \cdot)$ continuous for a.e. x), provided that f holds inequalities (iv) and (v) uniformly in x. We have chosen the current presentation just for simplicity.

5. Application to the existence of guided waves in optical systems

In this section we analyse the consequences of our main results in the problem of the existence of guided waves crossing an optical stratified medium composed of several layers of homogeneous material. As usual, given a dielectric response function ε, the nonlinear contribution is isolated by the decomposition

$$\varepsilon_{\rm L}(x) = \varepsilon(x, 0), \qquad \varepsilon_{\rm NL}(x, s) = \varepsilon(x, s) - \varepsilon_L(x).$$

For simplicity, we are limited to the case $\varepsilon_{\rm NL}(x, s) = B(x)F(s)$, although more general situations could be studied (see remark 4.1). This case covers most of the examples found in the literature, as we will see later. Then, equation (2) is

$$-\ddot{u}(x) + \left(k^2 - \frac{\omega^2}{c^2}\varepsilon_{\rm L}(x)\right) u(x) = \frac{\omega^2}{c^2} B(x) F\left(\frac{1}{2}u(x)^2\right) u(x). \tag{14}$$

This corresponds to equation (1) with $a(x) = k^2 - (\omega^2/c^2)\varepsilon_{\rm L}(x)$, $b(x) = (\omega^2/c^2)B(x)$ and $f(s) = F((1/2)s^2)s$. Therefore, $f(s)$ is always an odd function, whereas in applications a, b are typically piecewise constant functions, each constant corresponding to a different layer.

We will say that a dielectric function $\varepsilon(x, s) = \varepsilon_{\rm L}(x) + B(x)F(s)$ is of Kerr type if F is increasing and

$$F(0) = 0, \qquad \lim_{s\to+\infty} F(s) = +\infty. \tag{15}$$

This definition covers in particular the classical Kerr nonlinearity $\varepsilon(x, s) = \varepsilon_L(x) + B(x)s$.

Corollary 5.1. *Let us assume that* $k^2 > \omega^2/c^2\|\varepsilon_{\mathrm{L}}\|$, $B \succeq 0$ *and* Supp (B) *is a non-empty compact set. If the dielectric function is of Kerr type, there exists a positive solution* $u \in H^1(\mathbb{R})$ *of equation (14). If moreover* $\varepsilon_{\mathrm{L}}(x)$, $B(x)$ *are even functions and* $0 \notin$ Supp (B), *there exists an odd solution* $\tilde{u} \in H^1(\mathbb{R})$ *of equation (14).*

The proof is direct. From the point of view of physics, the requirement of the compactness of Supp (B) means that the nonlinear contribution of the dielectric response is confined to the internal layers.

Kerr nonlinearities are by far the most common in the related references. For instance, [2, 12, 18] consider the equation

$$-\ddot{u} + \lambda^2 u = \chi_A(x)u(x) + (1 - \chi_A(x))|u(x)|^{p-1}u(x),$$

where $p \geqslant 2$, A is a closed interval or the union of two closed intervals and χ_A is the characteristic function. The conditions of the first part of corollary 5.1 hold when $\overline{\mathbb{R} \setminus A}$ is compact. In particular, this includes the 'reversed' case of that studied by Akhmediev [2], that is,

$$\varepsilon(x, s) = \begin{cases} q^2 + s & \text{if } |x| < d, \\ q^2 + p^2 & \text{if } |x| > d, \end{cases} \tag{16}$$

without restriction in the width d of the internal layer. For this three-layered optical medium, corollary 5.1 provides a positive solution $u \in H^1(\mathbb{R})$; however, we do not know if there exists a second (odd) solution since $0 \in$ Supp (B). To get this kind of solution the medium should have at least five layers, a case studied in [12, 18]. Again, our results cover the case when the nonlinearity is confined to the internal layers. For instance, for the dielectric response

$$\varepsilon(x, s) = \begin{cases} q^2 + s & \text{if } d_1 < |x| < d_2, \\ q^2 + p^2 & \text{if } |x| < d_1 \text{ or } |x| > d_2, \end{cases} \tag{17}$$

with $q, p, d_1, d_2 > 0$, we get two different guided waves.

The study of dielectric responses attempting to model materials with saturation is of high interest. Some simple models cited in [15] are

$$\varepsilon(x, s) = \varepsilon_{\mathrm{L}}(x) + B(x)\frac{s}{1+s},$$

$$\varepsilon(x, s) = \varepsilon_{\mathrm{L}}(x) + B(x)(1 - \mathrm{e}^{-s}).$$

In general, we will say that a given dielectric function $\varepsilon(x, s) = \varepsilon_{\mathrm{L}}(x) + B(x)F(s)$ is of non-Kerr type if F is increasing and

$$F(0) = 0, \qquad \lim_{s\to+\infty} F(s) = F_\infty < +\infty.$$

By using theorems 3.1 and 4.1, it is easy to prove the following result.

