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STABILITY INVARIANCE OF A PERIODIC LINEAR SWITCHED SYSTEM

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Abstract. In this paper we study the stability character of the linear differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \mathbf{A}(t)$ is a piecewise constant matrix function of period $T > 0$, with $\mathbf{A}(t) \in \mathcal{C}_n$ for all t and a fixed class $\mathcal{C}_n$ of matrices of order n. Concretely, we are interested in the characterization of the permutations of the pieces of $\mathbf{A}$ which do not change the stability character of the equation. We completely solve the problem for a generalized version of Meissner equation which is also of interest from the physical point of view.
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1 Introduction

Let $\mathcal{C}_n$ be a certain fixed class of matrices of order $n \in \mathbb{N}$, and let $\mathbf{A}(t)$ be a piecewise-constant periodic matrix function of period $T > 0$,

$$\mathbf{A}(t) = \mathbf{A}_i,\ t \in \mathbf{I}_i = (h(i-1), hi];\ \ \mathbf{A}_i \in \mathcal{C}_n \text{ for all } i \leq N; \tag{1}$$

where $Nh = T$. Let us consider the linear system of differential equations

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t). \tag{2}$$

Such a system is a special case of the so-called *switched systems*. A switched system is typically defined by a family of continuous systems and a *switching signal* describing the jumps between them (in our case, the switching signal is periodic). This topic has become very popular, specially in the frame of control theory, and a considerable amount of references are avaiable (see for instance [1,3,9,10,14,15] only to mention some of them). In applications, switched systems arise in a natural way from processes which present abrupt changes of the conditions. In this sense, the survey [10] presents an extensive bibliogaphy with big number of applications.

We say that (2) is stable if all its solutions are bounded and we say that it is asymptotically stable if

$$\lim_{t \to \infty} ||x(t)|| = 0$$

for all solution of (2). Finally, we say that the equation is unstable if it is not stable. In this paper we are mainly interested in the problem:

(P) Characterize the sets of permutations $\sigma \in \mathbf{S}_N$ such that the stability character of $\mathbf{x}'(t) = \mathbf{A}_\sigma(t)\mathbf{x}(t)$ (where $\mathbf{A}_\sigma(t) = \mathbf{A}_{\sigma(i)}$, $t \in \mathbf{I}_i$, $i \leq N$ is also T-periodic) is the same as the stability character of (2).

Definition 1 *We say that $\sigma \in \mathbf{S}_N$ is an admissible permutation for the class C_n if the stability character of $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ is the same as the stability character of $\mathbf{x}'(t) = \mathbf{A}_\sigma(t)\mathbf{x}(t)$ for all choices of the matrix function $\mathbf{A}$ given by (1). We say that σ is a general admissible permutation if it is admissible for the classes $C_n = M_n(\mathbb{R})$, $n \geq 1$.*

A classical example in which (P) can be completely solved (see [2]) is given by Meissner's equation

$$x'' + \alpha(t)x = 0 \tag{3}$$

where α is a T-periodic piecewise constant function defined as

$$\alpha(t) = w_i \qquad t \in ((i-1)h, ih], \; i = 1, ..., N,$$

$h = T/N$ and $w_i > 0$ for all i. In Section 2 some general stability considerations are used in order to caracterize the set of general admissible permutation. In Section 3, problem (P) is solved for a generalized Meissner equation which is the n-dimensional version of the classical one [12].

2 Characterizations of stability

It is clear that we can prove a Floquet type theorem for the equations we have in consideration, since the continuity of $\mathbf{A}(t)$ is used in the proof of Floquet theorem only to guarantee the unicity of the solutions of (2) (with a general T-periodic function $\mathbf{A}(t)$) for each initial condition $\mathbf{x}(0) = \mathbf{x}_0$ (see [4]), but this is also true in our case, since we will understand that our solution is constructed by glueing the solutions of $\mathbf{x}' = \mathbf{A}_i\mathbf{x}$, $t \in (h(i-1), hi]$ where the initial condition is

$$\mathbf{x}((i-1)h) = \lim_{\varepsilon \to 0+} \mathbf{x}((i-1)h - \varepsilon).$$

Hence it is easy to prove the following

Theorem 1 ***(Floquet-type Theorem).*** *Let $\Phi(t)$ be a fundamental matrix solution of (2), and let $\mathbf{B} \in \mathbf{M}_n(\mathbb{R})$ be such that* $\exp(TB) = \Phi^{-1}(0)\Phi(T)$. *Then there exists a T-periodic matrix function $\mathbf{P}(\mathbf{t})$ such that*

$$\Phi(t) = \mathbf{P}(t)\exp(t\mathbf{B})$$

holds for all $t \in \mathbb{R}$.

