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PERIODIC MOTIONS OF LINEAR IMPACT OSCILLATORS VIA THE SUCCESSOR MAP*

DINGBIAN QIAN† AND PEDRO J. TORRES‡

Abstract. We investigate the existence and multiplicity of nontrivial periodic bouncing solutions for linear and asymptotically linear impact oscillators by applying a generalized version of the Poincaré–Birkhoff theorem to an adequate Poincaré section called the successor map. The main theorem includes a generalization of a related result by Bonheure and Fabry and provides a sufficient condition for the existence of periodic bouncing solutions for Hill's equation with obstacle at $x \neq 0$.

Key words. impact oscillator, Hill's equation, periodic solution, successor map, Poincaré–Birkhoff twist theorem

AMS subject classifications. 34C15, 34C25, 34B30, 54H25

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1. Introduction and main results.

In this paper, we study the existence of $2m\pi$-periodic bouncing solutions for the following linear impact oscillator:

$$\begin{cases} x'' + a(t)x = p(t) \quad \text{for } x(t) > 0; \\ x(t) \geq 0; \\ x(t_0) = 0 \ \Rightarrow x'(t_0+) = -x'(t_0-), \end{cases} \tag{1.1}$$

where $a(t),\ p(t)$ are 2π-periodic continuous functions and $p(t)$ satisfies

$$p(t) \leq 0 \quad \text{and} \quad \overline{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)dt < 0. \tag{1.2}$$

This system is included in a larger family of impact oscillators given by

$$\begin{cases} x'' + f(t,x,x') = 0 \quad \text{for } x(t) > q(t); \\ x(t) \geq q(t); \\ x(t_0) = q(t_0) \ \Rightarrow x'(t_0+) = -x'(t_0-) + 2q'(t_0), \end{cases} \tag{1.3}$$

where f is continuous and 2π-periodic with respect to t and $q \in C^2(\mathbb{R})$ is also 2π-periodic. From the viewpoint of mechanics this equation models the motion of a particle attached to a nonlinear spring and bouncing elastically against the barrier described by $q(t)$. Thus (1.3) is a model of dynamical system with discontinuity [23] that can be included in the wide family of *vibro-impact systems* [3]. Because of the range of applications in physics and engineering, vibro-impact systems have attracted the attention of a lot of researchers and in consequence the number of papers related to this topic is huge; see [4, 8, 10, 21, 22, 14] and their bibliographies only to mention

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†Department of Mathematics, Suzhou University, Suzhou 215006, People's Republic of China (dbqian@suda.edu.cn). This author's research was supported by NNSF of China (10271085) and NSF of Jiangsu Province, China (BK2002037, 02KJB110003).

‡Universidad de Granada, Departamento de Matemática Aplicada, 18071 Granada, Spain (ptorres@ugr.es). This author's research was supported by D.G.I. BFM2002-01308, Ministerio de Ciencia y Tecnología, Spain.

some of them. There are also interesting relations with Fermi accelerator [15, 35], dual billiards [7], and celestial mechanics [9].

In spite of this, even for the simple case of a one-degree-of-freedom linear oscillator with impacts, the dynamics is far from being understood, although some results are known [6, 24, 25, 33]. Our purpose in this paper is to investigate the existence of nontrivial periodic bouncing solutions with prescribed number of impacts for linear and asymptotically linear impact oscillators. As it is known, the existence of subharmonics of arbitrary order is usually a hint of a complex dynamics. The following definition clarifies the concept of bouncing solution we mean here.

DEFINITION 1.1. *A continuous function $x : \mathbb{R} \to \mathbb{R}$ is a* bouncing solution *for problem* (1.3) *if the following conditions hold:*

1. $x(t) \geq q(t)$ *for all* $t \in \mathbb{R}$*;*
2. *the set* $W = \{t : x(t) = q(t)\}$ *is discrete and not empty;*
3. $x'(t_0+) = -x'(t_0-) + 2q'(t_0)$ *for any* $t_0 \in W$*;*
4. *given an interval* I*, if* $I \cap W = \emptyset$*, then* $x \in C^2(I, \mathbb{R}^+)$ *and it is a classical solution of* (1.3).

Note that the change of variables $y(t) = x(t) - q(t)$ enables to assume without loss of generality that the barrier is fixed at zero. In this context, Lazer and McKenna [25] proved the existence of 2π-periodic bouncing solution for a linear oscillator with small amplitude forcing term and small viscous damping. Recently, Bonheure and Fabry [6] proved the existence of a 2π-periodic bouncing solution for the linear oscillator

$$x'' + \lambda x = p(t), \tag{1.4}$$

where $\lambda > 0$ is a constant and $p(t) < 0$. They also introduced the concept of *admissible* solution in [6] to treat the case where $p(t)$ changes its sign and showed some existence results for perturbations of a linear oscillator. The main feature of an admissible solution is that it can vanish on a whole interval. This is physically equivalent to an attachment of the particle to the barrier $x = 0$ during a whole interval of time. Due to the condition (1.2), we are able to work directly with the more specific concept of bouncing solution, which constitutes a particular class of admissible solutions.

Obviously, our model (1.1) includes (1.4) and also the bouncing problem for the Hill's equation

$$x'' + a(t)x = 0 \tag{1.5}$$

with obstacle $q(t) = d > 0$. Note that $x(t)$ is a bouncing solution of the problem

$$\begin{cases} x'' + a(t)x = 0 \quad \text{for } x(t) > d; \\ x(t) \geq d; \\ x(t_0) = d \;\Rightarrow x'(t_0+) = -x'(t_0-) \end{cases} \tag{1.6}$$

if and only if $y(t)$ is a solution of (1.1) with $p(t) = -a(t)d$ by means of the change $y(t) = x(t) - d$.

The approach of this paper is different from that in [25, 6]. We apply a new generalized version of Poincaré–Birkhoff twist theorem to the so-called successor map, defined as follows. For a given $\tau \in \mathbb{R}$ and $v \in \mathbb{R}^+$, let us denote by $x(t; \tau, v)$ the unique solution of the bouncing problem (1.1) with initial conditions $x(\tau; \tau, v) = 0,\ x'(\tau; \tau, v) = v > 0$. We assume conditions such that this solution is well defined and vanishes at some time $\hat{\tau} > \tau$. Thus $\hat{\tau}$ is the time of the next impact. As the bouncing is elastic, the velocity after this impact is

$$\hat{v} = -x'(\hat{\tau}; \tau, v).$$

If $\hat{v}$ is finite and positive, then the map

$$\mathcal{S} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+,$$
$$\mathcal{S}(\tau, v) = (\hat{\tau}, \hat{v})$$

is well defined, continuous, and one to one. Following [1, 31, 32, 33], this function is called *successor map*, although in this context "impact map" would be also adequate.

Let us state some notation to be used in the rest of the paper: given a 2π-periodic function $p(t)$, $\bar{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)\mathrm{d}t$ is the mean value of p and $\|p\|_\infty = \max_{0\leq t\leq 2\pi} |p(t)|$. The projection for the component i of a given vector is denoted by Π_i. All along the paper, the iteration of the successor map is denoted by $\mathcal{S}^n(\tau, v) = (\hat{\tau}^n(\tau, v),\ \hat{v}^n(\tau, v))$ and we will use $\hat{\tau}^n = \hat{\tau}^n(\tau, v)$, $\hat{v}^n = \hat{v}^n(\tau, v)$ for short. Therefore, $\Pi_1(\mathcal{S}^n(\tau, v)) = \hat{\tau}^n$, $\Pi_2(\mathcal{S}^n(\tau, v)) = \hat{v}^n$. Both notations are used without distinction.

