Journal of Dynamics and Differential Equations, Vol. 17, No. 1, January 2005 (© 2005)
DOI: 10.1007/s10884-005-2937-4

Twist Character of the Fourth Order Resonant Periodic Solution

Jinzhi Lei[1], **Pedro J. Torres**[2], **and Meirong Zhang**[3,4]

Received October 17, 2003

In this paper, we will give, for the periodic solution of the scalar Newtonian equation, some twist criteria which can deal with the fourth order resonant case. These are established by developing some new estimates for the periodic solution of the Ermakov–Pinney equation, for which the associated Hill equation may across the fourth order resonances. As a concrete example, the least amplitude periodic solution of the forced pendulum is proved to be twist even when the frequency acroses the fourth order resonances. This improves the results in Lei *et al*. (2003). Twist character of the least amplitude periodic solution of the forced pendulm. *SIAM J. Math. Anal.* **35**, 844–867.

KEY WORDS: Twist character; periodic solution; fourth order resonance; third order approximation; forced pendulum.

1. INTRODUCTION

In this paper we are concerned with the twist character of the 2π-periodic solution of the scalar Newtonian equation

$$\ddot{x} + g(t, x) = 0, \tag{1.1}$$

where $g = g(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is 2π-periodic in t and of class $C^{0,4}$ in (t, x). Special attention is paid to the case of the fourth order resonant periodic solutions.

[1] Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing 100084, China. E-mail: jzlei@mail.tsinghua.edu.cn
[2] Universidad de Granada, Departamento de Matemática Aplicada, 18071 Granada, Spain. E-mail: ptorres@ugr.es
[3] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China. E-mail: mzhang@math.tsinghua.edu.cn
[4] To whom correspondence should be addressed.

1040-7294/05/0100-0021/0 © 2005 Springer Science+Business Media, Inc.

Suppose that $x = u(t)$ is a 2π-periodic solution of (1.1). The first order approximation, or the linearization, of (1.1) near $u(t)$ is the Hill equation

$$\ddot{x} + a(t)x = 0, \tag{1.2}$$

where

$$a(t) = g_x(t, u(t)).$$

Let $\lambda_{1,2}$ denote the Floquet multipliers of (1.2). We say that (1.2) is *hyperbolic*, *elliptic*, or *parabolic* if $|\lambda_{1,2}| \neq 1$, $|\lambda_{1,2}| = 1$ but $\lambda_{1,2} \neq \pm 1$, or $\lambda_{1,2} = \pm 1$, respectively.

Since (1.1) is a conservative system, the stability of the periodic solution $u(t)$ of (1.1) cannot be deduced directly from that of (1.2). However, the stability of $u(t)$ 'almost' implies that (1.2) is stable, i.e., (1.2) is elliptic, although some exceptional cases do happen [7].

When (1.2) is elliptic, we say that $u(t)$ is *resonant* of the order k if $\lambda_{1,2}^k = 1$ but $\lambda_{1,2}^j \neq 1$ for all $1 \leqslant j < k$. We say that $u(t)$ is *strongly resonant* if $\lambda_{1,2}^k = 1$ for some $1 \leqslant k \leqslant 4$. When $u(t)$ is not strongly resonant, or 4-elementary [12], or has no resonances up to order 4, the stability of $u(t)$ can be studied using higher order approximation of (1.1) along $u(t)$. In a series of papers [10–13], R. Ortega gave an extensive study for the third order approximation of (1.1) along $u(t)$. That is, one considers the equation

$$\ddot{x} + a(t)x + b(t)x^2 + c(t)x^3 + \cdots = 0 \tag{1.3}$$

with $a(t)$ is as above and $b(t), c(t)$ are

$$b(t) = \frac{1}{2} g_{xx}(t, u(t)), \quad c(t) = \frac{1}{6} g_{xxx}(t, u(t)).$$

Here the solution $u(t)$ of (1.1) has been transformed to the origin $x(t) = 0$ of (1.3). Based on the theory of Birkhoff normal form [1, Section 3.6] and the Moser twist theorem [5, 14], he has derived out the (first) twist coefficient of (1.3) and then finds many interesting applications, especially when (1.2) is within the first stability zone (for the definition, see Section 3 below). Later, some refinement and other applications of these results are given in [3, 6, 8, 9, 18]. However, these applications deal with only the cases that (1.2) is in the lower order stability zones.

In a recent work [2], motivated by the least amplitude periodic solution of the forced pendulum

$$\ddot{x} + \omega^2 \sin x = p(t), \tag{1.4}$$

where the frequency $\omega > 0$ and the forcing $p(t)$ is 2π-periodic, the authors have developed some interesting estimates so that (1.2) may be in arbitrarily higher order stability zones. It is also found there that the estimates of the (unique) positive 2π-periodic solution $r(t)$ of the Ermakov–Pinney equation [16]

$$\ddot{r} + a(t)r = \frac{1}{r^3} \tag{1.5}$$

are crucial in studying the twist character, because $r(t)$ represents the growth of the Floquet solutions of (1.2).

However, these works mentioned above are mainly concerned with the non-strongly resonant case. The main aim of this paper is to develop some new twist criteria so that the fourth order resonant case can be studied following the ideas in [12] and some satisfactory applications to (1.4) can be obtained.

In order to describe the results, let us introduce some notation. We use $C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ to denote the space of continuous 2π-periodic real-valued functions. For $x \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ and an exponent $\alpha \in [1, \infty]$, we use $\|x\|_\alpha$ to denote the L^α norm of $x(t)$ over $[0, 2\pi]$. For any $0 < \sigma_1 \leqslant \sigma_2$, let

$$C_{\sigma_1,\sigma_2} = \{a(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}) | a(t) \text{ satisfies } \sigma_1^2 \leqslant a(t) \leqslant \sigma_2^2 \text{ for all } t\}.$$

For any $1 \leqslant k \leqslant 4$, define the set

$$\Omega_k = \{\omega > 0 | \omega \neq p/q \text{ for all } p, q \in \mathbb{N} \text{ with } 1 \leqslant q \leqslant k\}.$$

Then each Ω_k is a countable union of intervals.

Recall that a basic case for (1.2) to have no strong resonances is that the coefficient $a(t) \in C_{\sigma_1,\sigma_2}$ for some σ_1, σ_2, where σ_1, σ_2 are from the same interval of Ω_4. In this case, it is proved in [2] that the L^4 norm of $r(t)$, the solution of (1.5), can be estimated using a connection between (1.5) and the Riccati equation on the complex plane. See [2, Lemma 3.6].

In Section 2, we will find a new domain $\mathscr{D}_2$ of (σ_1, σ_2) so that $\|r\|_4$ can be estimated when $a(t) \in C_{\sigma_1,\sigma_2}$. See Theorems 2.4 and 2.8. The domain $\mathscr{D}_2$ contains the fourth order resonances $\sigma_1 = \sigma_2 = (2k-1)/4, k \in \mathbb{N}$. These estimates are very accurate in some sense and are crucial in the latter discussions because they enable us to study the cases where σ_1, σ_2 may across the fourth order resonances $(2k-1)/4, k \in \mathbb{N}$.

In Section 3, we will present some new twist criteria which are based on the estimates in Section 2 and some ideas in [2, 12]. The difference between the results here and that in the previous work [2] is that we can now study the fourth order resonances. See Theorems 3.1 and 3.2.

In Section 4, we will apply the results to the twist character of periodic solution of the scalar Newtonian equation (1.1). A particular example is the forced pendulum (1.4). In this case, we will find an explicit bound $P(\omega)$, which is defined on Ω_3, such that if $\|p\|_1 < P(\omega)$ then the least amplitude periodic solution $x_\omega(t)$ (see [2]) is of twist type, see Theorem 4.1. Here the region Ω_3 contains the fourth order resonances $(2k-1)/4, k\in\mathbb{N}$. The bound $P(\omega)$ is of order $O(\omega^{1/2})$ when ω is bounded away from strong resonances. Moreover, at the fourth order resonances $(2k-1)/4$, $P((2k-1)/4)$ has a positive limit $\sqrt{2}/\pi$ as $k\to\infty$.

2. ESTIMATES OF THE PERIODIC SOLUTION OF THE ERMAKOV–PINNEY EQUATION

We begin with an abstract result which will be useful for the existence and estimates of periodic solutions.

Proposition 2.1. *Let X be a Banach space. Suppose that $P, Q{:}X\to X$ are completely continuous operators. Consider the operator T from X to itself decomposed as*

$$Tx = Px + Qx. \tag{2.1}$$

Define for $r\geqslant 0$,

$$\Delta_P(r) = \sup_{x\in X, \|x\|\leqslant r} \|P(x)\|, \quad \Delta_Q(r) = \sup_{x\in X, \|x\|\leqslant r} \|Q(x)\|.$$

If the equation

$$\Delta_P(\rho) + \Delta_Q(\rho) = \rho \tag{2.2}$$

has non-negative solutions, then the operator T in (2.1) has at least one fixed point x_* satisfying $\|x_*\|\leqslant\rho_*$, where ρ_* is the least non-negative solution of (2.2). Moreover, if P and Q are Lipschitz continuous in $\bar{B}_{\rho_*} = \{x\in X \mid \|x\|\leqslant\rho_*\}$, with the Lipschitz constants L_1 and L_2, respectively, then T has a unique fixed point in $\bar{B}_{\rho_*}$ provided that $L_1 + L_2 < 1$.

Proof. Under the assumption, the operator maps $\bar{B}_{\rho_*}$ into itself. Thus the existence and the estimate of the fixed points can be obtained by applying the Schauder Fixed Point Theorem on the domain $\bar{B}_{\rho_*}$. □

We remark that $\Delta_P(r)$ and $\Delta_Q(r)$ can be replaced by any functions which are greater than $\Delta_P(r)$ and $\Delta_Q(r)$, respectively.