Corollary 2 *The following claims are equivalent:*

(*a*) *all solutions of (2) are uniformly bounded*

(*b*) *There exists a constant* $C > 0$ *such that* $||\exp(t\mathbf{B})||_\infty \leq C$ *for all* $t \in \mathbb{R}$.

(*c*) $\mathbf{spec}(\mathbf{B}) \subseteq \mathbb{H} = \{z \in \mathbb{C} : Re z \leq 0\}$ *and, if* $\lambda \in \partial\mathbb{H}$ *is an eigenvalue of* $\mathbf{B}$*, its multiplicity is one.*

(*d*) $\rho(\Phi(T)) \leq 1$ *and, if* $\lambda \in \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ *is an eigenvalue of* $\Phi(T)$*, its multiplicity is one.*

Proof. $\mathbf{P}(t) = \Phi(t)\exp(-t\mathbf{B})$ is clearly a (T-periodic) continuous function. Furthermore, $\mathbf{P}(t)$ is nonsingular for all t. This implies that there are constants $\alpha, \beta > 0$ such that

$$\alpha \leq ||\mathbf{P}(t)||_\infty \leq \beta \text{ for all } t \in \mathbb{R}$$

and proves the equivalence of (*a*) and (*b*). Obviously, (*b*) is equivalent to say that all solutions of $\mathbf{x}'(t) = \mathbf{B}\mathbf{x}(t)$ are uniformly bounded, which is known to be equivalent to (*d*) (see [4]). Now, it follows from the equality $\mathbf{spec}(\exp(\mathbf{B})) = \exp(\mathbf{spec}(\mathbf{B}))$ for all matrix $\mathbf{B}$, where multiplicities are also preserved (see [4]), that (*c*) and (*d*) are equivalent claims. ■

Condition (*d*) of Corollary 2 appears in the study of stability of certain numerical methods for the solution of ODE. Concretely, the Linear Multistep Method given by

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}$$

is known to be stable if and only if the roots of the first characteristic polynomial associated to the method, $\rho(z) = \sum_{i=0}^{k} \alpha_i z^i$ all belong to the unit disc, and, if $|z_0| = 1$, $\rho(z_0) = 0$, then $\rho'(z_0) \neq 0$. This property was discovered by Dahlquist in [5] (see also [7]).

Definition 2 *We say that the matrix* $\mathbf{A}$ *satisfy the Dahlsquist condition if all its eigenvalues belong to the unit disc* $\mathbb{D}$ *and those eigenvalues of* $\mathbf{A}$ *which belong to the unit circle* $\partial\mathbb{D}$ *are simple.*

It follows from Corollary 2 that the study of admissible permutations associated to a certain class $\mathcal{C}_n$ of matrices is precisely the study of permutations $\sigma \in \mathbf{S}_N$ such that the monodromy matrix $\Phi(T)$ of $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ satisfies the Dahlquist condition if and only if the monodromy matrix $\Phi_\sigma(T)$ of $\mathbf{x}'(t) = \mathbf{A}_\sigma(t)\mathbf{x}(t)$ satisfies the Dahlquist condition, for all choices of $\mathbf{A}$ with the aditional restriction $\mathbf{A}_i \in \mathcal{C}_n$ for all i.

It is well known (see [11]) the fact that $\mathbf{p}(AB) = \mathbf{p}(BA)$ for all matrices $A, B \in M_n(\mathbb{R})$, where $\mathbf{p}(A)(z) = \det(A - z\mathbf{I}_n)$ denotes the characteristic polynomial of A. This does not imply the equality of the polynomials

$$\mathbf{p}(A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)})$$

for arbitrary matrices $A_i \in M_n(\mathbb{R})$ and arbitrary permutations $\sigma \in \mathbf{S}_N$ ($N \geq 3$, $n \geq 2$), as it is proved by the following example

Example 1 *Set* $A = \begin{pmatrix} 1 & 1/2 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ *and* $C = \frac{3}{4}\begin{pmatrix} 0 & 1 \\ 1 & \frac{-1}{2} \end{pmatrix}$. *Then*

$$\mathbf{p}\,(ABC) = \mathbf{p}\begin{pmatrix} \frac{3}{8} & \frac{3}{16} \\ 0 & \frac{-3}{4} \end{pmatrix} = (z+\frac{3}{4})(z-\frac{3}{8})$$

and

$$\mathbf{p}\,(ACB) = \mathbf{p}\begin{pmatrix} \frac{-3}{16} & \frac{9}{16} \\ \frac{3}{4} & \frac{-3}{4} \end{pmatrix} = (z+\frac{3(5+\sqrt{57})}{32})(z-\frac{3(-5+\sqrt{57})}{32})$$

This proves the existence of matrices $A, B, C \in M_2(\mathbb{R})$ *such that* $\rho(A)$, $\rho(B)$, $\rho(C) \leq 1$, $\rho(ACB)$ *is less than* 1 *but* $\rho(ACB) > 1$. *In particular,* $\mathbf{p}(ABC) \neq \mathbf{p}(ACB)$.