Our main result is the following.

THEOREM 1.2. *Assume that the successor map $\mathcal{S}$ is well defined for all $(\tau, v) \in \mathbb{R} \times \mathbb{R}^+$ and $p(t) \leq 0$ for all t, $\bar{p} < 0$. Then for any $m, n \in \mathbb{N}$ such that $n > 2m(\sqrt{\|a\|_\infty})$, there exists at least one $2m\pi$-periodic bouncing solution of* (1.1) *with exactly n impacts in each period. Moreover, for any $m \in \mathbb{N}$ such that $2m(\sqrt{\|a\|_\infty}) < 1$, there exist at least two $2m\pi$-periodic solutions with one bouncing in each period.*

The following corollaries present two concrete situations where the successor map is well defined and the previous result applies.

COROLLARY 1.3. *Assume that $p(t) \leq 0$ for all t, $\bar{p} < 0$, and $\bar{a} > 0$. Then, the conclusion of Theorem* 1.2 *holds.*

COROLLARY 1.4. *Assume that $p(t) \leq 0$ for all t, $\bar{p} < 0$, and $a(t) \equiv 0$. Then for any $m, n \in \mathbb{N}$, $n \geq 2$, there exists at least one $2m\pi$-periodic bouncing solution of* (1.1) *with exactly n impacts in each period. Moreover, for any $m \in \mathbb{N}$, there exist at least two $2m\pi$-periodic solutions with one bouncing in each period.*

Remark 1.5. In our opinion, the application of the Poincaré–Birkhoff twist theorem to the successor map instead of the Poincaré map (as it is done in [6]) is more natural and direct. For the linear impact oscillator (1.4) we can obtain at least two $2m\pi$-periodic bouncing solutions for (1.4) with exactly 1 impact in each period if $2m\sqrt{\lambda} < 1$, whereas in [6] only one solution is found. Moreover, we can deal with a nonconstant coefficient $a(t)$, in contrast with [6].

In order to understand some of the new phenomena arising in vibro-impact systems, it is interesting to consider in detail the Hill's equation with impacts (1.6) as a particular case. Note that if the obstacle is placed at $d = 0$, then a classical solution x of Hill's equation generates a bouncing solution $|x|$ of (1.6). Hence, in this case (1.6) inherits the dynamics of Hill's equation without impacts and in consequence its resonant or nonresonant character. However, if the obstacle is $d > 0$, the situation is different. Physically, this model corresponds to a kind of offset impact oscillator [18], consisting of a linear spring-mass system with a displaced wall with respect to the origin (see Figure 1(a)). The time-dependence of the stiffness coefficient $a(t)$ of the spring can be produced by periodic changes of the temperature or other physical variables. A periodic bouncing solution corresponds to a nontrivial periodic motion with prescribed impacts in one period. The following result holds.

COROLLARY 1.6. *Assume that $d > 0$, $a(t) \geq 0$ for all t, and $\bar{a} > 0$. Then for any $m, n \in \mathbb{N}$ such that $n > 2m(\sqrt{\|a\|_\infty})$, there exists at least one $2m\pi$-periodic bouncing solution of* (1.6) *with exactly n impacts in each period.*

The proof follows from Corollary 1.3 by means of the change of variables $y = x-d$. Thus, the Hill's equation could be unstable (equivalently, all nontrivial solutions are

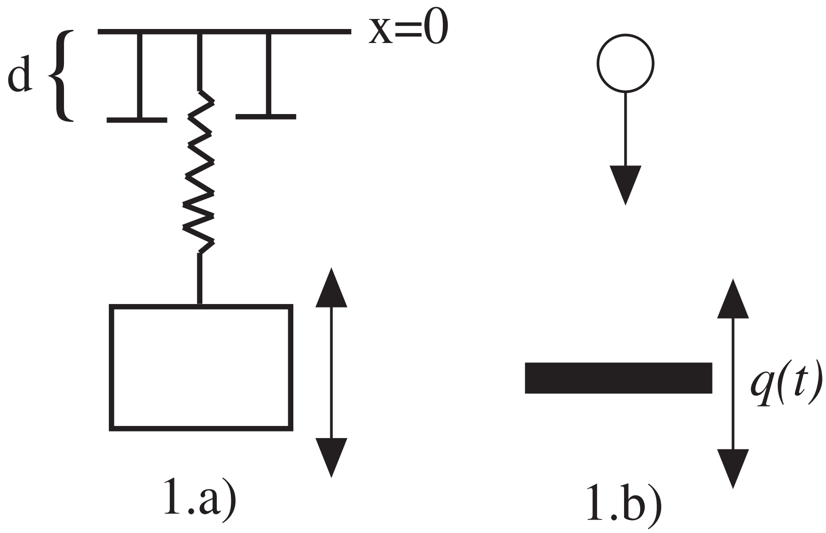

FIG. 1. (a) *The offset oscillator.* (b) *The "ping-pong" model.*

unbounded [26]) but nevertheless (1.6) has periodic bouncing solutions. In other words, possible parametric resonances are killed by the presence of an obstacle. This fact is a good example of the obstacle's influence in the dynamics of a given system.

Another simple but physically interesting model is the *"ping-pong" problem*, that is, a free ball moving in a vertical line subjected to gravity force and bouncing against a barrier or racket describing a periodic movement $q(t)$ (see Figure 1(b)). If G is the acceleration of gravity, the motion of the ball is described by

$$\begin{cases} x'' + G = 0 \quad \text{for } x(t) > q(t); \\ x(t) \geq q(t); \\ x(t_0) = q(t_0) \ \Rightarrow x'(t_0+) = -x'(t_0-) + 2q'(t_0). \end{cases}$$

This is a simple variation of Fermi's model that have deserved the attention of many researchers (see [19, 5, 13] and their references). After the change $y(t) = x(t) - q(t)$, the problem is transformed in (1.1) with $a(t) \equiv 0$ and $p(t) = -G - q''(t)$. Then, if $q''(t) > -G$ for any t, the ball experiences a diversity of periodic motions with a prescribed number of impacts as a consequence of Corollary 1.4.

Remark 1.7. The concept of bouncing solution could involve other new features and strong differences with the situation when working with differential equations without impacts. An interesting open problem is to prove or disprove the validity of Massera's theorem for impact oscillators. Massera's theorem asserts that in the framework of periodic differential equations the existence of a bounded solution implies the existence of a periodic solution [28]. This classical result is false in the context of equations with impacts in the sense that a bounded bouncing solution (using the definition in this paper) does not imply a periodic bouncing solution. To prove this, consider the Mathieu equation $a(t) = \gamma + \delta \cos t$ with obstacle $d = 0$ and parameters γ, δ placed in a stability region with irrational rotation number. Then any nontrivial solution of (1.5) is quasi-periodic (but not periodic) and in consequence every bouncing solution of (1.5) is bounded but there are no periodic bouncing solutions. Of course,

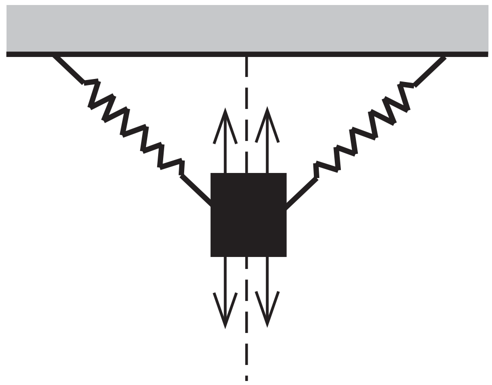

Fig. 2. *A spring-mass impact system.*

this is just an effect of the definition chosen here, since the trivial solution is excluded. Note that the trivial solution is not a bouncing solution but it is an admissible solution in the sense of [6]. So the exciting question of the validity of Massera's theorem for impact oscillators is still open: does the existence of a bounded bouncing solution imply the existence of a periodic admissible solution (including trivial solution)? We do not know the answer.