Suppose that $a(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$. In the following we are going to study the Hill equation (1.2). Associated with (1.2) is the Ermakov–Pinney equation (1.5) and the Ricatti equation (on the extended complex plane $\overline{\mathbb{C}}$)

$$\dot{z} = z^2 + a(t), \quad z(0) = z(2\pi). \tag{2.3}$$

It is well known that (2.3) always has 2π-periodic solutions, with values possibly in the extended complex plane $\bar{\mathbb{C}}$, see for instance the survey [4]. When (1.2) is elliptic, the 2π-periodic solutions of (2.3) take values in the complex plane $\mathbb{C}$. In this case, (1.5) has a unique positive 2π-periodic solution denoted by $r(t)$. See [2, Lemma 3.3].

From [2], it is known that the estimate of $r(t)$ is crucial to study the twist condition of the third order approximation (1.3) of a non-linear Newtonian equation. In the following, our goal is to get adequate bounds on these solutions $r(t)$ when (1.2) may across the fourth order resonances.

We always assume that $a(t)$ has positive mean value. Let us define

$$\sigma^2 = \frac{1}{2\pi} \int_0^{2\pi} a(t)\mathrm{d}t > 0, \quad \tilde{a}(t) = a(t) - \sigma^2. \tag{2.4}$$

Hereinafter, $\sigma = \sigma(a) > 0$. Under the change of variables $y = z - \sigma i$, problem (2.3) is transformed into

$$\dot{y} = 2\sigma i y + y^2 + \tilde{a}(t), \quad y(0) = y(2\pi). \tag{2.5}$$

Assume further that $\sigma \in \Omega_2$, i.e., $2\sigma \notin \mathbb{N}$. Then $y(t)$ is a 2π-periodic solution of (2.5) with value in $\mathbb{C}$ if and only if $y \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C})$ satisfies

$$y(t) = \int_0^{2\pi} \chi(t,s)(y^2(s) + \tilde{a}(s))\mathrm{d}s, \tag{2.6}$$

where the kernel $\chi(t,s)$ is

$$\chi(t,s) = \begin{cases} \frac{e^{2i\sigma(t-s)}}{1-e^{-i4\sigma\pi}} & \text{if } 0 \leqslant s < t \leqslant 2\pi, \\ \frac{e^{2i\sigma(t-s)}}{e^{-i4\sigma\pi}-1} & \text{if } 0 \leqslant t \leqslant s \leqslant 2\pi. \end{cases}$$

In the notation of Proposition 2.1, let us consider the Banach space $X = (C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C}), \|\cdot\|_\infty)$ and define then operators $P, Q : X \to X$ by

$$Py \equiv L\tilde{a}, \quad Qy = Ly^2,$$

where

$$(Lh)(t) = \int_0^{2\pi} \chi(t,s)h(s)\mathrm{d}s.$$

Then problem (2.6) is equivalent to a fixed point equation

$$y = Py + Qy \tag{2.7}$$

in the space X. It is easy to verify that P and Q are completely continuous. Since

$$|\chi(t,s)| \equiv \frac{1}{2|\sin 2\pi\sigma|},$$

we have, for any $y \in X$ with $\|y\|_\infty \leqslant \rho$,

$$\Delta_P(\rho) \equiv \|L\tilde{a}\|_\infty \leqslant \frac{\|\tilde{a}\|_1}{2|\sin 2\pi\sigma|},$$

$$|(LQ)(y)(t)| = \left|\int_0^{2\pi} \chi(t,s) y^2(s)\mathrm{d}s\right| \leqslant \rho^2 \cdot \max_t \int_0^{2\pi} |\chi(t,s)|\mathrm{d}s = \frac{\pi\rho^2}{|\sin 2\pi\sigma|}$$

and, consequently,

$$\Delta_Q(\rho) \leqslant \frac{\pi\rho^2}{|\sin 2\pi\sigma|}.$$

Proposition 2.2. *Consider the periodic problem (2.3). Let σ and $\tilde{a}(t)$ be as in (2.4). If*

$$\sigma \in \Omega_2 \quad \textit{and} \quad \|\tilde{a}\|_1 = \int_0^{2\pi} |a(t) - \sigma^2|\mathrm{d}t \leqslant \frac{\sin^2 2\pi\sigma}{2\pi}, \tag{2.8}$$

then (2.3) has solutions $z_1(t)$ and $z_2(t) (= \overline{z_1(t)})$, which take values in $\mathbb{C}$ and, for all t,

$$|z_1(t) - i\sigma| \leqslant \tau(a) := \frac{|\sin 2\pi\sigma| - (\sin^2 2\pi\sigma - 2\pi\|\tilde{a}\|_1)^{1/2}}{2\pi} \leqslant \frac{\|\tilde{a}\|_1}{|\sin 2\sigma\pi|}. \tag{2.9}$$

Proof. We will apply Proposition 2.1 to (2.7) by using the estimates on $\Delta_P(\rho)$ and $\Delta_Q(\rho)$ as above. Note that the quadratic equation of ρ

$$\frac{\|\tilde{a}\|_1}{2|\sin 2\pi\sigma|} + \frac{\pi\rho^2}{|\sin 2\pi\sigma|} = \rho \tag{2.10}$$

has non-negative solutions if and only if (2.8) is satisfied. In this case, the least non-negative solution of (2.10) is $\rho_* = \tau(a)$ which is defined in (2.9). The second inequality in (2.9) is elementary. Thus the estimate (2.9) follows immediately from Proposition 2.1. □

Corollary 2.3. *Let $\sigma = \sigma(a)$ and $\tau = \tau(a)$ be as in (2.4) and (2.9). Assume that (2.8) is satisfied. Then the unique (positive) 2π-periodic solution $r(t)$ of the Ermakov–Pinney equation (1.5) satisfies*

$$(\sigma + \tau)^{-1/2} \leqslant r(t) \leqslant (\sigma - \tau)^{-1/2}. \tag{2.11}$$

Proof. Let

$$z(t) = -\frac{\dot{r}(t)}{r(t)} + \frac{i}{r^2(t)}. \tag{2.12}$$

Then $z(t)$ is a 2π-periodic solution of the Riccati equation (2.3). From (2.9), we have

$$\left|\frac{1}{r^2(t)} - \sigma\right| \leqslant |z(t) - i\sigma| \leqslant \tau(a). \tag{2.13}$$

Note that when (2.8) is satisfied, one has

$$\tau(a) \leqslant \frac{\|\tilde{a}\|_1}{|\sin 2\sigma\pi|} \leqslant \frac{|\sin 2\sigma\pi|}{2\pi} < \sigma.$$

Now (2.11) is obvious. □

When we study the twist coefficients in the latter sections, we need to estimate the upper bounds of the L^4 norm of $r(t)$. As a direct consequence of the L^∞ estimates (2.11) for $r(t)$, one has the following L^4 estimates:

$$\frac{(2\pi)^{1/2}}{\sigma + \tau} \leqslant \|r\|_4^2 \leqslant \frac{(2\pi)^{1/2}}{\sigma - \tau}. \tag{2.14}$$

The estimates (2.14) can be improved using more sophisticated techniques.

Theorem 2.4. *Assume that $a(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ satisfies $a(t) \geqslant 0$ for all t and conditions (2.8). Then we have the following estimates on the L^4 norm of the periodic solution $r(t)$ of (1.5):*

$$\frac{(2\pi)^{1/2}}{\sigma} \frac{(1+16\sigma^2)^{1/2}}{(1+16\sigma^2)^{1/2} + 4\tau} \leqslant \|r\|_4^2 \leqslant \frac{(2\pi)^{1/2}}{\sigma} \frac{(1+16\sigma^2)^{1/2}}{(1+16\sigma^2)^{1/2} - 4\tau}. \tag{2.15}$$

Since $0 \leqslant \tau < \sigma$, it is easy to check that the estimates (2.15) are always better than (2.14) when $\tau(a) > 0$.

Proof. In order to obtain (2.15), using (2.12) we can obtain from (2.13) a differential inequality on $r(t)$

$$\dot{r}^2/r^2+(\sigma-1/r^2)^2\leqslant\tau^2,$$

i.e.,

$$\dot{r}^2\leqslant\frac{2\sigma r^2-1-(\sigma^2-\tau^2)r^4}{r^2}. \tag{2.16}$$

By applying the comparison theorem for solutions of the first order ordinary differential equations, one may find from (2.16) estimates on $r(t)$ for all t. However, we will not develop this.

As for the bounds of $\|r\|_4$, we rewrite the inequality (2.16) as

$$\frac{1}{4}((r^2)')^2\leqslant 2\sigma r^2-1-(\sigma^2-\tau^2)r^4. \tag{2.17}$$

Let $m=\min_t r(t)$ and $M=\max_t r(t)$. Since $a(t)\geqslant 0$ for all t, by integrating (1.5) over $[0,2\pi]$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}a(t)r(t)\mathrm{d}t=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r^3(t)\mathrm{d}t}.$$

Thus $m\sigma^2\leqslant 1/m^3$ and $m^2\leqslant 1/\sigma$. Similarly, we have $M^2\geqslant 1/\sigma$. Consequently, $r^2(t_0)=1/\sigma$ for some $t_0\in[0,2\pi]$. Applying the Poincaré inequality to the function $R(t):=r^2(t)-1/\sigma$ on the interval $[t_0,t_0+2\pi]$, we have

$$\int_0^{2\pi}\dot{R}^2\mathrm{d}t=\int_{t_0}^{t_0+2\pi}\dot{R}^2\mathrm{d}t\geqslant\frac{1}{4}\int_{t_0}^{t_0+2\pi}R^2\mathrm{d}t=\frac{1}{4}\int_0^{2\pi}R^2\mathrm{d}t.$$

This yields

$$\begin{aligned}\frac{1}{4}\int_0^{2\pi}((r^2)')^2\mathrm{d}t &\geqslant \frac{1}{16}\int_0^{2\pi}(r^2(t)-1/\sigma)^2\mathrm{d}t\\ &=\frac{1}{16}\|r\|_4^4-\frac{1}{8\sigma}\|r\|_2^2+\frac{2\pi}{16\sigma^2}.\end{aligned}$$

Integrating (2.17) over $[0,2\pi]$, we have

$$\frac{1}{4}\int_0^{2\pi}((r^2)')^2\mathrm{d}t\leqslant 2\sigma\|r\|_2^2-2\pi-(\sigma^2-\tau^2)\|r\|_4^4.$$