So, it is of interest to know for which permutations σ we can guarantee that

$$\mathbf{p}(A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)}) = \mathbf{p}(A_1A_2\cdots A_N)$$

for arbitrary matrices $A_i \in M_n(\mathbb{R})$.

We will only consider the case $n = 2$ since if $\mathbf{p}(A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)}) \neq \mathbf{p}(A_1A_2\cdots A_N)$ for some permutation $\sigma \in \mathbf{S}_N$ and some matrices $A_i \in M_2(\mathbb{R})$, then for all $n > 2$ we can take

$$B_i = \begin{pmatrix} A_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-2} \end{pmatrix}, \; i = 1, \dots, N,$$

(where $\mathbf{I}_k$ denotes the identity matrix of order k) and it is clear that

$$\mathbf{p}(B_{\sigma(1)}B_{\sigma(2)}\cdots B_{\sigma(n)}) \neq \mathbf{p}(B_1B_2\cdots B_n)$$

Theorem 3 *Let* $N \geq 3$ *be fixed. Then* $\sigma \in \mathbf{S}_N$ *satisfies*

$$\mathbf{p}(A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)}) = \mathbf{p}(A_1A_2\cdots A_N)$$

for arbitrary matrices $A_i \in M_n(\mathbb{R})$ *if and only if*

$$\sigma \in \mathbf{span}\{\tau\} = \{\mathbf{Id}, \tau, \tau^2, \dots, \tau^{N-1}\},$$

where τ *is the* N*-cycle* $\tau = (1\ 2\ 3\ \cdots\ N\ )$.

Proof First of all, we note that the set

$$G = \{\sigma \in \mathbf{S}_N : \mathbf{p}(A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(n)}) = \mathbf{p}(A_1A_2\cdots A_n) \text{ for all } A_i \in M_n(\mathbb{R})\}$$

is a subgroup of $\mathbf{S}_N$. This follows from the fact that if $\sigma_1, \sigma_2 \in G$, then

$$\begin{aligned} \mathbf{p}(A_{\sigma_2(\sigma_1(1))}A_{\sigma_2(\sigma_1(2))}\cdots A_{\sigma_2(\sigma_1(n))}) &= \mathbf{p}(A_{\sigma_1(1)}A_{\sigma_1(2)}\cdots A_{\sigma_1(n)}) \\ &= \mathbf{p}(A_1A_2\cdots A_n) \end{aligned}$$

for arbitrary matrices $A_i \in M_n(\mathbb{R})$ and that all elements of $\mathbf{S}_N$ are of finite order. What we want to prove is that $G = \mathbf{span}\{\tau\}$, where $\tau(k) = k+1$ for all $k < N$ and $\tau(N) = 1$.

It is clear that $\mathbf{span}\{\tau\}$ is a subset of G since

$$\begin{aligned} \mathbf{p}(A_1(A_2 \cdots A_N)) &= \mathbf{p}((A_2 \cdots A_N)A_1) \\ &= \mathbf{p}(A_{\tau(1)}A_{\tau(2)} \cdots A_{\tau(n)}) \end{aligned}$$

for arbitrary matrices $A_i \in M_n(\mathbb{R})$.

On the other hand, let us assume that $\sigma \notin \mathbf{span}\{\tau\}$. Let $p = \sigma(1)$. Then $\tau^{N-p+1}\sigma(1) = \tau^{N-p+1}(p) = 1$. Now, it is clear that $\tau^{N-p+1}\sigma \neq \mathbf{Id}$ since $\sigma \notin \mathbf{span}\{\tau\}$. Let i_0 be the least integer such that $\tau^{N-p+1}\sigma(i) \neq i$ and let $j_0 = \tau^{N-p+1}\sigma(i_0)$. Then $j_0 > i_0$ and $\tau^{N-p+1}\sigma$ is of the form

$$\tau^{N-p+1}\sigma = \begin{pmatrix} 1 & \ldots & i_0 - 1 & i_0 & i_0+1 & \ldots & k & \ldots & N \\ 1 & \ldots & i_0 - 1 & j_0 & \tau^{N-p+1}\sigma(i_0+1) & \ldots & i_0 & \ldots & \tau^{N-p+1}\sigma(N) \end{pmatrix}$$