Corollary 5.2. *Let us assume that* $k^2 > \omega^2/c^2\|\varepsilon_{\mathrm{L}}\|$, $B \succeq 0$ *and* Supp (B) *is a non-empty compact set. If the dielectric function is of non-Kerr type and the following condition holds*

$$\omega^2 F_\infty \min_{x\in \text{Supp }(B)} \int_{\text{Supp }(B)} G(x, s)B(s)\mathrm{d}s > \frac{c^2}{m_1 p_0}, \tag{18}$$

then there exists a positive solution $u \in H^1(\mathbb{R})$ *of equation (14). If moreover* $\varepsilon_{\mathrm{L}}(x)$, $B(x)$ *are even functions and* $0 \notin$ Supp (B), *there exists an odd solution* $\tilde{u} \in H^1(\mathbb{R})$ *of equation (14).*

Consequently, we have proved the existence of guided waves in non-Kerr media for high values of ω, k.

Our results also provides some information on the localization of the solutions that can be of interest in the study of branches of solutions in systems controlled by parameters. As a basic example, we consider

$$\varepsilon(x,s) = \varepsilon_{\mathrm{L}}(x) + \lambda B(x)F(s),$$

where B is fixed and λ is a positive parameter.

Corollary 5.3. *Let us assume that* $k^2 > \omega^2/c^2\|\varepsilon_{\mathrm{L}}\|$, $B \succeq 0$ *and* Supp (B) *is a non-empty compact set. If the dielectric function is of Kerr type, then for all* $\lambda > 0$ *there exists a positive solution* $u_\lambda \in H^1(\mathbb{R})$ *of equation (14). Moreover,*

$$\lim_{\lambda\to 0^+}\|u_\lambda\| = +\infty, \qquad \lim_{\lambda\to+\infty}\|u_\lambda\| = 0.$$

If moreover $\varepsilon_{\mathrm{L}}(x)$, $B(x)$ *are even functions and* $0 \notin$ Supp (B)*, there exists a second branch of odd solutions* $\tilde{u}_\lambda \in H^1(\mathbb{R})$ *with the same limiting properties.*

Proof. The application of theorem 3.1 requires the existence of r_λ, R_λ such that

$$F\left(\frac{1}{2}r_\lambda^2\right) \leqslant \left(\frac{1}{c^2}\lambda\omega^2 \max_{x\in\,\mathrm{Supp}\,(B)}\int_{\mathrm{Supp}\,(B)} G(x,s)B(s)\mathrm{d}s\right)^{-1}$$

$$\leqslant \left(\frac{1}{c^2}\lambda\omega^2 m_1 p_0 \min_{x\in\,\mathrm{Supp}\,(B)}\int_{\mathrm{Supp}\,(B)} G(x,s)B(s)\mathrm{d}s\right)^{-1} \leqslant F\left(\frac{1}{2}R_\lambda^2\right).$$

Using (15), such $r_\lambda < R_\lambda$ exist and can be chosen so that

$$\lim_{\lambda\to 0^+} r_\lambda = +\infty, \qquad \lim_{\lambda\to+\infty} R_\lambda = 0.$$

Hence, we obtain a branch of solutions u_λ such that $r_\lambda \leqslant \|u_\lambda\| \leqslant R_\lambda/m_1 p_0$, and now a passing to the limit finishes the proof, since the arguments for the branch of odd solutions are analogous. ■

The proof of the following result is similar.

Corollary 5.4. *Let us assume that* $k^2 > \omega^2/c^2\|\varepsilon_{\mathrm{L}}\|$, $B \succeq 0$ *and* Supp (B) *is a non-empty compact set. If the dielectric function is of non-Kerr type, then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *there exists a positive solution* $u_\lambda \in H^1(\mathbb{R})$ *of equation (14). Moreover,*

$$\lim_{\lambda\to+\infty}\|u_\lambda\| = 0.$$

If moreover $\varepsilon_{\mathrm{L}}(x)$, $B(x)$ *are even functions and* $0 \notin$ Supp (B)*, there exists a second branch of odd solutions* $\tilde{u}_\lambda \in H^1(\mathbb{R})$ *for* $\lambda > \tilde{\lambda}_0 > 0$ *with the same property.*

We emphasize that in the concrete model of a multi-layered optical structure, the hypothesis $0 \notin$ Supp (B) means that in order to assure the existence of odd bound states at least five layers are needed. Moreover, the layer containing 0 must have a linear response. The existence of this kind of solution in the remaining symmetric cases is an interesting open problem.

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