From these two inequalities, we have

$$\left(\frac{1}{16}+\sigma^2-\tau^2\right)\|r\|_4^4+2\pi\left(\frac{1}{16\sigma^2}+1\right)\leqslant\left(\frac{1}{8\sigma}+2\sigma\right)\|r\|_2^2.$$

By the Hölder inequality, $\|r\|_2^2 \leqslant (2\pi)^{1/2}\|r\|_4^2$. Thus we have a quadratic inequality on $\|r\|_4^2$

$$\left(\frac{1}{16}+\sigma^2-\tau^2\right)\|r\|_4^4+2\pi\left(\frac{1}{16\sigma^2}+1\right)\leqslant(2\pi)^{1/2}\left(\frac{1}{8\sigma}+2\sigma\right)\|r\|_4^2.$$

Finally we can obtain (2.15) by solving the inequality above. □

Remark 2.5. When $\sigma=\sigma(a)$ is large, by the definition of $\tau(a)$ we have $\tau(a)\leqslant 1/(2\pi)$. It follows from (2.15) that

$$\|r\|_4^2=(2\pi)^{1/2}/\sigma+O(1/\sigma^2).$$

This shows that the estimates (2.15) are very accurate, because for $a(t)\equiv\sigma^2$, one has $r(t)\equiv 1/\sigma^{1/2}$ and

$$\|r\|_4^2=(2\pi)^{1/2}/\sigma.$$

Lemma 2.6. *Suppose that $a(t)\in C_{\sigma_1,\sigma_2}$ for some $0<\sigma_1\leqslant\sigma_2$. Then*

$$\sigma_1\leqslant\sigma\leqslant\sigma_2, \tag{2.18}$$

$$\|\tilde{a}\|_1\leqslant\frac{4\pi\left(\sigma_2^2-\sigma^2\right)\left(\sigma^2-\sigma_1^2\right)}{\sigma_2^2-\sigma_1^2}, \tag{2.19}$$

where the right-hand side of (2.19) is understood as 0 when $\sigma_1=\sigma_2$ and

$$\|\tilde{a}\|_1\leqslant\pi\left(\sigma_2^2-\sigma_1^2\right), \tag{2.20}$$

where the inequality is strict if $\sigma_1<\sigma_2$.

Proof. The estimate (2.18) is obvious. Let us now prove the estimates (2.19) and (2.20) on $\|\tilde{a}\|_1$. Define

$$I=\{t\in[0,2\pi]|a(t)\leqslant\sigma^2\},\quad J=[0,2\pi]\backslash I.$$

Let μ be the measure of I. Then

$$\int_I(\sigma^2-a(t))\mathrm{d}t=\int_J(a(t)-\sigma^2)\mathrm{d}t\leqslant \min\{(\sigma^2-\sigma_1^2)\mu,(\sigma_2^2-\sigma^2)(2\pi-\mu)\}.$$

Thus

$$\begin{aligned}\|\tilde{a}\|_1 &= \int_I (\sigma^2 - a(t))\mathrm{d}t + \int_J (a(t) - \sigma^2)\mathrm{d}t \\ &\leqslant\ 2\ \ \min\{(\sigma^2 - \sigma_1^2)\mu, (\sigma_2^2 - \sigma^2)(2\pi - \mu)\} \\ &\leqslant 2 \max_{0\leqslant\mu\leqslant 2\pi} \min\{(\sigma^2 - \sigma_1^2)\mu, (\sigma_2^2 - \sigma^2)(2\pi - \mu)\} \\ &= \frac{4\pi(\sigma_2^2 - \sigma^2)(\sigma^2 - \sigma_1^2)}{\sigma_2^2 - \sigma_1^2}.\end{aligned}$$

This proves (2.19). The estimate (2.20) can be obtained from (2.19), because

$$\|\tilde{a}\|_1 \leqslant \max_{\sigma_1\leqslant\sigma\leqslant\sigma_2} \frac{4\pi(\sigma_2^2 - \sigma^2)(\sigma^2 - \sigma_1^2)}{\sigma_2^2 - \sigma_1^2} = \pi(\sigma_2^2 - \sigma_1^2). \qquad \square$$

Remark 2.7. *For the estimate (2.20), if we consider a discontinuous step-potential*

$$A(t) = \begin{cases} \sigma_1^2 & \textit{if } 0 \leqslant t < \pi, \\ \sigma_2^2 & \textit{if } \pi \leqslant t < 2\pi, \end{cases}$$

where $0 < \sigma_1 < \sigma_2$, one has

$$\sigma^2 = \frac{\sigma_2^1 + \sigma_2^2}{2}, \quad \|\tilde{A}\|_1 = \pi(\sigma_2^2 - \sigma_1^2).$$

Thus the estimate (2.20) is optimal. When $\sigma_1 < \sigma_2$ and $a(t)$ is continuous, (2.20) is strict. This can be seen easily from the proof above.

Theorem 2.8. *(i) There exists a positive function $G_2(\omega)$ defined on Ω_2 such that, if $\omega \in ((n-1)/2, n/2) \subset \Omega_2$, then*

$$(n-1)/2 < G_2(\omega) < \omega, \tag{2.21}$$

and, if $G_2(\sigma_2) \leqslant \sigma_1 \leqslant \sigma_2$, then (2.8) is satisfied for all $a(t) \in C_{\sigma_1,\sigma_2}$.

(ii) There exists a positive function $N_2(\sigma_1, \sigma_2)$ defined on the domain

$$\mathscr{D}_2 = \{(\sigma_1, \sigma_2) | G_2(\sigma_2) \leqslant \sigma_1 \leqslant \sigma_2, \sigma_2 \in \Omega_2\} \tag{2.22}$$

such that if $a(t) \in C_{\sigma_1,\sigma_2}$ for some $(\sigma_1, \sigma_2) \in \mathscr{D}_2$, then the solution $r(t)$ of (1.5) satisfies

$$\|r\|_4^2 \leqslant N_2(\sigma_1, \sigma_2). \tag{2.23}$$

Moreover, both of the functions $G_2(\omega)$ and $N_2(\sigma_1, \sigma_2)$ can be computed using (2.29) and (2.30), respectively.

Proof. (i) Using estimate (2.19), condition (2.8) can be implied by the following assumption on σ_1 and σ_2:

$$\frac{4\pi(\sigma_2^2-\sigma^2)(\sigma^2-\sigma_1^2)}{\sigma_2^2-\sigma_1^2} \leqslant \frac{\sin^2 2\pi\sigma}{2\pi} \quad \text{for all } \sigma\in[\sigma_1,\sigma_2]. \tag{2.24}$$

Let us define

$$F(\sigma_1,\sigma_2;\sigma)=\left(\frac{\sin 2\pi\sigma}{2\pi}\right)^2-\frac{2(\sigma_2^2-\sigma^2)(\sigma^2-\sigma_1^2)}{\sigma_2^2-\sigma_1^2} \tag{2.25}$$

and

$$F(\sigma_1,\sigma_2):=\min_{\sigma\in[\sigma_1,\sigma_2]} F(\sigma_1,\sigma_2;\sigma). \tag{2.26}$$

Then (2.24) is equivalent to

$$F(\sigma_1,\sigma_2)\geqslant 0. \tag{2.27}$$

In the following we will prove that condition (2.27) can be rewritten as $G_2(\sigma_2)\leqslant\sigma_1\leqslant\sigma_2$, where the function $G_2(\cdot)$ satisfies (2.21).

Note that when $(n-1)/2<\sigma_1<\sigma_2<n/2$ for some $n\in\mathbb{N}$, we have

$$\frac{\partial F}{\partial\sigma}(\sigma_1,\sigma_2;\sigma)=\frac{\sin\, 4\pi\sigma}{2\pi}-\frac{4\sigma(\sigma_1^2+\sigma_2^2-2\sigma^2)}{\sigma_2^2-\sigma_1^2}.$$

Thus

$$\left.\frac{\partial F}{\partial\sigma}(\sigma_1,\sigma_2;\sigma)\right|_{\sigma=\sigma_1}=\frac{\sin\, 4\pi\sigma_1}{2\pi}-4\sigma_1<-2\sigma_1<0,$$

and

$$\left.\frac{\partial F}{\partial\sigma}(\sigma_1,\sigma_2;\sigma)\right|_{\sigma=\sigma_2}=\frac{\sin\, 4\pi\sigma_2}{2\pi}+4\sigma_2>2\sigma_2>0.$$

Therefore

$$F(\sigma_1,\sigma_2)=F(\sigma_1,\sigma_2;\sigma(\sigma_1,\sigma_2))$$

for some $\sigma(\sigma_1,\sigma_2)\in(\sigma_1,\sigma_2)$.

Fix a $\sigma_2\in((n-1)/2,n/2), n\in\mathbb{N}$. Note that when $\sigma_1=\sigma_2$,

$$F(\sigma_1,\sigma_2)=F(\sigma_2,\sigma_2)=\left(\frac{\sin\, 2\pi\sigma_2}{2\pi}\right)^2>0. \tag{2.28}$$

In the following we prove that $F(\sigma_1,\sigma_2)<0$ when $\sigma_1\to(n-1)/2+0$. To this end, let us prove that $F((n-1)/2,\sigma_2)<0$. So let us assume that

$\sigma_1 = (n-1)/2$ and $0 < \sigma - (n-1)/2 \ll 1$. We consider the following two cases.

Case 1. $n > 1$. We have

$$\left(\frac{\sin 2\pi\sigma}{2\pi}\right)^2 \approx (\sigma-(n-1)/2)^2, \quad \frac{2(\sigma_2^2-\sigma^2)(\sigma^2-\sigma_1^2)}{\sigma_2^2-\sigma_1^2} \approx 2(n-1)(\sigma-(n-1)/2).$$

Thus, by (2.25), $F((n-1)/2, \sigma_2; \sigma) < 0$ when $0 < \sigma - (n-1)/2 \ll 1$. So $F((n-1)/2, \sigma_2) < 0$ by (2.26).

Case 2. $n = 1$. In this case, if $0 < \sigma \ll 1$, we have

$$F(0, \sigma_2; \sigma) = \left(\frac{\sin 2\pi\sigma}{2\pi}\right)^2 - \frac{2(\sigma_2^2 - \sigma^2)\sigma^2}{\sigma_2^2} \approx -\sigma^2 < 0.$$

So $F(0, \sigma_2) < 0$ by (2.26).