Let $A, B, C \in M_2(\mathbb{R})$ be such that $\mathbf{p}(ABC) \neq \mathbf{p}(ACB)$ and set $A_1 = A$, $A_{i_0} = B$, $A_{j_0} = C$, and $A_k = \mathbf{I}_n$ for all $k \notin \{1, i_0, j_0\}$. Then

$$\begin{aligned} \mathbf{p}(ACB) &= \mathbf{p}(A_{\tau^{N-p+1}\sigma(1)}A_{\tau^{N-p+1}\sigma(2)}A_{\tau^{N-p+1}\sigma(3)} \cdots A_{\tau^{N-p+1}\sigma(N)}) \\ &= \mathbf{p}(A_{\tau^{p-1}(\tau^{N-p+1}\sigma(1))}A_{\tau^{p-1}(\tau^{N-p+1}\sigma(2))} \cdots A_{\tau^{p-1}(\tau^{N-p+1}\sigma(N))}) \\ &= \mathbf{p}(A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(N)}) \end{aligned}$$

and

$$\mathbf{p}(A_1A_2A_3 \cdots A_N) = \mathbf{p}(ABC) \neq \mathbf{p}(ACB) = \mathbf{p}(A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(N)})$$

so that $\sigma \notin G$. ■

Corollary 4 *$\sigma \in \mathbf{S}_N$ is a general admissible permutation if and only if $\sigma \in \mathbf{span}\{\tau\}$.*

Proof. It follows from Theorem 3 that the elements $\sigma \in \mathbf{span}\{\tau\}$ are general admissible permutations. Let $A, B, C \in M_2(\mathbb{R})$ be the matrices given in Example 1, and let $\sigma \in \mathbf{S}_N \setminus \mathbf{span}\{\tau\}$. We know from the proof of Theorem 3 that there exists a way to choose matrices $A_i \in M_2(\mathbb{R})$ $(i \leq N)$ such that $ABC = A_1A_2A_3 \cdots A_N$ and $ACB = A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(N)}$. For each $n \geq 2$, we set

$$\mathbf{A}_i = \begin{pmatrix} A_i & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{I}_{n-2} \end{pmatrix}, \; i = 1, 2, \ldots, N.$$

It is clear that

$$\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \cdots \mathbf{A}_N = \begin{pmatrix} ABC & \mathbf{0} \\ 0 & \frac{1}{2^N}\mathbf{I}_{n-2} \end{pmatrix}$$

satisfies Dahlquist condition, but $-\frac{3(5+\sqrt{57})}{32}$ is an eigenvalue of

$$\mathbf{A}_{\sigma(1)}\mathbf{A}_{\sigma(2)}\cdots\mathbf{A}_{\sigma(N)} = \begin{pmatrix} ACB & \mathbf{0} \\ \mathbf{0} & \frac{1}{2^N}\mathbf{I}_{n-2} \end{pmatrix}$$

Hence σ is not a general admissible permutation. ∎

Remark 1 *If $\sigma \in \mathbf{S}_N$ is not a general admissible permutation, then there are matrices $\{\mathbf{A}_i\}_{i=1}^N$ such that $\rho(\mathbf{A}_i) < 1$ for all i, and $\rho(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_N) < 1$ but $\rho(\mathbf{A}_{\sigma(1)}\mathbf{A}_{\sigma(2)}\cdots\mathbf{A}_{\sigma(N)}) > 1$. This proves that general admissible permutations are also the only permutations which preserve the spectral radius. This is also true in the Banach Algebra setting.*

Proof. We have already proved the existence of matrices $\mathbf{A}_i$ such that $\rho(\mathbf{A}_i) = 1$ for all i, $\rho(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_N) < 1$ and $\rho(\mathbf{A}_{\sigma(1)}\mathbf{A}_{\sigma(2)}\cdots\mathbf{A}_{\sigma(N)}) > 1$.

Now set $\mathbf{B}_i = (1-\varepsilon)\mathbf{A}_i$, $i \leq N$ for $\varepsilon > 0$ small enough. Then

$$\rho(\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\cdots\mathbf{B}_N) < (1-\varepsilon)^N < 1,$$

$\rho(\mathbf{B}_i) = 1-\varepsilon < 1$ for all i, and

$$\rho(\mathbf{B}_{\sigma(1)}\mathbf{B}_{\sigma(2)}\cdots\mathbf{B}_{\sigma(N)}) = (1-\varepsilon)^N\rho(\mathbf{A}_{\sigma(1)}\mathbf{A}_{\sigma(2)}\cdots\mathbf{A}_{\sigma(N)}) > 1. \blacksquare$$

3 Generalized Meissner Equation

Now we will study the second order linear differential equation

$$\mathbf{x}'' + \mathbf{\Lambda}(t)\mathbf{x} = 0 \tag{4}$$

where $\mathbf{\Lambda}(t)$ is a T-periodic piecewise constant matrix function defined as

$$\mathbf{\Lambda}(t) = \mathbf{\Lambda}_i \qquad t \in ((i-1)h, ih], \; i = 1, ..., N;$$

where $h = T/N$ and $0 \notin \mathbf{spec}(\mathbf{\Lambda}_i)$ for all i. From the physical point of view, this is a Hamiltonian system that can model several mechanical and electrical systems, for instance mechanical machines composed by several bodies connected by springs with variable coefficients or under abrupt changes in the masses of the bodies (see for instance [13]).