Our successor map approach is also suitable for use in nonlinear impact oscillators, as it is done in [34] for a singular equation. Here we include a result about the asymptotically linear impact oscillator.

Theorem 1.8. *Let us assume that $g(t,x)$ is continuous, 2π-periodic with respect to t, and satisfies*

$$\limsup_{x\to 0^+}\left|\frac{g(t,x)}{x}\right| < +\infty, \qquad \lim_{x\to+\infty}\frac{g(t,x)}{x} = 0. \tag{1.7}$$

Besides, let us suppose that the successor map $\mathcal{S}$ of the bouncing problem

$$\begin{cases} x'' + a(t)x + g(t,x) = p(t) \quad \textit{for } x(t) > 0; \\ x(t) \geq 0; \\ x(t_0) = 0 \ \Rightarrow x'(t_0+) = -x'(t_0-) \end{cases} \tag{1.8}$$

is well defined for all $(\tau, v) \in \mathbb{R}\times\mathbb{R}^+$ and $p(t) \leq 0$ for all t, $\bar{p} < 0$. Then, the conclusion of Theorem 1.2 *holds.*

A corollary of the previous result is the following.

Corollary 1.9. *Assume that $p(t) \leq 0$ for all t, $\bar{p} < 0$, $g(t,x)$ satisfies* (1.7) *and $a(t)x + g(t,x) \geq 0$ for any $x \geq 0$. Then, the conclusion of Theorem* 1.2 *holds.*

This result can be illustrated by a simple physical model presented in Figure 2. This mechanical system is a modification of the model presented in [2, 17] and consists of a single mass moving in a straight line, attached to the wall by two linear springs of constant k and natural length L and perturbed periodically by an external force $p(t)$. If it is assumed that the impacts between the mass and the wall are perfectly elastic, then the motion of the mass is governed by

$$\begin{cases} mx'' + 2k\left[x - \dfrac{Lx}{(c^2+x^2)^{1/2}}\right] = p(t) \quad \text{for } x(t) > 0; \\ x(t) \geq 0; \\ x(t_0) = 0 \ \Rightarrow x'(t_0+) = -x'(t_0-), \end{cases}$$

where $c > 0$ is the distance between the point of impact and the attachments of the springs (see [2, 17] for more details). If $p(t) \leq 0$ for all t, $\bar{p} < 0$, it is easy to verify that this problem is under the assumptions of Corollary 1.9 when $c > L$.

The rest of the paper is organized as follows. In section 2, the proof of Theorem 1.2 is given. It relies on a generalized version of Poincaré–Birkhoff theorem. Section 3 collects some auxiliary lemmas which are needed in the mentioned proof, more specifically the twist property of some iteration of the successor map is proved. Finally, section 4 is devoted to the study of the asymptotically linear impact oscillator.

2. Existence of periodic bouncing solutions. We will apply the Poincaré–Birkhoff twist theorem to the successor map $\mathcal{S}$ for proving the existence of 2π-periodic bouncing solutions for impact oscillators (1.1). The successor map was used recently by Ortega [33] for investigation of the boundedness of all the solutions for a linear impact oscillator by using Moser's twist theorem and the authors [34] for investigation of the periodic bouncing solutions for some singular impact oscillator. As a general idea, this successor map is just a different section of the flux and it goes back at least to Alekseev [1] and Moser [30].

The following generalized version of Poincaré–Birkhoff twist theorem is based on the theorems of Franks [16] and Ding [11] and is slightly different from the version used by others (see, for example, [20], [6], and [27]).

Let A and B be two annuli

$$A := \mathbf{S}^1 \times [a_1, a_2], \qquad B := \mathbf{S}^1 \times [b_1, b_2]$$

with $0 < b_1 < a_1 < a_2 < b_2 < +\infty$. A map $f : A \to B$ possesses a lift $\tilde{f} : \mathbb{R} \times [a_1, a_2] \to \mathbb{R} \times [b_1, b_2]$ with the form

$$\theta' = \theta + h(\theta, \rho), \quad \rho' = g(\theta, \rho),$$

where $h,\ g$ are continuous and 2π-periodic in θ. We say that $\tilde{f}$ satisfies the boundary twist condition if

$$h(\theta, a_1) \cdot h(\theta, a_2) < 0 \quad \text{for } \theta \in [0,\ 2\pi].$$

THEOREM 2.1. *Assume that $f : A \to B$ is an area-preserving homeomorphism homotopic to the inclusion such that $f(A) \cap \partial B = \emptyset$. Moreover, f possesses a lift $\tilde{f}$ satisfying the boundary twist condition and the area of the two connected components of the complement of $f(A)$ in B is the same as the area of the corresponding connected components of the complement of A in B. Then, f has at least two geometrically distinct fixed points (θ_i, ρ_i), $(i = 1, 2)$ satisfying $h(\theta_i, \rho_i) = 0$ for $i = 1, 2$.*

Proof. The proof basically combines the proofs from Franks [16] and Ding [11]. In [16], Franks showed that by using a result from Oxtoby and Ulam, one can extend f to an area-preserving homeomorphism $F : B \to B$ such that F is the identity on the boundary of B (see the proof and the remark of Theorem 4.2 in [16]). Then, we can assume further that F is an area-preserving homeomorphism of $D := \{(\theta, \rho) : \rho \leq b_2\}$ to its image such that $O \in F(D \backslash B)$. Now we meet all the assumptions of the argument in [11]. According to the argument of [11], we can prove that F, and then f, has at least two fixed points in A. Moreover, the fixed points (θ_i, ρ_i) satisfy $h(\theta_i, \rho_i) = 0$ for $i = 1, 2$ (see [11] and [12] for more details). Figure 3 illustrates the geometrical meaning of the hypotheses. □