The fact $F((n-1)/2, \sigma_2) < 0$, together with (2.28), implies that for any fixed $\sigma_2 \in ((n-1)/2, n/2)$, the equation $F(\sigma_1, \sigma_2) = 0$ has a (unique) solution $\sigma_1 = G_2(\sigma_2) \in ((n-1)/2, \sigma_2)$, which means that $G_2(\sigma_2)$ satisfies (2.21). Thus the function $G_2(\sigma_2)$ is determined by

$$F(G_2(\sigma_2), \sigma_2) = 0. \tag{2.29}$$

Now the inequality (2.27) can be expressed as $G_2(\sigma_2) \leqslant \sigma_1 \leqslant \sigma_2$.

(ii) For any given $(\sigma_1, \sigma_2) \in \mathscr{D}_2$, where $\mathscr{D}_2$ is defined by (2.22), we use (2.9), (2.15), and (2.19) to define

$$N_2(\sigma_1, \sigma_2; \sigma) = \frac{(2\pi(1+16\sigma^2))^{1/2}}{\sigma\left((1+16\sigma^2)^{1/2} - \frac{2|\sin 2\pi\sigma|}{\pi} + 4\left(\frac{\sin^2 2\pi\sigma}{4\pi^2} - \frac{2(\sigma_2^2-\sigma^2)(\sigma^2-\sigma_1^2)}{\sigma_2^2-\sigma_1^2}\right)^{1/2}\right)}$$

for $\sigma \in [\sigma_1; \sigma_2]$ and

$$N_2(\sigma_1, \sigma_2) = \max_{\sigma \in [\sigma_1, \sigma_2]} N_2(\sigma_1, \sigma_2; \sigma), \quad (\sigma_1, \sigma_2) \in \mathscr{D}_2. \tag{2.30}$$

It follows from (2.15) that the solution $r(t)$ satisfies (2.23). □

Remark 2.9. Let us compare the estimate (2.23) with some other estimates to the solutions $r(t)$ of (1.5) in literature.

(i) When $0 < \sigma_1 \leqslant \sigma_2 \leqslant 1/4$, which implies that (1.2) is in the first stability zone, it is proved in [6] that

$$\sigma_2^{-1/2} \leqslant r(t) \leqslant \sigma_1^{-1/2}.$$

Hence we have

$$\|r\|_4^2 \leqslant (2\pi)^{1/2}/\sigma_1 \tag{2.31}$$

in this case. Estimate (2.31) is better than (2.23). However, (2.31) does not hold in higher order stability zones.

(ii) If $a(t)\in C_{\sigma_1,\sigma_2}$ for some (σ_1,σ_2) in the following domain

$$\mathscr{D}_4 := \{(\sigma_1,\sigma_2)|(n-1)/4<\sigma_1\leqslant\sigma_2<n/4, n\in\mathbb{N}\},$$

it is proved in [2, Lemma 3.6] that $r(t)$ has the following estimate:

$$\|r\|_4^2 \leqslant N_4(\sigma_1,\sigma_2) := \max\left\{\left(\frac{2\pi}{\sigma_1\sigma_2}\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2}, \left(\frac{2\pi}{\sigma_1\sigma_2}\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^{1/2}\right\}. \tag{2.32}$$

Note that the domain $\mathscr{D}_4$ does not contain the fourth order resonances and the domain $\mathscr{D}_2$ does contain neighborhoods of the fourth order resonances $\sigma_1=\sigma_2=(2k-1)/4, k\in\mathbb{N}$. It is this improvement that enables us to develop the twist criteria for the fourth order resonant case. The comparison between the domains $\mathscr{D}_2$ and $\mathscr{D}_4$ is given in Fig. 1.

In the applications below, both the estimates (2.23) and (2.32) can be used. For simplicity, let us define

$$N(\sigma_1,\sigma_2) = \begin{cases} N_2(\sigma_1,\sigma_2) & \text{if } (\sigma_1,\sigma_2)\in\mathscr{D}_2\setminus\mathscr{D}_4, \\ N_4(\sigma_1,\sigma_2) & \text{if } (\sigma_1,\sigma_2)\in\mathscr{D}_4\setminus\mathscr{D}_2, \\ \min\{N_2(\sigma_1,\sigma_2), N_4(\sigma_1,\sigma_2)\} & \text{if } (\sigma_1,\sigma_2)\in\mathscr{D}_2\cap\mathscr{D}_4. \end{cases} \tag{2.33}$$

Then, if $a(t)\in C_{\sigma_1,\sigma_2}$ for some $(\sigma_1,\sigma_2)\in\mathscr{D}_2\cup\mathscr{D}_4$,

$$\|r\|_4^2 \leqslant N(\sigma_1,\sigma_2). \tag{2.34}$$

3. TWIST CRITERIA

Let us consider the equation

$$\ddot{x}+a(t)x+b(t)x^2+c(t)x^3+\cdots=0, \tag{3.1}$$

where $a(t), b(t), c(t) \in C(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R})$, and the dots denote the terms of order higher than 3. We will derive some new twist criteria for the equilibrium $x(t)\equiv 0$ (as a 2π-periodic solution of (3.1)).

The first-order approximation of (3.1) is the Hill equation (1.2). For the Hill equation (1.2), by the change of variables $x=-r\sin\varphi, \dot{x}=r\cos\varphi, \varphi$ satisfies the following equation:

$$\dot{\varphi}=a(t)\cos^2\varphi+\sin^2\varphi. \tag{3.2}$$

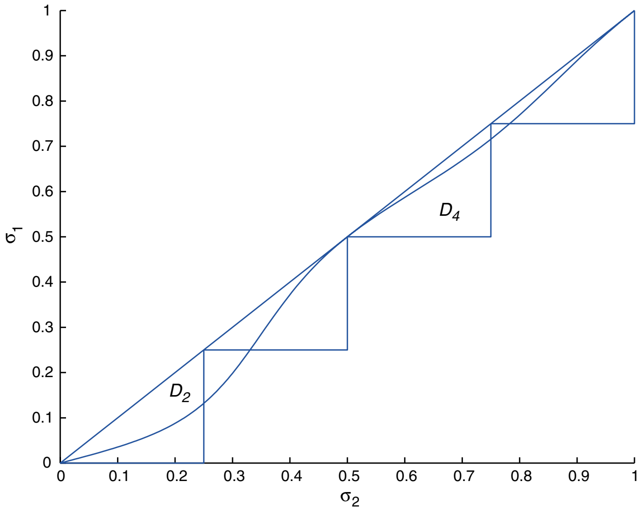

Figure 1. Domains $\mathscr{D}_2$ and $\mathscr{D}_4$.

Since (3.2) is a periodic equation on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, or even on the circle $\mathbb{R}/\pi\mathbb{Z}$, the *rotation number* of (3.2), or that of (1.2), is well defined:

$$\theta = \theta(a) = \lim_{|t|\to\infty} \frac{\varphi(t)}{t},$$

where $\varphi(t)$ is *any* solution of (3.2). It is known that $\theta \geqslant 0$.

In the following, we always assume that $a(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ satisfies $a(t) \geqslant 0$ for all t and (1.2) is elliptic. It is well known that this is equivalent to

$$\theta \in \Omega_2.$$

See, e.g., [17]. In this case, the Floquet multipliers of (1.2) are $\lambda_{1,2} = \exp(\pm i 2\pi\theta)$. When $\theta \in ((n-1)/2, n/2), n \in \mathbb{N}$, we say that Eq. (1.2) is in the nth *stability zone*. The third-order resonances mean that the rotation number

$$\theta = (3k-2)/3 \quad \text{or} \quad (3k-1)/3, \quad k \in \mathbb{N},$$

while the rotation numbers which do not correspond to the resonances up to the order 3 are

$$\theta \in \Omega_3.$$

Note that Ω_3 is the union of countable open intervals. Since (1.2) is elliptic, the Ermakov–Pinney equation (1.5) has a unique positive 2π-periodic solution denoted by $r(t) = r(t; a)$. Note that both of $\theta(a)$ and $r(t; a)$ depend upon the functions $a(t)$ in a nonlinear way, which cannot be given explicitly.

Let

$$\hat{F}(x_0, y_0) = (\hat{F}_1(x_0, y_0), \hat{F}_2(x_0, y_0))$$

be the Poincaré map of (3.1). Write $\hat{F}$ in the complex form, with $z = x_0 + iy_0$,

$$F(z, \bar{z}) = \hat{F}_1((z+\bar{z})/2, (z-\bar{z})/(2i)) + i\hat{F}_2((z+\bar{z})/2, (z-\bar{z})/(2i)).$$

When $\lambda = e^{i2\pi\theta}$ is not strongly resonant, $F(z, \bar{z})$ is C^∞ conjugate, in the group of area-preserving diffeomorphisms, to

$$N(z, \bar{z}) = \lambda(z + i\beta|z|^2 z + \cdots)$$

and when $\theta = (2k-1)/4, k \in \mathbb{N}$, i.e., $\lambda = \pm i$, $F(z, \bar{z})$ is C^∞ conjugate to

$$N(z, \bar{z}) = \lambda(z + i\beta|z|^2 z + \gamma\bar{z}^3 + \cdots),$$

where $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, and the dots denote the terms of order higher than 3. See [12, Proposition 2.1]. These $N(z, \bar{z})$ are called the Birkhoff normal form of F. The coefficients β and γ, which depend only on a, b, c and are invariant under conjugacies of area-preserving diffeomorphisms, are called the *twist coefficients* of (3.1). When $\beta \neq 0$ in the first case and $|\beta| > |\gamma|$ in the second case, we say that the solution $x(t) = 0$ of (3.1) (as a 2π-periodic solution) is of *twist type* (see [12, Proposition 2.2]). In this case, the Moser twist theorem is applicable and will yield the typical dynamical behavior near 0 such as $x(t) = 0$ is stable in the sense of Lyapunov, and (3.1) has, in a neighborhood of $x(t) = 0$, infinitely many subharmonics with periods tending to infinity and infinitely many quasi-periodic solutions.