Of course (4) is equivalent to the first order linear differential equation

$$\mathbf{X}'(t) = \begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{\Lambda}(t) & \mathbf{0} \end{pmatrix} \mathbf{X}(t).$$

It is very easy to verify that the monodromy matrix is

$$\mathbf{\Phi}(T) = \mathbf{B}_N\mathbf{B}_{N-1}\cdots\mathbf{B}_1,$$

where

$$\mathbf{B}_i = \exp\begin{pmatrix} \mathbf{0} & h\mathbf{I}_n \\ -h\boldsymbol{\Lambda}_i & \mathbf{0} \end{pmatrix},\ i = 1, 2, ..., N$$

If $\mathbf{x}(t)$ is solution of $\mathbf{x}''+\boldsymbol{\Lambda}(t)\mathbf{x} = 0$ then $\mathbf{x}(-t)$ is solution of $\mathbf{x}''+\boldsymbol{\Lambda}(-t)\mathbf{x} = 0$ and, in consecuence both equations have the same stability character. Moreover, the monodromy matrix of the second equation is

$$\boldsymbol{\Psi}(T) = \mathbf{B}_1\mathbf{B}_2\cdots\mathbf{B}_N,$$

where $\boldsymbol{\Phi}(T) = \mathbf{B}_N\mathbf{B}_{N-1}\cdots\mathbf{B}_1$ is the monodromy matrix of the original equation. Hence Dahlquist condition is satisfied by $\mathbf{B}_1\mathbf{B}_2\cdots\mathbf{B}_N$ if and only if it is satisfied by $\mathbf{B}_N\mathbf{B}_{N-1}\cdots\mathbf{B}_1$ for all matrices $\mathbf{B}_i$ of the form

$$\mathbf{B}_i = \exp\begin{pmatrix} \mathbf{0} & h\mathbf{I}_n \\ -h\boldsymbol{\Lambda}_i & \mathbf{0} \end{pmatrix} \tag{5}$$

This is not in contradiction with Corollary 4. It only claims that the set of permutations which preserve Dahlquist condition of a product of matrices of the form (5) is bigger than $\mathbf{span}\{\tau\}$. In what follows, we will denote by $\mathcal{N}_n$ the set of all matrices of the form (5) and by $\mathcal{D}_N$ the set of permutations $\sigma \in \mathbf{S}_N$ such that Dahlquist condition is satisfied by $\mathbf{B}_1\mathbf{B}_2\cdots\mathbf{B}_N$ if and only if it is satisfied by $\mathbf{B}_{\sigma(1)}\mathbf{B}_{\sigma(2)}\cdots\mathbf{B}_{\sigma(N)}$ for all matrices $\mathbf{B}_i \in \mathcal{N}_n$.

Remark 2 *We will use in the rest of the paper the following notation:*

$$\mathbf{B}_1\mathbf{B}_2\cdots\mathbf{B}_N \sim \mathbf{C}_1\mathbf{C}_2\cdots\mathbf{C}_N$$

means that there exists a certain $\sigma \in \mathcal{D}_N$ such that

$$\mathbf{C}_1\mathbf{C}_2\cdots\mathbf{C}_N = \mathbf{B}_{\sigma(1)}\mathbf{B}_{\sigma(2)}\cdots\mathbf{B}_{\sigma(N)}.$$

It is clear that $\mathbf{span}\{\tau, \zeta\}$ is a subset of $\mathcal{D}_N$, where τ is the N-cycle $\tau = (1\ 2\ 3\ \cdots N\ )$ and $\zeta(k) = N - k + 1$ for all k. Furthermore, if $N \leq 3$ then $\mathbf{span}\{\tau, \zeta\} = \mathcal{D}_N = \mathbf{S}_N$. In what follows we will assume that $N \geq 4$.

Now we can write the main result of this section, which is the following

Theorem 5 *The following claims hold true*

(*A*) $\mathcal{D}_N = \mathbf{span}\{\tau, \zeta\}$. *Moreover, $\mathcal{D}_N$ is a group with the composition of functions as operation, which is isomorphic to the diedral group of order N (and this justifies our notation).*

(*B*) *For all $\sigma \in \mathbf{S}_N \setminus \mathbf{span}\{\tau, \zeta\}$ there exists a generalized Meissner's equation $\mathbf{x}'' + \boldsymbol{\Lambda}\mathbf{x} = \mathbf{0}$ which is stable, such that $\mathbf{x}'' + \boldsymbol{\Lambda}_\sigma\mathbf{x} = 0$ is not stable.*

To prove this Theorem, we will first prove the Lemmas:

Lemma 6 $\mathbf{span}\{\tau, \zeta\}$ *is isomorphic to the diedral group of order N.*