Now, we apply the above Poincaré–Birkhoff theorem to the successor map $\mathcal{S}$. From the discussion in the next section we know that our successor map $\mathcal{S}$

$$\mathcal{S} : (\tau,\ v) \mapsto (\hat{\tau},\ \hat{v})$$

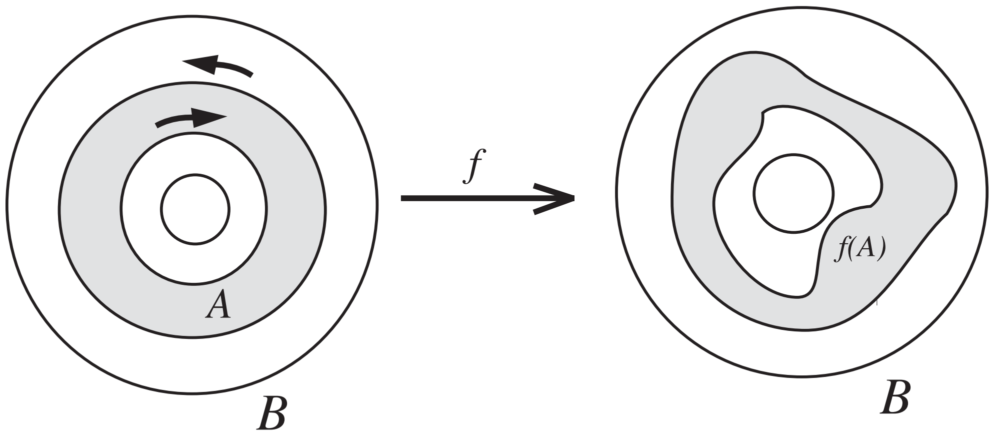

FIG. 3. *The Poincaré–Birkhoff theorem.*

is well defined, one to one, and continuous in its domain $\mathbb{R}\times\mathbb{R}^+$. Moreover, it satisfies

$$\mathcal{S}(\tau+2\pi,v)=\mathcal{S}(\tau,v)+(2\pi,0).$$

Thus, we can interpret τ and v as polar coordinates and $\mathcal{S}$ is an embedding homeomorphism on $\mathbf{S}^1\times\mathbb{R}^+$. It is easy to show that for any $n,m\in\mathbb{N}$, a fixed point of the map $\mathcal{S}^n(\tau,v)-(2m\pi,0)$ corresponds a $2m\pi$-periodic bouncing solution of the equation with n impacts in each period. We have the following lemma.

LEMMA 2.2. *$\mathcal{S}$ is an area-preserving map with the area element $vdvd\tau$. Moreover, $\mathcal{S}$ is area-preserving homotopic to the inclusion, and for any annuli $A\subset B\subset\mathbf{S}^1\times\mathbb{R}^+$ with $\mathcal{S}(A)\subset\overset{\circ}{B}$, the area of the two connected components of the complement of $\mathcal{S}(A)$ in B is the same as the area of the corresponding components of the complement of A in B.*

The proof of this lemma is similar to the proof of Lemma 1 in [20] and the proof of Proposition 2.3 in [31]. At first we can prove, under the assumption of the C^1-smoothness of a and p which implies the C^1-smoothness of $\mathcal{S}$, that $\mathcal{S}$ is an exact symplectic map in its domain; that is, for any C^1-closed path γ in its domain

$$\int_\gamma \frac{v^2}{2}d\tau=\int_{\mathcal{S}\circ\gamma}\frac{v^2}{2}d\tau. \tag{2.1}$$

Moreover, note that $\mathcal{S}$ is an embedding homeomorphism on $\mathbf{S}^1\times\mathbb{R}^+$, then from Jordan separation theorem (see, for instance, [29]), we know that for any annuli $A\subset B\subset\mathbf{S}^1\times\mathbb{R}^+$ with $\mathcal{S}(A)\subset\overset{\circ}{B}$, there are two connected components of the complement of $\mathcal{S}(A)$ in B. Such components are the images of the two components of the complement of A in B. Hence, the geometric meaning of (2.1) is that the area of the components of the complement of $\mathcal{S}(A)$ in B are the same as the area of the corresponding components of the complement of A in B. The conclusion for the case of continuous functions a and p follows from an approximation argument.

Moreover, Lemmas 3.4 and 3.6 (see section 3) imply that, under the assumptions of Theorem 1.2, we can choose $v_-^{(n)}<v_+^{(n)}$ such that

$$\begin{aligned}\Pi_1(\mathcal{S}^n(\tau,v_-^{(n)}))-\tau&<2m\pi,\\ \Pi_1(\mathcal{S}^n(\tau,v_+^{(n)}))-\tau&>2m\pi\quad\text{for }\tau\in[0,2\pi].\end{aligned}$$

Hence, let A be the annulus bounded by $\mathbf{S}^1 \times \{v_-^{(n)}\}$ and $\mathbf{S}^1 \times \{v_+^{(n)}\}$ and let B be the annulus bounded by $\mathbf{S}^1 \times \{v_*\}$ and $\mathbf{S}^1 \times \{v^*\}$. We can prove, as showed in section 3, that $f(A) \subset \overset{\circ}{B}$ for $v_* > 0$ sufficiently small and v^* sufficiently large, where $f : A \to B$ is defined by

$$f(\tau, v) = \mathcal{S}^n(\tau, v) - (2m\pi, 0).$$

It is easy to see that f is an area-preserving homeomorphism homotopic to the inclusion and $\tilde{f}$ satisfies the boundary twist condition. Thus the conclusion of Theorem 1.2 follows by a direct application of Theorem 2.1. Note that in any case we get two fixed points of $\mathcal{S}^n(\tau, v) - (2m\pi, 0)$. However, if the number of bouncings n is greater than or equal to 2, these two fixed points provided by Theorem 2.1 may correspond to the same bouncing solution, so we can only assure the existence of two different $2m\pi$-periodic bouncing solutions when there is only one impact in each period.

3. Twist property for the successor map. The aim of this section is to provide the necessary properties for the application of the Poincaré–Birkhoff theorem yet done in section 2. Basically, our goal is to prove that the rotation for some iteration of the successor map is *slow* for small velocities and *fast* for large velocities. This will be done through some auxiliary lemmas concerning the asymptotic dynamics of the solutions for (1.1). Lemma 3.1 gives a second-order differential inequality to be used later. Lemma 3.2 shows that $\mathcal{S}$ is well defined for $v \ll 1$, Lemma 3.3 shows that the impact velocity $\hat{v}$ is small if the initial velocity v is small enough and in consequence, Lemma 3.4 gives the slow rotation for some iteration of $\mathcal{S}$ for small initial velocities. Lemma 3.5 discusses, under the assumption that the successor map is well defined, the fast rotation of $\mathcal{S}$ for large velocities by using the Sturm comparison theorem. This fact implies (Lemma 3.6) the fast rotation for some iteration of $\mathcal{S}$ for large initial velocities. At the end of this section, we discuss, in Lemmas 3.7 and 3.8, when the successor map $\mathcal{S}$ is well defined by using some oscillatory properties of the solutions of the Hill's equation.