Before stating the main results, let us define the functions on

$$\Theta_3 = \{\theta > 0 | 3\theta \notin \mathbb{N}\}$$

by

$$K_1(\theta) = \frac{\sqrt{2}}{8} + \max\left\{-\frac{3}{16}\cot(\pi\theta), 0\right\} + \max\left\{-\frac{1}{16}\cot(3\pi\theta), 0\right\}, \theta \in \Theta_3,$$

$$K_2(\theta) = \begin{cases} |2 + 3\cos(2\pi\theta)|/(8|\sin(3\pi\theta)|) & \text{if } \theta \in (0, 1/3) \cup (2/3, 1), \\ |\cos(2\pi\theta)|\sqrt{-2\cos(2\pi\theta)}/(8|\sin(3\pi\theta)|) & \text{if } \theta \in (1/3, 2/3) \end{cases}$$

and $K_2(\theta)$ is extended to Θ_3 by 1-periodicity. Define then $K{:}\Theta_3 \longrightarrow \mathbb{R}$ by

$$K(\theta) = \min\{K_1(\theta), K_2(\theta)\}.$$

In following, we will give a twist condition of (3.1) when $c(t)$ is negative and $b(t)$ may change sign. This case is motivated by the forced pendulum (1.4). However, the statement of the results make use of $\theta(a)$ and $r(t;a)$.

Theorem 3.1. *Assume that $a(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ satisfies $a(t) \geqslant 0$ for all t and (1.2) is elliptic without resonances up to order 3. Then there exists a constant $\mu = \mu(\theta) > 0$, depending on $\theta = \theta(a)$, such that $x(t) = 0$ (as a 2π-periodic solution of (3.1)) is of twist type provided that b(t) and c(t) satisfy*

$$\max_{t\in\mathbb{R}} c(t) < -\mu \|r\|_4^2 \|b\|_4^2. \tag{3.3}$$

Proof. Since (1.2) is elliptic, by [10, Proposition 7], (1.2) can be transformed into a *R-elliptic* equation after rescaling [12]. This means that the corresponding Poincaré mapping is a rigid rotation [10]. Note that the condition for $x(t) \equiv 0$ to be of twist type is invariant after rescaling [12, Lemma 2.5].

Since (1.2) has no resonances up to order 3, the rotation number θ is in

$$\Omega_3 = \Omega_4 \cup \{(2k-1)/4 | k \in \mathbb{N}\}.$$

Let $\lambda = e^{\pm i2\pi\theta}$ be the Floquet multipliers of (1.2). Recall the arguments in refs. [12] and [2, Section 3.2]. Suppose that t_0 is some critical point of $r(t)$ and $\varphi{:}\mathbb{R} \to \mathbb{R}$ is defined by

$$\varphi(t) = \int_{t_0}^{t} \frac{\mathrm{d}t}{r^2(t)}.$$

By [12, Propositions 2.4 and 4.4] and [2, Propositions 3.1 and 3.2], let us introduce

$$\begin{aligned}\beta \; &= (r(t_0))^{-1/2} \left[-\frac{3}{8} \int_{t_0}^{t_0+2\pi} c(t) r^4(t) \mathrm{d}t \right. \\ &+ \int\int_{[t_0, t_0+2\pi]^2} b(t) b(s) r^3(t) r^3(s) \chi_1(|\varphi(t) - \varphi(s)|) \mathrm{d}t \;\, \mathrm{d}s \\ &+ \frac{3}{16} \cot(\pi\theta) \left| \int_{t_0}^{t_0+2\pi} b(t) r^3(t) e^{-i\varphi(t)} \mathrm{d}t \right|^2\end{aligned}$$

$$+\frac{1}{16}\cot(3\pi\theta)\left|\int_{t_0}^{t_0+2\pi} b(t)r^3(t)e^{3i\varphi(t)}\mathrm{d}t\right|^2\Bigg], \tag{3.4}$$

or, equivalently,

$$\begin{aligned}\beta &= (r(t_0))^{-1/2}\left[-\frac{3}{8}\int_{t_0}^{t_0+2\pi} c(t)r^4(t)\mathrm{d}t\right.\\ &\quad\left.+\int\int_{[t_0,t_0+2\pi]^2} b(t)b(s)r^3(t)r^3(s)\chi_2(|\varphi(t)-\varphi(s)|)\mathrm{d}t\ \mathrm{d}s\right],\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\chi_1(x) &= \frac{3\sin x - 2\sin^3 x}{8}, \quad x\in[0,2\pi\theta],\\ \chi_2(x) &= \frac{3}{16}\frac{\cos(x-\pi\theta)}{\sin(\pi\theta)}+\frac{1}{16}\frac{\cos 3(x-\pi\theta)}{\sin 3\pi\theta}, \quad x\in[0,2\pi\theta]\end{aligned}$$

and the factor $(r(t_0))^{-1/2}>0$ is not of importance in the estimates below.

When $\theta=(2k-1)/4$, $k\in\mathbb{N}$, or $\lambda=\exp(i2\pi\theta)=\pm i$, we have an additional coefficient

$$\begin{aligned}|\gamma| &= (r(t_0))^{-1/2}\left|\frac{1}{8}\int_{t_0}^{t_0+2\pi} c(t)r(t)^4e^{4i\varphi(t)}\mathrm{d}t\right.\\ &\quad\left.-\frac{1}{8}\int\int_{[t_0,2\pi+t_0]^2} b(t)b(s)r^3(t)r^3(s)\chi_3(t,s)\mathrm{d}s\ \mathrm{d}t\right|,\end{aligned}$$

where

$$\chi_3(t,s)=e^{i(2\varphi(t)+\varphi(s))}\left(\sin|\varphi(t)-\varphi(s)|\pm e^{-i(\varphi(t)-\varphi(s))}\right)t, \quad s\in[0,2\pi].$$

Recall that $x(t)\equiv 0$ is of twist type, if

$$\beta\neq 0 \text{ for } \theta\in\Omega_4,$$

or

$$|\beta|>|\gamma| \quad \text{for } \theta=(2k-1)/4, \quad k\in\mathbb{N}.$$

Using the formulas (3.4) and (3.5), it is proved in [2, Section 3.5] that if $\theta\in\Omega_3$, then

$$|\beta|>(r(t_0))^{-1/2}\left[\frac{3}{8}\int_0^{2\pi}|c(t)|r^4(t)\mathrm{d}t-K(\theta)\left(\int_0^{2\pi}|b(t)|r^3(t)\mathrm{d}t\right)^2\right].$$

When $\theta=(2k-1)/4$ for some $k\in\mathbb{N}$, since

$$
\begin{aligned}
|\chi_3(t,s)| &= \left|\sin|\varphi(t)-\varphi(s)| \pm e^{-i(\varphi(t)-\varphi(s))}\right| \\
&\leqslant \max_{x\in\mathbb{R}} |\sin|x| \pm e^{-ix}| \\
&= \max_{x\in\mathbb{R}} \left|\frac{3}{2} \pm \sin(2|x|) - \frac{1}{2}\cos(2|x|)\right|^{1/2} \\
&= \left(\frac{3+\sqrt{5}}{2}\right)^{1/2} =: c_0,
\end{aligned}
$$

we have, by noticing that $b(t), c(t), r(t)$ are 2π-periodic,

$$
\begin{aligned}
|\gamma| &\leqslant (r(t_0))^{-1/2}\left[\frac{1}{8}\int_0^{2\pi}|c(t)r^4(t)|\mathrm{d}t+\frac{1}{8}\iint_{[0,2\pi]^2}|\chi_3(t,s)\|b(t)\|b(s)|r^3(t)r^3(s)\mathrm{d}t\mathrm{d}s\right], \\
&\leqslant (r(t_0))^{-1/2}\left[\frac{1}{8}\int_0^{2\pi}|c(t)|r^4(t)\mathrm{d}t+\frac{c_0}{8}\left(\int_0^{2\pi}|b(t)|r^3(t)\mathrm{d}t\right)^2\right].
\end{aligned}
$$

Thus $x(t)=0$ is of twist type when

$$
\frac{3}{8}\int_0^{2\pi}|c(t)|r^4(t)\mathrm{d}t > K(\theta)\left(\int_0^{2\pi}|b(t)|r^3(t)\mathrm{d}t\right)^2 \quad \text{if } \theta\in\Omega_4, \tag{3.6}
$$

$$
\frac{1}{4}\int_0^{2\pi}|c(t)|r^4(t)\mathrm{d}t > \left(K(\theta)+\frac{c_0}{8}\right)\left(\int_0^{2\pi}|b(t)|r^3(t)\mathrm{d}t\right)^2 \quad \text{if } \theta=\frac{2k-1}{4}, \quad k\in\mathbb{N}. \tag{3.7}
$$

Note that

$$
\left(\int_0^{2\pi}|b(t)|r^3(t)\mathrm{d}t\right)^2 \leqslant \|r\|_4^6\|b\|_4^2
$$

and if $c(t)<0$ for all t, then

$$
\int_0^{2\pi}|c(t)|r^4(t)\mathrm{d}t \geqslant (-\max_t c(t))\|r\|_4^4.
$$

Thus these conditions (3.6) and (3.7) can be guaranteed, respectively by

$$
\begin{aligned}
&\max_t c(t) < -\frac{8}{3}K(\theta)\|r\|_4^2\|b\|_4^2 \qquad \text{if } \theta\in\Omega_4, \\
&\max_t c(t) < -\left(4K(\theta)+\frac{c_0}{2}\right)\|r\|_4^2\|b\|_4^2 \qquad \text{if } \theta=\frac{2k-1}{4}, \quad k\in\mathbb{N}.
\end{aligned}
$$

As a result, one can take μ in (3.3) as

$$\mu = \begin{cases} \frac{8}{3}K(\theta) & \text{if } \theta \in \Omega_4, \\ 4K(\theta) + \left(\frac{3+\sqrt{5}}{8}\right)^{1/2} & \text{if } \theta = \frac{2k-1}{4}, \quad k \in \mathbb{N}. \end{cases} \tag{3.8}$$

□

In the next result, we assume that $a(t)$ in (3.1) is in C_{σ_1,σ_2} for some (σ_1, σ_2) and will derive a twist result from Theorem 3.1 for (3.1).