Proof It follows from the fact that τ is an n-cycle that it has order N. On the other hand,

$$\zeta^2(k) = \zeta(N-k+1) = N - N + k - 1 + 1 = k$$

for all k, so that ζ is an element of order two. Hence we only need to prove that the relation $\zeta\tau = \tau^{N-1}\zeta$ holds. Now, taking in consideration that

$$\tau^{N-1} = \tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ N & 1 & 2 & \cdots & N-1 \end{pmatrix},$$

it is easy to check the relation

$$\tau^{N-1}\zeta = \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ N-1 & N-2 & N-3 & \cdots & N \end{pmatrix} = \zeta\tau . \blacksquare$$

Lemma 7 *Let* $\mathbf{H} \in M_n(\mathbb{C})$. *Then*

$$\lambda \in \mathbf{spec}\begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{H} & \mathbf{0} \end{pmatrix} \Longleftrightarrow -\lambda^2 \in \mathbf{spec}(\mathbf{H})$$

Proof The proof follows from the relation

$$\begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{H} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \Longleftrightarrow \left\{ \begin{array}{ccc} \mathbf{b} & = & \lambda\mathbf{a} \\ \lambda^2\mathbf{a} & = & -\mathbf{H}\mathbf{a} \end{array} \right.$$

Lemma 8 *Let* $\mathbf{D}_n = \mathbf{diag}[1-\varepsilon_1, 1-\varepsilon_2, ..., 1-\varepsilon_n]$ *be a perturbation of the identity, where we assume that the* ε_i *are pairwise distinct,* $\varepsilon_i \in [0,1]$ *for all* i. *Let us denote by* $\mathbf{A}$ *and* $\mathbf{B}$ *the matrices*

$$\begin{aligned} \mathbf{A} &= \exp\begin{pmatrix} \mathbf{0} & \frac{\pi}{N^2}\mathbf{I}_n \\ -\frac{\pi}{N^2}\mathbf{D}_n^2 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{N^2}\mathbf{D}_n) & \mathbf{D}_n^{-1}\sin(\frac{\pi}{N^2}\mathbf{D}_n) \\ -\mathbf{D}_n\sin(\frac{\pi}{N^2}\mathbf{D}_n) & \cos(\frac{\pi}{N^2}\mathbf{D}_n) \end{pmatrix} \\ \mathbf{B} &= \exp\begin{pmatrix} \mathbf{0} & \frac{\pi}{N^2}\mathbf{I}_n \\ -\frac{\pi}{N^2}\cdot\frac{N^4}{4}\mathbf{I}_n & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \frac{2}{N^2}\mathbf{I}_n \\ -\frac{N^2}{2}\mathbf{I}_n & \mathbf{0} \end{pmatrix} \end{aligned}$$

where N *is supposed to be an integer* $N > 4$, *and let* $p, q \in \{1, ..., N-3\}$ *be such that* $p+q = N-2$. *Then* $\mathbf{A}^{N-2}\mathbf{B}^2$ *satisfies the Dahlquist condition but* $\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}$ *doest not satisfy the Dahlquist condition.*

Proof It follows from the relation $\mathbf{B}^2 = -\mathbf{I}_{2n}$ and from Lemma 6 that $\mathbf{A}^{N-2}\mathbf{B}^2 = -\mathbf{A}^{N-2}$ has all its eigenvalues simple and in the boundary of the unit disc, since

$$\begin{aligned} \mathbf{spec}(-\mathbf{A})^{N-2} &= -\left(\mathbf{spec}\exp\begin{pmatrix} \mathbf{0} & \frac{\pi}{N^2}\mathbf{I}_n \\ -\frac{\pi}{N^2}\mathbf{D}_n & \mathbf{0} \end{pmatrix}\right)^{N-2} \\ &= -\left(\exp\mathbf{spec}\begin{pmatrix} \mathbf{0} & \frac{\pi}{N^2}\mathbf{I}_n \\ -\frac{\pi}{N^2}\mathbf{D}_n & \mathbf{0} \end{pmatrix}\right)^{N-2} \\ &= \left\{-\exp\left(\frac{\pm(N-2)\sqrt{\pi(1-\varepsilon_1)}}{N}\mathbf{i}\right), \cdots, -\exp\left(\frac{\pm(N-2)\sqrt{\pi(1-\varepsilon_n)}}{N}\mathbf{i}\right)\right\} \end{aligned}$$

Hence $\mathbf{A}^{N-2}\mathbf{B}^2$ satisfy the Dahlquist condition.