LEMMA 3.1. *Suppose that $x_1(t)$ is a solution of the equation $x'' = Mx$ for $t \in I$, where $M > 0$, and $x_2(t)$ satisfies the differential inequality $x'' \leq Mx$ for $t \in I$, with the same initial conditions $x_1(\tau) = x_2(\tau),\ x_1'(\tau) = x_2'(\tau)$. Then $x_1(t) \geq x_2(t)$ for $t \in I$.*

Proof. Let $z_n(t) = x_n(t) - x_2(t)$, where $x_n(t)$ is the solution of $x'' = Mx$ with the initial condition $x_n(\tau) = x_2(\tau),\ x_n'(\tau) = x_2'(\tau) + \frac{1}{n}$. Then $z_n(\tau) = 0,\ z_n'(\tau) = \frac{1}{n} > 0$ which implies that $z_n(t) > 0$ for $t > \tau$ and t close to τ. Moreover, $z_n''(t) \geq Mz_n(t)$ for $t > \tau$. Thus $z_n'(t) > z_n'(\tau) > 0$ and $z_n(t)$ increases strictly for $t > \tau$. Hence $x_n(t) > x_2(t)$ for $t > \tau$. Let $n \to \infty$. Then $x_n(t) \to x_1(t)$ in any compact interval by using the continuous dependence on initial values. Therefore, $x_1(t) \geq x_2(t)$ for $t \in I$. ∎

LEMMA 3.2. *If $p(t) \leq 0$ and $\overline{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)dt < 0$, then every solution $x(t; \tau, v)$ of* (1.1) *starting from $x(\tau; \tau, v) = 0,\ x'(\tau; \tau, v) = v > 0$ does not satisfy $x(t; \tau, v) = x'(t; \tau, v) = 0$ for any t in its domain. Moreover, $\mathcal{S}$ is well defined and one to one for $v \ll 1$.*

Proof. Note that every solution of (1.1) starting from $x(\tau; \tau, v) = 0,\ x'(\tau; \tau, v) = v > 0$ satisfies

$$x' = y, \quad y' = -a(t)x + p(t)$$

in $(x,\ y)$-plane before it meets $x = 0$ again. Then $x'(t; \tau, v) > 0$ when it is in the half-plane $y > 0$ which implies that $x(t; \tau, v) > 0$ for $t > \tau$ and close to τ. Moreover, if there are $\tau_1,\ \tau_2$ such that

$$x'(\tau_1;\tau,v) = x'(\tau_2;\tau,v) = 0, \qquad x'(s;\tau,v) > 0 \quad \text{for } s \in (\tau_1,\tau_2),$$

then

$$x(\tau_2;\tau,v) > x(\tau_1;\tau,v). \tag{3.1}$$

If $x'(\tau_3;\tau,v) = 0$ and $x'(s;\tau,v) < 0$ for $s \in (\tau_3,t)$, then by using polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

in the half-plane $y \le 0$ we get

$$r' = (1-a(t))r\cos\theta\sin\theta + p(t)\sin\theta \ge -Kr,$$

where $K = \max_{0\le t\le 2\pi}|1-a(t)|$. Thus

$$r(t) \ge r(\tau_3)\exp(-K(t-\tau_3)). \tag{3.2}$$

Therefore, either $x(t;\tau,v)$ has no any impact in $t>\tau$ or $x(t;\tau,v)$ has its next impact at $t=\hat\tau$. In this case, (3.1), (3.2) imply that

$$x'(\hat\tau;\tau,v) \le -x(\tilde\tau;\tau,v)\exp(-K(\hat\tau-\tau)) < 0,$$

where $t=\tilde\tau$ is the first time $x(t;\tau,v)$ meets $y=0$ after τ. The conclusion of the first part of the lemma is thus proved.

Next, note that when $x(t;\tau,v)$ is remaining in half-plane $x>0$,

$$x''(t;\tau,v) = -a(t)x(t;\tau,v) + p(t) \le Mx(t;\tau,v),$$

where $M = \max_{0\le t\le 2\pi}|a(t)|$. Then Lemma 3.1 implies that

$$x(t;\tau,v) \le M_0 = \frac{v}{2\sqrt{M}}(\exp(2\pi\sqrt{M}) - \exp(-2\pi\sqrt{M})) \tag{3.3}$$

for $t\in(\tau,\tau+2\pi)$. Thus,

$$x'(t;\tau,v) = v - \int_\tau^t (a(s)x(s;\tau,v)-p(s))ds \le O(v) + \int_\tau^t p(s)ds.$$

Because $\bar p < 0$, there must be $\tilde\tau \in (\tau,\tau+2\pi)$ such that

$$x(\tilde\tau;\tau,v) > 0, \qquad x'(\tilde\tau;\tau,v) = 0, \qquad x'(s;\tau,v) > 0 \quad \text{for } s\in(\tau,\tilde\tau),$$

provided that $v \ll 1$. Moreover, for $t\in(\tilde\tau,\tau+2\pi)$, we have

$$x(t;\tau,v) = x(\tilde\tau;\tau,v) - \int_{\tilde\tau}^t\int_{\tilde\tau}^w (a(s)x(s;\tau,v)-p(s))dsdw = O(v) + \int_{\tilde\tau}^t\int_{\tilde\tau}^w p(s)dsdw$$

from which it follows that there exists $\hat\tau\in(\tilde\tau,\tau+2\pi)$ such that

$$x(\hat\tau;\tau,v) = 0, \qquad x'(\hat\tau;\tau,v) < 0, \qquad x(t;\tau,v) > 0 \quad \text{for } t\in[\tilde\tau,\hat\tau),$$

provided that $v \ll 1$ and $\bar p < 0$. The lemma is thus proved. ∎

The next lemma clarifies the behavior of the next impact velocity $\hat{v}$ for small v.

LEMMA 3.3. *If $p(t) \le 0$ and $\overline{p} < 0$, then the next velocity $\hat{v}$ of the successor map satisfies*

$$\lim_{v \to 0^+} \hat{v}(\tau, v) = 0$$

uniformly for $\tau \in [0, 2\pi)$.

Proof. As it is shown in Lemma 3.2, for $v > 0$ small enough, we have a well-defined $\hat{\tau} \in (\tau, \tau + 4\pi)$. Moreover,

$$\max_{\tau \le t \le \hat{\tau}} x(t; \tau, v) = O(v) \quad \text{as } v \to 0^+. \tag{3.4}$$

By contradiction, let us assume that there exist $\{\tau_n\}$ belonging to $[0, 2\pi)$ and $\{v_n\}$ with $v_n \to 0^+$ as $n \to \infty$, such that $\hat{v}(\tau_n, v_n) \le -\delta < 0$. Then there exist $t_n \in (\tau_n, \hat{\tau}(\tau_n, v_n))$ satisfying

$$x'(t_n; \tau_n, v_n) = -\frac{\delta}{2}, \qquad x'(s; \tau_n, v_n) \le -\frac{\delta}{2} \quad \text{for } s \in [t_n, \hat{\tau}(\tau_n, v)].$$

Denote by $P = \|p\|_\infty$, then

$$\begin{aligned} -\frac{\delta}{2} \ge \hat{v}(\tau_n, v_n) - x'(t_n; \tau_n, v_n) &= -\int_{t_n}^{\hat{\tau}(\tau_n, v_n)} (a(s)x(s; \tau_n, v_n) - p(s))ds \\ &\ge -(M + P)(\hat{\tau}(\tau_n, v_n) - t_n), \end{aligned}$$

provided that $\max_{t_n \le t \le \hat{\tau}(\tau_n, v_n)} x(t; \tau_n, v_n) \le 1$ (this is guaranteed for v_n small by (3.4)). Thus we can estimate

$$\begin{aligned} \max_{\tau_n \le t \le \hat{\tau}_n} x(t; \tau_n, v_n) &\ge x(t_n; \tau_n, v_n) - x(\hat{\tau}(\tau_n, v_n); \tau_n, v_n) \\ &= -\int_{t_n}^{\hat{\tau}(\tau_n, v_n)} x'(s; \tau_n, v_n) \mathrm{d}s \ge \frac{\delta}{2} \cdot (\hat{\tau}(\tau_n, v_n) - t_n) \ge \frac{\delta^2}{4(M + P)}, \end{aligned}$$

which contradicts (3.4). The result is thus proved. ∎

Let us recall that we write $\mathcal{S}^n(\tau, v) = (\hat{\tau}^n(\tau, v),\ \hat{v}^n(\tau, v))$ and we will use the abbreviation $\hat{\tau}^n = \hat{\tau}^n(\tau, v),\ \hat{v}^n = \hat{v}^n(\tau, v)$. Then, it is deduced from Lemma 3.3 that for all $n \in \mathbb{N}$ and $v_n > 0$, there exists $v_0 > 0$, such that

$$|x'(t; \tau, v)| \le v_n \quad \text{for } v \in (0, v_0],\ t \in [\tau, \hat{\tau}^n].$$