Let $I_n = (a_n, b_n), n \in \mathbb{N}$, denote all intervals of Ω_3

$$\Omega_3 = \bigcup_{n \in \mathbb{N}} I_n.$$

Explicitly, for $k \in \mathbb{N}$,

$$I_{4k-3} = \left(k-1, k-\frac{2}{3}\right), \quad I_{4k-2} = \left(k-\frac{2}{3}, k-\frac{1}{2}\right),$$

$$I_{4k-1} = \left(k-\frac{1}{2}, k-\frac{1}{3}\right), \quad I_{4k} = \left(k-\frac{1}{3}, k\right).$$

Note that when $a(t) \in C_{\sigma_1,\sigma_2}$, where (σ_1, σ_2) is in the following domain

$$\mathscr{D}_3 = \left\{(\sigma_1, \sigma_2) | a_n < \sigma_1 \leqslant \sigma_2 < b_n, n \in \mathbb{N}\right\},$$

(1.2) has no resonances of order $\leqslant 3$. Meanwhile, the solution $r(t)$ of the corresponding Eq. (1.5) can be estimated when $(\sigma_1, \sigma_2) \in \mathscr{D}_2 \cup \mathscr{D}_4$. Thus we introduce the following domain:

$$\mathscr{D} = \mathscr{D}_3 \cap (\mathscr{D}_2 \cup \mathscr{D}_4).$$

This domain $\mathscr{D}$ is plotted in Fig. 2. It is easy to see that $\mathscr{D}$ can be written as

$$\mathscr{D} = \left\{(\sigma_1, \sigma_2) | G(\sigma_2) < \sigma_1 \leqslant \sigma_2, \sigma_2 \in \Omega_3\right\},$$

where the function $G{:}\Omega_3 \to [0, \infty)$ is

$$G(\sigma_2) = \max\{[3\sigma_2]/3, \min\{G_2(\sigma_2), [4\sigma_2]/4\}\}, \tag{3.9}$$

where $[x]$ denotes the greatest integer $\leqslant x$.

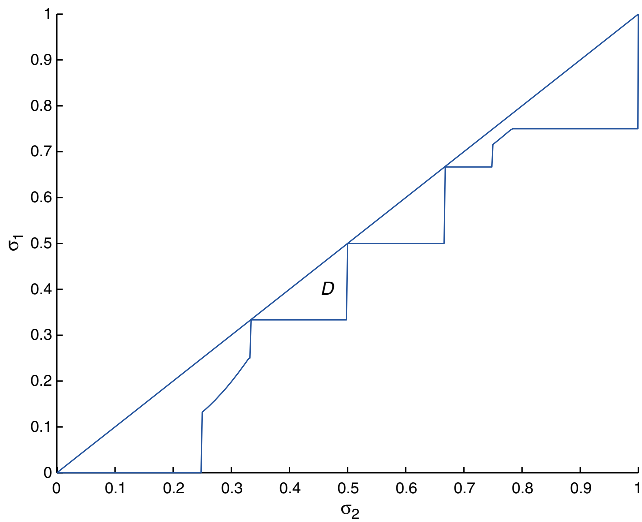

Figure 2. Domain $\mathscr{D}$.

Theorem 3.2. *Assume that* $a(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ *is in* C_{σ_1,σ_2} *for some* $(\sigma_1, \sigma_2) \in \mathscr{D}$. *Then there exists a constant* $\nu = \nu(\sigma_1, \sigma_2) > 0$, *depending only on* (σ_1, σ_2), *such that* $x(t) = 0$ *(as a* 2π*-periodic solution of (3.1)) is of twist type provided that* $b(t)$ *and* $c(t)$ *satisfy*

$$\max_{t\in\mathbb{R}} c(t) < -\nu(\sigma_1, \sigma_2)\|b\|_4^2. \tag{3.10}$$

Explicitly, $\nu(\sigma_1, \sigma_2)$ *can take*

$$\nu(\sigma_1, \sigma_2) = \begin{cases} \left(4K(\sigma_2) + \left(\frac{3+\sqrt{5}}{8}\right)^{1/2}\right) N(\sigma_1, \sigma_2) & \text{if } \frac{2k-1}{4} \in [\sigma_1, \sigma_2] \text{ for some } k \in \mathbb{N}, \\ \frac{8}{3} K(\sigma_2) N(\sigma_1, \sigma_2) & \text{otherwise}, \end{cases} \tag{3.11}$$

where $N(\sigma_1, \sigma_2)$ *is defined by (2.33).*

Proof. If $a(t) \in C_{\sigma_1,\sigma_2}$, it is well known the rotation number θ of (1.2) satisfies

$$\sigma_1 \leqslant \theta \leqslant \sigma_2.$$

Note that, on each interval I from Θ_3, $K(\theta)$ is non-decreasing. Thus

$$K(\theta) \leqslant K(\sigma_2).$$

Using the estimates $\|r\|_4^2 \leqslant N(\sigma_1, \sigma_2)$ in (2.33) and (2.34), we know from (3.8) that ν in (3.10) can be as in (3.11). □

Remark 3.3. If $c(t) > 0$ for all t, one can work out a twist criterion as in Theorem 3.1. If

$$\min_{t\in\mathbb{R}} c(t) > \mu\|r\|_4^2\|b\|_4^2, \tag{3.12}$$

where μ is as in (3.3), then $x(t)=0$ is of twists type. Such a case does happen in some singular repulsive equations [15]. One may use the twist condition (3.12) to develop some practical twist conditions for singular equations, as argued in Theorem 3.2. In fact, basing on the estimates (2.4), one may develop also some twist criteria when $b(t)$ and $c(t)$ may change sign, following some ideas in [2,6]. However, we will not do this because we are here mainly interested in the forced pendulum (1.4).

4. APPLICATIONS TO THE NEWTONIAN EQUATIONS

4.1. Application to the Forced Pendulum

In this Section, we consider the concrete example (1.4) and will apply the results above to improve the twist results on the least amplitude periodic solution given in [2].

Let us recall some results of [2, Section 2]. Define for $\omega \in \Omega_1 = (0,\infty)\backslash\mathbb{N}$,

$$\alpha = \alpha(\omega) = \frac{\int_0^{\omega\pi} |\cos s|\mathrm{d}s}{6|\sin\omega\pi|}, \quad \gamma=\gamma(\omega)=\frac{1}{2\omega|\sin\omega\pi|}. \tag{4.1}$$

If $p(t)\in C(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R})$ satisfies the condition

$$27\alpha\gamma^2\|p\|_1^2 \leqslant 4, \tag{4.2}$$

then (1.4) has a 2π-periodic solution $x=x_\omega(t)$ such that $\|x_\omega\|_\infty$ is smallest among all of 2π-periodic solutions of (1.4). (The solution $x_\omega(t)$ is called *the least amplitude periodic solution* of (1.4) in ref. [2].) Moreover, $x_\omega(t)$ satisfies

$$\|x_\omega\|_\infty \leqslant X^*(\alpha,\gamma), \tag{4.3}$$

where $X^*(\alpha,\gamma)$ is the minimal non-negative solution of the cubic equation

$$\alpha X^3 + \gamma\|p\|_1 = X. \tag{4.4}$$

Explicitly,

$$X^*(\alpha,\gamma) = \frac{2}{(3\alpha)^{1/2}}\cos\frac{\vartheta+\pi}{3}, \left(\vartheta = \arccos\ \left(\frac{3}{2}\gamma\|p\|_1(3\alpha)^{1/2}\right)\in\left(0,\frac{\pi}{2}\right)\right).$$

One sees that $\|x_\omega\|_\infty$ is small when $\omega \gg 1$. The existence condition (4.2) can be rewritten as

$$\|p\|_1 \leqslant \frac{4\sqrt{2}}{3} \frac{\omega|\sin\omega\pi|^{3/2}}{\left(\int_0^{\pi\omega}|\cos s|\mathrm{d}s\right)^{1/2}} =: P_1(\omega). \tag{4.5}$$

In the following we will prove that $x_\omega(t)$ is of twist type under more restriction on $\|p\|_1$ than (4.5).

Theorem 4.1. *There exists a positive function $P(\omega)$ defined on Ω_3 such that*

(i) If $\omega \in \Omega_3$ and $p(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ satisfies

$$\|p\|_1 < P(\omega),$$

then the least amplitude 2π-periodic solution $x_\omega(t)$ of (1.4) is of twist type.

(ii) At fourth order resonances, we have $\lim_{k\to+\infty} P((2k-1)/4) = \sqrt{2}/\pi$.

(iii) $P(\omega)$ is of order $O(\omega^{1/2})$ when ω is bounded away from the resonances of order $\leqslant 4$ and tends to $+\infty$.

Proof. (i) Let $\omega \in \Omega_3$ and $p(t) \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$. Suppose that (4.2), or (4.5), is satisfied. We consider the third order approximation of (1.4) to $x_\omega(t)$

$$\ddot{x} + a_\omega(t)x + b_\omega(t)x^2 + c_\omega(t)x^3 + \cdots = 0, \tag{4.6}$$

where

$$a_\omega(t) = \omega^2 \cos x_\omega(t), \quad b_\omega(t) = -\frac{\omega^2}{2}\sin x_\omega(t), \quad c_\omega(t) = -\frac{\omega^2}{6}\cos x_\omega(t).$$

In order to apply Theorem 3.2 to (4.6), introduce a parameter

$$\eta = (\cos X^*(\alpha, \gamma))^{1/2}. \tag{4.7}$$

The condition

$$(\eta\omega, \omega) \in \mathscr{D} = \mathscr{D}_3 \cap (\mathscr{D}_2 \cup \mathscr{D}_4)$$

can be rewritten as

$$G(\omega)/\omega < \eta \leqslant 1. \tag{4.8}$$

See (3.9).