On the other hand,

$$\begin{aligned}\mathbf{A}^k &= \exp\begin{pmatrix} \mathbf{0} & \frac{k\pi}{N^2}\mathbf{I}_n \\ -\frac{k\pi}{N^2}\mathbf{D}_n^2 & \mathbf{0}\end{pmatrix} \\ &= \begin{pmatrix} \cos(\frac{k\pi}{N^2}\mathbf{D}_n) & \mathbf{D}_n^{-1}\sin(\frac{k\pi}{N^2}\mathbf{D}_n) \\ -\mathbf{D}_n\sin(\frac{k\pi}{N^2}\mathbf{D}_n) & \cos(\frac{k\pi}{N^2}\mathbf{D}_n)\end{pmatrix}\end{aligned}$$

for all integer k. Hence

$$\begin{aligned}&\mathbf{tr}(\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}) \\ &= \mathbf{tr}\left((\tfrac{N^4}{4}\mathbf{D}_n^{-2} + \tfrac{4}{N^4}\mathbf{D}_n^2)\sin(\tfrac{p\pi}{N^2}\mathbf{D}_n)\sin(\tfrac{q\pi}{N^2}\mathbf{D}_n) - 2\cos(\tfrac{p\pi}{N^2}\mathbf{D}_n)\cos(\tfrac{q\pi}{N^2}\mathbf{D}_n)\right) \\ &= \textstyle\sum_{k=1}^n \left((\tfrac{N^4}{4}(1-\varepsilon_k)^{-2} + \tfrac{4}{N^4}(1-\varepsilon_k)^2)\sin(\tfrac{p\pi}{N^2}(1-\varepsilon_k))\sin(\tfrac{q\pi}{N^2}(1-\varepsilon_k))\right) \\ &\quad -2\textstyle\sum_{k=1}^n \cos(\tfrac{p\pi}{N^2}(1-\varepsilon_k))\cos(\tfrac{q\pi}{N^2}(1-\varepsilon_k))\end{aligned}$$

It is easy to check that, (where we need to use that $\sin \pi x \geq 2\sqrt{2}x$ for all $x \in [0, \frac{\pi}{4}]$, and $\max\{\frac{p\pi}{N^2}(1-\varepsilon_k), \frac{q\pi}{N^2}(1-\varepsilon_k)\} \leq \frac{\pi}{4}$, since $\max\{\frac{p\pi}{N^2}, \frac{q\pi}{N^2}\} \leq \frac{\pi}{4}$ if and only if $N^2 \geq 4\max\{p,q\}$, which obviously follows from the relation $p+q = N-2$)

$$\begin{aligned}\mathbf{tr}(\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}) &\geq \sum_{k=1}^n \left((\frac{N^4}{4}(1-\varepsilon_k)^{-2} + \frac{4}{N^4}(1-\varepsilon_k)^2)\frac{8pq}{N^4}(1-\varepsilon_k)^2\right) - 2n \\ &= \sum_{k=1}^n \frac{2pqN^8 + 32(1-\varepsilon_k)^4 pq}{N^8} - 2n \\ &= \sum_{k=1}^n (2(pq-1) + \frac{32(1-\varepsilon_k)^4 pq}{N^8}) > 2n \text{ (since } pq - 1 \geq 1).\end{aligned}$$

This proves that the modulus of at least one of the eigenvalues of $\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}$ is greater than 1. Hence this matrix does not satisfy the Dahlquist condition. ■

Lemma 9 *There are matrices $\mathbf{A}, \mathbf{B} \in \mathcal{N}_n$ such that $\mathbf{A}^2\mathbf{B}^2$ satisfies the Dahlquist condition but $\mathbf{ABAB}$ does not.*

Proof. We set

$$\begin{aligned}\mathbf{A} &= \exp\begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{D}_n^2 & \mathbf{0}\end{pmatrix} = \begin{pmatrix} \cos(\mathbf{D}_n) & \mathbf{D}_n^{-1}\sin(\mathbf{D}_n) \\ -\mathbf{D}_n\sin(\mathbf{D}_n) & \cos(\mathbf{D}_n)\end{pmatrix} \\ \mathbf{B} &= \exp\begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ -\frac{\pi^2}{4}\mathbf{I}_n & \mathbf{0}\end{pmatrix} = \begin{pmatrix} \mathbf{0} & \frac{2}{\pi}\mathbf{I}_n \\ -\frac{\pi}{2}\mathbf{I}_n & \mathbf{0}\end{pmatrix},\end{aligned}$$

where $\mathbf{D}_n = \mathbf{diag}[d_1, ..., d_n]$, $d_k = \frac{\pi}{2} + 2k\pi$; $k \leq n$. Then $\mathbf{A}^2\mathbf{B}^2$ satisfies the Dahlsquist condition since its spectrum

$$\mathbf{spec}(\mathbf{A}^2\mathbf{B}^2) = \{-e^{\pm 2i\sqrt{\frac{\pi}{2}+2k\pi}} : k \in \{1, 2, ..., n\}\}$$

has all its eigenvalues simple and all of them belong to the unit circle. On the other hand, the spectral radius of $\mathbf{AB}$ is greater than 1 since

$$\begin{aligned} |\mathbf{tr}(\mathbf{AB})| &= \left|\frac{\pi}{2}\mathbf{tr}(\mathbf{D}_n^{-1}\sin(\mathbf{D}_n)) + \frac{2}{\pi}\mathbf{tr}(\mathbf{D_n}\sin(\mathbf{D_n}))\right| \\ &= \sum_{k=1}^{n}\left(\frac{2}{(2k+1)\pi}\frac{\pi}{2} + \frac{2}{\pi}\frac{(2k+1)\pi}{2}\right) > 2n. \end{aligned}$$