Now, suppose that there are $c > 0$ and $\delta > 0$ such that $p(t) \le -c$ for $t \in [\tau_0 - 2\delta, \tau_0 + 2\delta]$. Then,

$$\hat{\tau}^n - \tau < \delta \quad \text{for } v \ll 1 \text{ and } \tau \in [\tau_0 - \delta, \tau_0 + \delta]. \tag{3.5}$$

Actually,

$$|\hat{v}^j + \hat{v}^{j-1}| = |x'(\hat{\tau}^j; \tau, v) - x'(\hat{\tau}^{j-1}; \tau, v)| = \int_{\hat{\tau}^{j-1}}^{\hat{\tau}^j} (a(t)x(t; \tau, v) - p(t))dt \ge \frac{c}{2}(\hat{\tau}^j - \hat{\tau}^{j-1}),$$

provided that (3.4) and $[\hat{\tau}^{j-1}, \hat{\tau}^j] \subset [\tau_0 - 2\delta, \tau_0 + 2\delta]$. Then

$$\hat{\tau}^n - \tau \le \frac{4}{c} \sum_{j=1}^{n} \hat{v}^j < \delta \tag{3.6}$$

if we choose $v \ll 1$ and $\tau \in [\tau_0 - \delta, \tau_0 + \delta]$.

Now, we can prove the twist property of the successor map for $v \ll 1$.

LEMMA 3.4. *Let us suppose that $p(t) \leq 0$ and $\bar{p} < 0$. Then, for all $n, m \in \mathbb{N}$, there exists $v_n > 0$ such that*

$$\Pi_1(\mathcal{S}^n(\tau, v)) - \tau < 2m\pi \quad \text{for } v \in (0, v_n] \text{ and } \tau \in [0, 2\pi].$$

Proof. Since $p(\cdot)$ is continuous and $\bar{p} < 0$, there are $c > 0$, $\delta > 0$, and $\tau_0 \in [0, 2\pi]$ such that $p(t) \leq -c$ for $t \in [\tau_0 - 2\delta, \tau_0 + 2\delta]$. Then, there exists $v \ll 1$ such that

$$\hat{\tau}^n(\tau, v) - \tau < \delta \quad \text{for } \tau \in [\tau_0 - \delta, \tau_0 + \delta]. \tag{3.7}$$

For $\tau \in (\tau_0 + \delta, 2\pi + \tau_0 - \delta)$ either $\hat{\tau}^n(\tau, v) \leq 2\pi + \tau_0 - \delta$ which implies that

$$\hat{\tau}^n - \tau < 2\pi - 2\delta, \tag{3.8}$$

or there exists $t \in (\hat{\tau}^{j-1}, \hat{\tau}^j) \cap [2\pi + \tau_0 - \delta, 2\pi + \tau_0]$ for some $j \in \{1, 2, \ldots, n\}$. Then, by estimating like in (3.6) it is proved that, if v small enough, $\hat{\tau}^j - t \leq \frac{\delta}{n}$. From here it is deduced that

$$\hat{\tau}^j \in \left(2\pi + \tau_0 - \delta, 2\pi + \tau_0 + \frac{\delta}{n}\right] \tag{3.9}$$

and then $\hat{\tau}^n - \hat{\tau}^j < \frac{n-1}{n}\delta$ which implies that

$$\hat{\tau}^n - \tau = \hat{\tau}^n - \hat{\tau}^j + \hat{\tau}^j - t + t - \tau < \frac{n-1}{n}\delta + \frac{\delta}{n} + 2\pi + \tau_0 - (\tau_0 + \delta) = 2\pi. \tag{3.10}$$

Since $\mathcal{S}$ is continuous on $\mathbb{R} \times \mathbb{R}^+$ (this is a consequence of the uniqueness of the solution for the initial value problem for linear equation), the above estimations are uniform for the compact interval $[0, 2\pi]$. Therefore, (3.7)–(3.10) complete the proof of the lemma. ◻

Under the assumption that the successor map $\mathcal{S}$ is well defined, our next result proves the twist property for large velocities.

LEMMA 3.5. *Assume that $\mathcal{S} : (\tau, v) \mapsto (\hat{\tau}, \hat{v})$ for $(\tau, v) \in \mathbb{R} \times \mathbb{R}^+$ is well defined and $p(t) \leq 0$, $\bar{p} < 0$. Then*

$$\liminf_{v \to +\infty} [\hat{\tau}(\tau, v) - \tau] \geq \frac{\pi}{\sqrt{\|a\|_\infty}} \tag{3.11}$$

uniformly for $\tau \in [0, 2\pi)$. If $a(t) \equiv 0$, then

$$\lim_{v \to +\infty} [\hat{\tau}(\tau, v) - \tau] = +\infty \tag{3.12}$$

uniformly for $\tau \in [0, 2\pi)$.

Proof. Suppose firstly that $\|a\|_\infty > 0$ and there are $\tau_n \in [0, 2\pi)$ and $v_n > 0$ with $v_n \to +\infty$ as $n \to \infty$ such that $\hat{\tau}(\tau_n, v_n) - \tau_n \leq \pi/\sqrt{\|a\|_\infty} - \gamma$ with $\gamma > 0$. Then there are $\tau_* \in [0, 2\pi]$ and $\hat{\tau}_* \in (\tau_*, \tau_* + \pi/\sqrt{\|a\|_\infty} - \gamma]$ such that $\tau_n \to \tau_*$ and $\hat{\tau}(\tau_n, v_n) \to \hat{\tau}_*$ as $n \to \infty$. Moreover, $y_n(t) = x(t; \tau_n, v_n)/v_n$ is the solution of the equation

$$x'' + a(t)x = \frac{1}{v_n} p(t)$$

with the initial conditions $y_n(\tau_n) = 0,\ y_n'(\tau_n) = 1$. By continuous dependence of the solutions with respect to initial value and parameters we have that

$$\lim_{n\to\infty} y_n(t) = y_0(t) \quad \text{and} \quad \lim_{n\to\infty} y_n'(t) = y_0'(t) \tag{3.13}$$

uniformly on compact intervals, where $y_0(t)$ is the solution of Hill's equation (1.5) with the initial condition $y_0(\tau_*) = 0,\ y_0'(\tau_*) = 1$. Thus $y_0(\hat{\tau}_*) = 0$ due to the continuous dependence of the solutions with respect to initial values and parameters. On the other hand, by using Sturm comparison theorem, it is proved that

$$\tau' - \tau \geq \frac{\pi}{\sqrt{\|a\|_\infty}},$$

where τ' and τ are two consecutive zeros of $y_0(t)$, so in consequence $\hat{\tau}_* - \tau_* \geq \pi/\sqrt{\|a\|_\infty}$. This is a contradiction. If $a(t) \equiv 0$, then any solution $x(t;\tau,v)$ of the equation $x'' = p(t)$ has the derivative $x'(t;\tau,v) = v + \int_\tau^t p(s)\mathrm{d}s$. Hence, for any fixed $v > 0$ there exists a $\hat{\tau} > \tau$ such that $x(\hat{\tau};\tau,v) = \int_\tau^{\hat{\tau}}(v + \int_\tau^t p(s)\mathrm{d}s)\mathrm{d}t = 0$ and $\lim_{v\to+\infty}(\hat{\tau} - \tau) = +\infty$. Therefore, the lemma is proved. ∎