Note from (4.3) that

$$-\max_t\ c_\omega(t) \geqslant \frac{\omega^2}{6}\cos X^*(\alpha,\gamma) = \frac{\omega^2\eta^2}{6},$$

$$\|b_\omega\|_4^2 \leqslant (2\pi)^{1/2}\left(\frac{\omega^2}{2}\right)^2 (\sin X^*(\alpha,\gamma))^2 = \frac{(2\pi)^{1/2}}{4}\omega^4(1-\eta^4).$$

When η satisfies (4.8), the twist condition (3.10) can be ensured by

$$\frac{\omega^2\eta^2}{6} > \nu(\eta\omega,\omega)\frac{(2\pi)^{1/2}}{4}\omega^4(1-\eta^4),$$

i.e.,

$$2\eta^2 > 3(2\pi)^{1/2}\omega^2\nu(\eta\omega,\omega)(1-\eta^4), \tag{4.9}$$

where $\nu(\cdot,\cdot)$ is defined by (3.11).

Note that (4.9) is always satisfied for $\eta = 1$. Thus we can combine conditions (4.8) and (4.9) into a single inequality

$$Q(\omega) < \eta \leqslant 1, \tag{4.10}$$

where the function $Q:\Omega_3 \to (0,1)$ is well defined.

In order to obtain the relationship between ω and $\|p\|_1$, we use relations (4.4) and (4.7). Let

$$X(\omega) = \arccos\ Q^2(\omega).$$

Note that the function $X - \alpha X^3$, $X \in [0,\infty)$, attains its maximum $2/(3\sqrt{3\alpha})$ at $X_0 = (3\alpha)^{-1/2}$. Following (4.4), we define a function $\tilde{P}:\Omega_3 \to (0,\infty)$ by

$$\tilde{P}(\omega) = \begin{cases} 2\omega|\sin\omega\pi|(X(\omega) - \alpha(\omega)X^3(\omega)) & \text{if } X(\omega) \leqslant (3\alpha(\omega))^{-1/2}, \\ \frac{4\omega|\sin\omega\pi|}{3\sqrt{3\alpha(\omega)}} & \text{if } X(\omega) > (3\alpha(\omega))^{-1/2}. \end{cases} \tag{4.11}$$

Consequently, if $\|p\|_1 < P(\omega) := \min\{P_1(\omega), \tilde{P}(\omega)\}$, then $x_\omega(t)$ is of twist type.

(ii) Now we consider the fourth order resonances. To this end, let $\omega_k = (2k-1)/4, k \in \mathbb{N}$. Then $Q(\omega_k)$ in (4.10) is determined by conditions (4.8) and (4.9).

Let us first consider condition (4.8). Since $(\eta\omega_k, \omega_k) \notin \mathscr{D}_4$, it is necessary that $(\eta\omega_k, \omega_k) \in \mathscr{D}_3 \cap \mathscr{D}_2$ and $G(\omega_k) = G_2(\omega_k)$, where $G_2(\omega_k)$ is determined by (2.29). Let $\eta_k = G_2(\omega_k)/\omega_k$. It is easy to see that $0 < \eta_k < 1$ and $\eta_k \to 1$ as $k \to \infty$. Introduce the function

$$F_k(\xi) = \frac{1-\eta_k^2}{\omega_k^2} F(\eta_k\omega_k, \omega_k; \xi\omega_k) = (1-\eta_k^2)\left(\frac{\sin 2\pi\xi\omega_k}{2\pi\omega_k}\right)^2 - 2(1-\xi^2)(\xi^2-\eta_k^2),$$

where F is given by (2.25) and $\xi \in [\eta_k, 1]$. By (2.29), there exists $\xi_k \in (\eta_k, 1)$ satisfying the equations

$$0 = \frac{\partial F_k(\xi)}{\partial \xi}\bigg|_{\xi=\xi_k} = (1-\eta_k^2)\frac{\sin 4\pi\xi_k\omega_k}{2\pi\omega_k} - 4\xi_k(\eta_k^2 + 1 - 2\xi_k^2) \tag{4.12}$$

$$0 = F_k(\xi_k) = (1-\eta_k^2)\left(\frac{\sin 2\pi\xi_k\omega_k}{2\pi\omega_k}\right)^2 - 2(1-\xi_k^2)(\xi_k^2-\eta_k^2). \tag{4.13}$$

In the following we will find an asymptotic formula for η_k from (4.12) and (4.13). Set $\eta_k = 1 - \zeta_k$, where $\zeta_k \to 0$. By (4.12), one has $\xi_k = 1 - \zeta_k/2 + o(\zeta_k)$. Thus

$$1-\eta_k^2 = 2\xi_k + o(\zeta_k), \quad 1-\xi_k^2 = \zeta_k + o(\zeta_k), \quad \xi_k^2 - \eta_k^2 = \zeta_k + o(\zeta_k)$$

and

$$\left(\frac{\sin 2\pi\xi_k\omega_k}{2\pi\omega_k}\right)^2 = \left(\frac{\sin((2k-1)\pi/2 - 2\pi\zeta_k\omega_k)}{2\pi\omega_k}\right)^2 = \frac{1}{4\pi^2\omega_k^2}(1 + O(\zeta_k^2\omega_k^2)).$$

Substituting these into (4.13), we obtain

$$\zeta_k = \frac{1}{4\pi^2\omega_k^2}(1+o(1)).$$

Thus

$$\eta_k = G_2(\omega_k)/\omega_k = 1 - \zeta_k = 1 - \frac{1}{4\pi^2\omega_k^2}(1+o(1)). \tag{4.14}$$

Next let us consider condition (4.9). Note that

$$4K(\omega_k) + \left((3+\sqrt{5})/8\right)^{1/2} =: K_*(\omega_k)$$

takes only two positive constants $K_*(1/4)$ and $K_*(3/4)$. When $\eta \geqslant \eta_k$, where η_k is as in (4.14) and k is large,

$$2\eta^2 = 2 + \mathrm{o}(1)$$

and

$$\begin{aligned}
&3(2\pi)^{1/2}\omega_k^2 \nu(\eta\omega_k, \omega_k)(1-\eta^4)\\
&\quad = 3(2\pi)^{1/2}\omega_k^2 N(\eta\omega_k, \omega_k) K_*(\omega_k)(1-\eta^4)\\
&\quad = 3(2\pi)^{1/2}\omega_k^2 \frac{(2\pi)^{1/2}}{\omega_k}(1+\mathrm{o}(1)) K_*(\omega_k)(1-\eta^4)\\
&\quad = 3(2\pi)^{1/2}\omega_k^2 N(\eta\omega_k, \omega_k) K_*(\omega_k)(1-\eta^4)\\
&\quad \leqslant 6\pi\omega_k(1+\mathrm{o}(1)) K_*(\omega_k)\frac{1}{\pi^2\omega_k^2}(1+\mathrm{o}(1))\\
&\quad = \frac{6K_*(\omega_k)}{\pi\omega_k}(1+\mathrm{o}(1)) = \mathrm{O}(1/\omega_k).
\end{aligned}$$

These show that (4.9) is always satisfied when $\eta \geqslant \eta_k, k \gg 1$. Thus, when $k \gg 1$, we have

$$Q(\omega_k) = \eta_k = 1 - \frac{1}{4\pi^2\omega_k^2}(1+\mathrm{o}(1))$$

and

$$X(\omega_k) = \arccos\ Q^2(\omega_k) = \frac{1}{\pi\omega_k}(1+\mathrm{o}(1)).$$

By (4.1),

$$\alpha(\omega_k) = \frac{\sqrt{2}}{3}\omega_k(1+\mathrm{o}(1)),$$

$$\gamma(\omega_k) = \frac{\sqrt{2}}{\omega_k}$$

and

$$(3\alpha(\omega_k))^{-1/2} = \frac{1}{2^{1/4}\omega_k^{1/2}}(1+\mathrm{o}(1)).$$

So $X(\omega_k) < (3\alpha(\omega_k))^{-1/2}$ for $k \gg 1$. By (4.11), we have

$$\tilde{P}(\omega_k) = 2\omega_k|\sin\omega_k\pi|(X(\omega_k) - \alpha(\omega_k)X^3(\omega_k)) = \frac{\sqrt{2}}{\pi} + \mathrm{o}(1).$$

Note from (4.5) that

$$P_1(\omega_k) = \frac{2^{11/4}\omega_k^{1/2}}{3}(1+\mathrm{o}(1)).$$

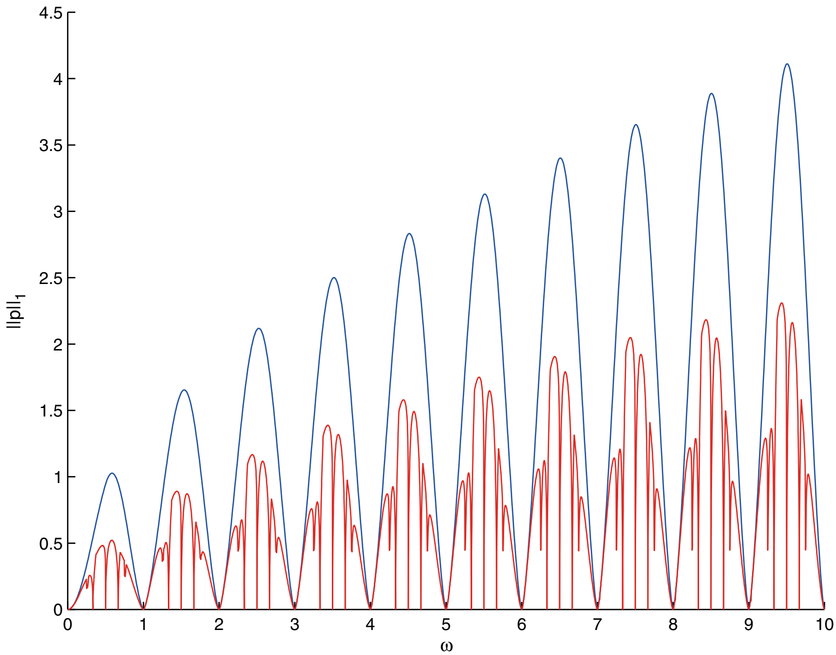

Figure 3. Twist result.

Thus

$$\lim_{k\to\infty} P(\omega_k) = \lim_{k\to\infty} \tilde{P}(\omega_k) = \frac{\sqrt{2}}{\pi}.$$

(iii) The conclusion can be obtained by finding the estimates on $\tilde{P}(\omega)$. □

The graph of $P(\omega)$ is plotted in Fig. 3, where the lower curve describes the twist result in Theorem 4.1. A comparison to the existence condition (4.5) is the upper curve shown in the figure. One may compare this figure with Fig. 2 in [2].