Hence

$$\rho(\mathbf{ABAB}) = \rho(\mathbf{AB})^2 > 1. \blacksquare$$

Proof of the main result We only need to prove claim (B) of the Theorem, since (B) implies (A) obviously. To do this, it will be enough to prove that for all $\sigma \in \mathbf{S}_N \setminus \mathbf{span}\{\tau, \zeta\}$ there are matrices $A_i \in \mathcal{N}_n$, $i = 1, ..., N$ such that $A_1A_2\cdots A_N$ satisfies the Dahlquist condition but $A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)}$ doest not satisfy the Dahlquist condition.

There is no lost of generality in assuming that $\sigma(1) = 1$, since $\mathbf{span}\{\tau\} \subseteq \mathcal{D}_N$ implies that

$$A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(n)} \sim A_{\tau^s\sigma(1)}A_{\tau^s\sigma(2)}\cdots A_{\tau^s\sigma(n)} \text{ for } s = 1, ..., N.$$

Let $k = \max\{i : \sigma(i) < i\}$. It is clear that $k > 2$ and that $\sigma_{|\{k+1,...,N\}} = 1_{|\{k+1,...,N\}}$. We make the following cases:

Case 1: $k < N$. Then there exists some $t \in \{2, ..., k-1\}$ such that $\sigma(t) = k$ and taking

$$A_i = \begin{cases} \mathbf{B} & i \in \{k, k+1\} \\ \mathbf{A} & \text{otherwise} \end{cases}$$

we have that

$$A_{\sigma(i)} = \begin{cases} \mathbf{B} & i \in \{t, k+1\} \\ \mathbf{A} & \text{otherwise} \end{cases}$$

where $\mathbf{A}, \mathbf{B}$ are defined in Lemma 4. Hence

$$A_1A_2\cdots A_N = \mathbf{A}^{k-1}\mathbf{B}^2\mathbf{A}^{N-k-1} \sim \mathbf{A}^{N-2}\mathbf{B}^2$$

satisfies the Dahlquist condition, since $\mathbf{A}^{N-2}\mathbf{B}^2$ satisfies this condition. On the other hand,

$$A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)} = \mathbf{A}^{t-1}\mathbf{B}\mathbf{A}^{k-t+1}\mathbf{B}\mathbf{A}^{N-2-k} \sim \mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}$$

where $p + q = N - 2$, $p = N + t - k - 3$, and $q = k - t + 1$; but Lemma 8 proves that $\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}$ does not satisfy Dahlquist condition. This proves our claim for the case 1.

Case 2: $k = N$. Then there exists some $t \in \{2, ..., N-1\}$ such that $\sigma(t) = N$. If $t = 2$, then

$$\begin{aligned} A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)} &\sim A_1 A_N A_{\sigma(2)}\cdots A_{\sigma(N)} \\ &\sim A_{\sigma(N)}\cdots A_{\sigma(2)}A_N A_1 \\ &\sim A_1 A_{\sigma(N)}\cdots A_{\sigma(2)}A_N \\ &\sim A_{\theta(1)}A_{\theta(2)}\cdots A_{\theta(N)} \end{aligned}$$

for a certain $\theta \in \mathbf{S}_N \setminus \mathbf{span}\{\tau, \zeta\}$ which is in the first case. If $t > 2$, taking

$$A_i = \begin{cases} \mathbf{B} & i \in \{1, N\} \\ \mathbf{A} & \text{otherwise} \end{cases}$$

we have that

$$A_{\sigma(i)} = \begin{cases} \mathbf{B} & i \in \{1, t\} \\ \mathbf{A} & \text{otherwise} \end{cases}$$

Hence

$$A_1 A_2 \cdots A_N = \mathbf{B}\mathbf{A}^{N-2}\mathbf{B} \sim \mathbf{A}^{N-2}\mathbf{B}^2$$

satisfies the Dahlquist condition and

$$A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)} \sim \mathbf{B}\mathbf{A}^p\mathbf{B}\mathbf{A}^q \sim \mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}$$

for certain integers p, q such that $p + q = N - 2$. We use again Lemma 8 to prove that $A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(N)}$ does not satisfies the Dahlquist condition ■.

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