From the above estimation we can prove the twist property of successor map for $v \gg 1$. Recall that $r(t) = (x^2(t;\tau,v) + (x'(t;\tau,v))^2)^{1/2}$ satisfies

$$-Kr(t) - P \leq r'(t) \leq Kr(t) + P \quad \text{for } t \in (\tau, \hat{\tau}),$$

where $K = \max_{0\leq t\leq 2\pi}|1 - a(t)|$ and $P = \max_{0\leq t\leq 2\pi}|p(t)|$. Then, by using Gronwall inequality,

$$\left(v + \frac{P}{K}\right)\exp(-KT) \leq \hat{v} + \frac{P}{K} \leq \left(v + \frac{P}{K}\right)\exp(KT), \tag{3.14}$$

provided that $\hat{\tau} - \tau \leq T$. Suppose that $\Pi_1(\mathcal{S}^n(\tau,v)) - \tau \leq 2m\pi$, then

$$\Pi_1(\mathcal{S}^{i+1}(\tau,v)) - \Pi_1(\mathcal{S}^i(\tau,v)) \leq 2m\pi \quad \text{for } i = 0,1,\ldots,n-1.$$

This implies that

$$\begin{aligned}\left(\Pi_2(\mathcal{S}^i(\tau,v)) + \frac{P}{K}\right)\exp(-2m\pi K) &\leq \Pi_2(\mathcal{S}^{i+1}(\tau,v)) + \frac{P}{K}\\ &\leq \left(\Pi_2(\mathcal{S}^i(\tau,v)) + \frac{P}{K}\right)\exp(2m\pi K)\end{aligned}$$

for $i = 0,1,\ldots,n-1$. Thus for a given $v_0^+ > 0$ we have $v_{n,m}^+ > 0$ such that

$$\text{if } v > v_{n,m}^+ \text{ and } \Pi_1(\mathcal{S}^n(\tau,v)) - \tau \leq 2m\pi, \text{ then } \Pi_2(\mathcal{S}^i(\tau,v)) > v_0^+ \tag{3.15}$$

for $i = 0,1,\ldots,n-1$. Hence, the following result is obtained.

Lemma 3.6. *Let us suppose that $p(t) \leq 0$ and $\bar{p} < 0$. Let $n, m \in \mathbb{N}$ be such that $n > 2m(\sqrt{\|a\|_\infty})$. Then, there exists $v_{n,m}^+ > 0$ such that*

$$\Pi_1(\mathcal{S}^n(\tau,v)) - \tau > 2m\pi \quad \textit{for } v \geq v_{n,m}^+ \textit{ and } \tau \in [0,2\pi].$$

Proof. From Lemma 3.5 we know that there exists $v_0^+ > 0$ such that

$$\Pi_1(\mathcal{S}(\tau,v)) - \tau \geq \frac{\pi}{\sqrt{\|a\|_\infty}} \quad \text{for } v \geq v_0^+ \text{ and } \tau \in [0,2\pi). \tag{3.16}$$

By the periodicity of the equation, it is verified that

$$\mathcal{S}(\tau+2\pi,v)=\mathcal{S}(\tau,v)+(2\pi,0).$$

This means that the function $f(\tau)=\Pi_1(\mathcal{S}(\tau,v))-\tau$ is 2π-periodic. Therefore, (3.16) holds for all $\tau\in\mathbb{R}$. Taking $v_{n,m}^+$ as in (3.15), if $v>v_{n,m}^+$ then either $\Pi_1(\mathcal{S}^n(\tau,v))-\tau>2m\pi$ or $\Pi_1(\mathcal{S}^n(\tau,v))-\tau\le 2m\pi$. In the second case, it follows from (3.15) that $\Pi_2(\mathcal{S}^i(\tau,v))>v_0^+$ for $i=0,1,\dots,n-1$, and in consequence for every $i=1,\dots,n-1$ we have

$$\Pi_1(\mathcal{S}^{i+1}(\tau,v))-\Pi_1(\mathcal{S}^i(\tau,v))\ge\frac{\pi}{\sqrt{\|a\|_\infty}}\quad\text{for } v\ge v_{n,m}^+ \text{ and } \tau\in[0,2\pi).$$

Adding the previous inequalities for $i=1,\dots,n-1$ with (3.16),

$$\Pi_1(\mathcal{S}^n(\tau,v))-\tau\ge n\frac{\pi}{\sqrt{\|a\|_\infty}}.$$

Now, taking into account that $n>2m(\sqrt{\|a\|_\infty})$, the result is done. ∎

The rest of this section is devoted to the discussion of the conditions implying that the successor map is well defined. At first we can prove as in the proof of Lemma 3.2 that successor map is well defined if $p(t)\le 0$ for all t, $\bar p<0$ and $a(t)\equiv 0$. We will prove in the following that $\bar a>0$ is also enough to assure that the successor map is well defined. With this, the proofs of Corollaries 1.3 and 1.4 are completed.

Consider the solution $x(t;\tau,v)$ of the impact oscillator (1.1) starting from

$$x(\tau;\tau,v)=0, x'(\tau;\tau,v)=v>0.$$

Lemma 3.2 implies that either there exists $\hat\tau>\tau$ such that $x(\hat\tau;\tau,v)=0$ and $x(t;\tau,v)>0$ for $t\in(\tau,\hat\tau)$, or

$$x(t;\tau,v)>0\quad\text{for all } t>\tau, \tag{3.17}$$

and in consequence $x(t;\tau,v)$ is a (classical) solution of the equation $x''+a(t)x=p(t)$, with $t>\tau$. If (3.17) holds, we will show that there is a constant $\delta>0$ such that $x(t;\tau,v)\ge\delta$ for sufficiently large $t>\tau$. Actually, we will show firstly that (3.17) implies that

$$|x(t;\tau,v)|+|x'(t;\tau,v)|\ge 2\delta. \tag{3.18}$$

By contradiction, let us suppose that there exists $\tau_1>\tau$ such that $x(\tau_1;\tau,v)=\alpha\ge 0$, $x'(\tau_1;\tau,v)=\beta$ with $|\alpha|+|\beta|<2\delta$. Then as in Lemma 3.2 it is shown that

$$x(t;\tau,v)\le\frac{1}{2\sqrt{M}}((\sqrt{M}\alpha+\beta)\exp(2\pi\sqrt{M})+(\sqrt{M}\alpha-\beta)\exp(-2\pi\sqrt{M}))$$

for $t\in(\tau_1,\tau_1+2\pi)$, being $M=\|a\|_\infty$. Thus,

$$\begin{aligned}x(t;\tau,v)&=\alpha+\int_{\tau_1}^t\left(\beta+\int_{\tau_1}^s(-a(\xi)x(\xi;\tau_1,v)+p(\xi)\mathrm{d}\xi)\mathrm{d}s\right)\\&=O(|\alpha|+|\beta|)+\int_{\tau_1}^t\int_{\tau_1}^s(p(\xi))\mathrm{d}\xi\mathrm{d}s.\end{aligned}$$

This implies that, using $\bar{p} < 0$, if δ is small enough, then there must be some $\hat{\tau} \in (\tau_1, \tau_1 + 2\pi)$ such that $x(\hat{\tau}; \tau, v) = 0$. This contradicts (3.17).