4.2. Application to Newtonian Equations

The ideas above apply also to the periodic problem (1.1), where $g(t, x)$ is 2π-periodic in t. We will give only a brief discussion.

Define

$$\bar{g}(x) = \frac{1}{2\pi}\int_0^{2\pi} g(t,x)\mathrm{d}t, \quad \tilde{g}(t,x) = g(t,x) - \bar{g}(x).$$

We assume here that there exists $\bar{u} \in \mathbb{R}$ such that $\bar{g}(\bar{u}) = 0$ and $\bar{g}'(\bar{u}) := \omega^2 > 0$. Set $y = x - \bar{u}$ and

$$\hat{g}(y) = \bar{g}(y + \bar{u}) - \omega^2 y.$$

Problem (1.1) is equivalent to

$$\begin{gathered} \ddot{y} + \omega^2 y + \hat{g}(y) + \tilde{g}(t,y) = 0, \\ y(0) = y(2\pi), \quad \dot{y}(0) = \dot{y}(2\pi), \end{gathered} \tag{4.15}$$

hereinafter we write $\tilde{g}(t,y)$ for $\tilde{g}(t, y+\bar{u})$ for short. Then $y(t)$ is a solution of (4.15) if and only if $y \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ and satisfies

$$y(t) = \int_0^{2\pi} \chi(t,s)(-\hat{g}(y(s)) - \tilde{g}(s, y(s)))\mathrm{d}s, \tag{4.16}$$

where the kernel $\chi(t,s)$ is given explicitly by

$$\chi(t,s) = \begin{cases} \frac{\cos\omega(\pi - t + s)}{2\omega\sin\omega\pi} & \text{if } 0 \leqslant s \leqslant t \leqslant 2\pi, \\ \frac{\cos\omega(\pi - s + t)}{2\omega\sin\omega\pi} & \text{if } 0 \leqslant t \leqslant s \leqslant 2\pi. \end{cases}$$

Here we have assumed that $\omega \notin \mathbb{N}$.

Let us consider the Banach space $X = (C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}), \|\cdot\|_\infty)$. Comparing (4.16) with Proposition 2.1, we have

$$(Px)(t) \equiv -L\tilde{g}(t, x(t)), \qquad (Qx)(t) = -L\hat{g}(x(t)),$$

where

$$(Lh)(t) = \int_0^{2\pi} \chi(t,s)h(s)\mathrm{d}s.$$

Define

$$L_1(\omega) := \frac{1}{2\omega|\sin\omega\pi|} = \max_{(t,s)}|\chi(t,s)|, \quad L_2(\omega) := \frac{\int_0^{\omega\pi}|\cos s|\mathrm{d}s}{\omega^2|\sin\omega\pi|} = \max_t \int_0^{2\pi} |\chi(t,s)|\mathrm{d}s.$$

Then

$$\sup_{x\in X, \|x\|_\infty \leqslant r} \|P(x)\|_\infty \leqslant L_1(\omega) \sup_{x\in X, \|x\|_\infty \leqslant r} \|\tilde{g}(t, x(t))\|_1 := \Delta_P(r),$$

$$\sup_{x\in X, \|x\|_\infty \leqslant r} \|Q(x)\|_\infty \leqslant L_2(\omega) \max_{|s|\leqslant r} |\hat{g}(s)| := \Delta_Q(r).$$

From Proposition 2.1, we have

Proposition 4.2. *Assume that* $\bar{g}(\bar{u})=0$ *for some* $\bar{u}$ *and* $\omega=(\bar{g}'(\bar{u}))^{1/2}\in\Omega_1$. *If the equation*

$$\Delta_P(\rho)+\Delta_Q(\rho)=\rho, \tag{4.17}$$

has non-negative solution, then (1.1) has at least one 2π*-periodic solution* $u(t)=\bar{u}+\delta u(t)$*,satisfying*

$$\|\delta u\|_\infty \leqslant \rho_*,$$

where ρ_* *is the minimal non-negative solution of (4.17). If, futhermore,*

$$L_1(\omega)\sup_{|s|\leqslant\rho_*,t\in\mathbb{R}}|g_x(t,s)|+L_2(\omega)\sup_{|s|\leqslant\rho_*}|\hat{g}_x(s)|<1$$

then the solution is unique in the ball $\bar{B}(\bar{u},\rho_*)$*.*

Let us consider the twist character of the periodic solution $u(t)=\bar{u}+\delta u(t)$ of (1.1) in Proposition 4.2. Define

$$\bar{a}(t)=g_x(t,\bar{u}),\quad \bar{b}(t)=\frac{1}{2}g_{xx}(t,\bar{u}),\quad \bar{c}(t)=\frac{1}{6}g_{xxx}(t,\bar{u}).$$

Let us express the coefficients in the third order approximation of (1.1) to $u(t)$ as

$$a(t)=\bar{a}(t)+\delta a(t),\quad b(t)=\bar{b}(t)+\delta b(t),\quad c(t)=\bar{c}(t)+\delta c(t).$$

Let

$$M_k(\rho_*)=\sup_{\|v\|_\infty\leqslant\rho_*}\left\|\frac{\partial^k g}{\partial x^k}(t,\bar{u}+v(t))\right\|_\infty,\quad k=2,3,4.$$

Then

$$\|\delta a\|_\infty\leqslant M_2(\rho_*)\rho_*,\quad \|\delta b\|_\infty\leqslant\frac{1}{2}M_3(\rho_*)\rho_*,\quad \|\delta c\|_\infty\leqslant\frac{1}{6}M_4(\rho_*)\rho_*.$$

If

$$M_2(\rho_*)\rho_*<\min_t \bar{a}(t), \tag{4.18}$$

we define σ_1 and σ_2 by

$$\sigma_1^2=\min_t(\bar{a}(t)-M_2(\rho_*)\rho_*),\quad \sigma_2^2=\max_t(\bar{a}(t)+M_2(\rho_*)\rho_*). \tag{4.19}$$

Since

$$\max_t c(t)\leqslant\max_t\bar{c}(t)+\|\delta c\|_\infty,\quad \|b\|_4^2\leqslant(\|\bar{b}\|_4+(2\pi)^{1/4}\|\delta b\|_\infty)^2,$$

we have the following twist result.

Theorem 4.3. *Consider the periodic problem (1.1). If (4.17) has non-negative solutions,(4.18) is satisfied, and* $(\sigma_1, \sigma_2) \in \mathscr{D}$*,where* σ_1, σ_2 *are defined in (4.19), then the periodic solution* $u(t)=\bar{u}+\delta u(t)$ *of (1.1) is of twist type, provided that the following twist condition is satisfied:*

$$\max_t \bar{c}(t) < -\left(\frac{1}{6}M_4(\rho_*)\rho_* + \nu(\sigma_1,\sigma_2)\left(||\bar{b}||_4 + (2\pi)^{1/4}M_3(\rho_*)\rho_*/2\right)^2\right).$$

ACKNOWLEDGMENTS

Supported by D.G.I. BFM2002-01308, Ministerio Ciencia y Tecnología, Spain. Supported by the NNSFC (Nos. 10325102, 10301006), China. TRAPOYT-M.O.E. (2001), and the National 973 Project (No. TG1999075108), China.

REFERENCES

1. Chow, S.-N., Li, C. Z., and Wang, D. (1994). *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge University Press, Cambridge.
2. Lei, J., Li, X., Yan, P., and Zhang, M. (2003). Twist character of the least amplitude periodic solution of the forced pendulem. *SIAM J. Math. Anal.* **35**, 844–867.
3. Lei J., and Zhang, M., (2002). Twist property of periodic motion of an atom near a charged wire. *Lett. Math. Phys*. **60**, 9–17.
4. Mawhin, J. (1998). Nonlinear complex-valued differential equations with periodic, Floquet or nonlinear boundary conditions. *International Conference on Differential Equations (Lisboa, 1995)*, World Scientific, River Edge, NJ, pp. 154–164.
5. Moser, J. (1962). On invariant curves of area preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II*, 1–20.
6. Núñez, D. (2002). The method of lower and upper solutions and the stability of periodic oscillations. *Nonlinear Anal.* **51**, 1207–1222.
7. Núñez D., and Ortega, R. (2000). Parabolic fixed points and stability criteria for nonlinear Hill's equations. *Z. Angew. Math. Phys.* **51**, 890–911.
8. Núñez D., and Torres, P. J. (2001). Periodic solutions of twist type of an earth satellite equation. *Discr. Cont. Dynam. Syst.* **7**, 303–306.
9. Núñez D., and Torres, P. J. (2003). Stable odd solutions of some periodic equations modeling satellite motion. *J. Math. Anal. Appl.* **279**, 700–709.
10. Ortega, R. (1992). The twist coefficient of periodic solutions of a time-dependent Newton's equation. *J. Dynam. Differential Equations* **4**, 651–665.
11. Ortega, R. (1994). The stability of the equilibrium of a nonlinear Hill's equation. *SIAM J. Math. Anal.* **25**, 1393–1401.
12. Ortega, R. (1996). Periodic solution of a Newtonian equation: Stability by the third approximation. *J. Differential Equations* **128**, 491–518.
13. Ortega, R. (2003). The stability of the equilibrium: a search for the right approximation,*Preprint*, Available at http://www.ugr.es/∼ecuadif/fuentenueva.htm
14. Siegel C. L., and Moser, J. K. (1971). *Lectures on Celestial Mechanics*, Springer-Verlag, Berlin.

15. Torres P. J., and Zhang, M. (2004). Twist periodic solutions of repulsive singular equations. *Nonlinear Anal.* **56**, 591–599.
16. Yan P., and Zhang, M. (2003). Higher order nonresonance for differential equations with singularities.*Math. Methods Appl. Sci.* **26**, 1067–1074.
17. Zhang, M. (2001). The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials. *J. London Math. Soc.* **64** (2), 125–143.
18. Zhang, M. (2003). The best bound on the rotations in the stability of periodic solutions of a Newtonian equation. *J. London Math. Soc.* **67** (2), 137–148.