Note that (3.18) implies that $v \geq 2\delta$. Moreover, there exists $t_1 > \tau$ such that $x(t_1; \tau, v) \geq \delta$. We claim that

$$x(t; \tau, v) \geq \delta \quad \text{for } t \geq t_1. \tag{3.19}$$

If (3.19) is not true, let $t_2 = \inf\{t : t \geq t_1,\ x(t; \tau, v) < \delta\}$. Then $x'(t_2; \tau, v) \leq 0$. If $x'(t; \tau, v) \leq 0$ for $t \geq t_2$, then $x(t; \tau, v) \leq x(t_2; \tau, v) \leq \delta$ for $t \geq t_2$, and (3.18) implies that $x'(t; \tau, v) < -\delta$ for $t \geq t_2$. Thus

$$x(t; \tau, v) = x(t_2; \tau, v) + \int_{t_2}^{t} x'(s; \tau, v)\mathrm{d}s \leq -\delta(t - t_2) + \delta < 0$$

for $t > t_2 + 1$ which contradicts (3.17). Hence, we can define $t_3 = \inf\{t : t \geq t_2,\ x'(t; \tau, v) > 0\}$. Clearly, $x'(t_3; \tau, v) = 0$ and

$$x(t_3; \tau, v) = x(t_2; \tau, v) + \int_{t_2}^{t_3} x'(s; \tau, v)\mathrm{d}s \leq x(t_2; \tau, v) \leq \delta$$

which contradicts (3.18). Therefore, we have proved the following result.

LEMMA 3.7. *There exists $\delta > 0$ independent of (τ, v) such that if $\mathcal{S}$ is not defined for some $(\tau, v) \in \mathbb{R} \times \mathbb{R}^+$, then $x(t; \tau, v) \geq \delta$ for $t \gg 1$.*

Now we assume that Hill's equation is oscillatory, that is, all nonzero solution of Hill's equation have infinitely many zeros. It is a known fact (see [26]) that these zeros correspond to a sequence tending to $+\infty$.

LEMMA 3.8. *Let us assume that Hill's equation* (1.5) *is oscillatory. Then there exist $\beta_0 > 0$ and $\varepsilon_0 > 0$ such that any solution $x(t; \tau, v)$ of* (1.1) *such that $x(\tau_1; \tau, v) = \alpha,\ x'(\tau_1; \tau, v) = \beta$ with $\beta \geq \beta_0$ and $0 \leq \alpha \leq \varepsilon_0 \beta$ will have a next zero $\hat{\tau} > \tau_1$.*

Proof. Let $y_\beta(t) := \frac{1}{\beta} x(t; \tau, v)$. Then, $y_\beta(t)$ is a solution of the equation

$$x'' + a(t)x = \frac{1}{\beta} p(t)$$

for $t > \tau_1$ and $y_\beta(s) > 0$ for $s \in (\tau_1, t)$ with initial conditions

$$y_\beta(\tau_1) = \frac{\alpha}{\beta}, \qquad y'_\beta(\tau_1) = 1.$$

If $y_0(t)$ is the solution of the Hill's equation $x'' + a(t)x = 0$ with initial conditions $y_0(\tau_1) = 0,\ y'_0(\tau_1) = 1$, by continuous dependence of the solutions with respect to initial value and parameters we have that

$$\lim_{\beta \to +\infty} y_\beta(t) = y_0(t) \quad \text{and} \quad \lim_{\beta \to +\infty} y'_\beta(t) = y'_0(t) \tag{3.20}$$

uniformly on compact interval. Let $\hat{\tau}_0(\tau_1)$ be the next zero of $y_0(t)$ after τ_1 (that is, $y_0(\hat{\tau}_0) = 0$ and $y_0(t) > 0$ for all $\tau_1 < t < \hat{\tau}_0$). Then, $\hat{\tau}_0(\tau_1)$ is a simple zero with $y'_0(\hat{\tau}_0) < 0$ independent of β. Thus (3.20) implies that for β large enough and $\frac{\alpha}{\beta}$ small enough there exists $\hat{\tau}(\tau_1)$ such that $y_\beta(\hat{\tau}) = 0$. The lemma is thus proved. ∎

A direct consequence of the above lemma is that the successor map $\mathcal{S}$ for the impact oscillator (1.1) is well defined for $v \gg 1$. As shown in [26], the condition $\bar{a} > 0$ implies that Hill's equation (1.5) is oscillatory. This condition is also enough to assure that our successor map is well defined. Actually, if $\mathcal{S}$ is not defined for some (τ, v) with $v > 0$, then by using Lemmas 3.7 and 3.8, the solution $x(t; \tau, v)$ will satisfy $x(t; \tau, v) \geq \delta$ and $|\frac{x'(t;\tau,v)}{x(t;\tau,v)}| < \max\{\frac{\beta_0}{\delta}, \frac{1}{\varepsilon_0}\}$ for t large enough. Now, by integrating (1.1) in $[2l\pi, 2k\pi]$ for $l, k \in \mathbb{N}$ we have

$$\int_{2l\pi}^{2k\pi} \left(\frac{x''}{x}\right) \mathrm{d}t + 2\pi(k-l)\bar{a} \leq 0,$$

but this implies that

$$2\pi(k-l)\bar{a} + \int_{2l\pi}^{2k\pi} \frac{(x')^2}{x^2} \mathrm{d}t - 2\max\left\{\frac{\beta_0}{\delta}, \frac{1}{\varepsilon_0}\right\} \leq 0.$$

It is clear that this is not possible if k is large enough. Therefore the successor map $\mathcal{S}$ is well defined for all (τ, v) with $v > 0$, provided that $p(t) \leq 0$ for all $t, \bar{p} < 0$ and $\bar{a} > 0$.

4. Asymptotically linear impact oscillators. Finally, we discuss the case of the asymptotically linear impact oscillator (1.8). Throughout this section, it is understood that the assumptions of Theorem 1.8 hold. Such assumptions imply that there exist $M, P > 0$ such that $|a(t)x + g(t, x) - p(t)| \leq Mx + P$ for $x \geq 0$ and for all t. Moreover, the successor map $\mathcal{S}$ of the problem (1.8) is well defined for all $(\tau, v) \in \mathbb{R} \times \mathbb{R}^+$ and $p(t) \leq 0$ for all t, $\bar{p} < 0$. Then, by using similar arguments as in Lemmas 3.2 and 3.3, it is easy to prove that the conclusions of Lemma 3.4 are still valid for the successor map of problem (1.8). Roughly speaking, Lemma 3.4 means that the rotation of some iteration of the successor map is slow for small velocities. On the other hand, it is necessary to control the behavior of the successor map for large velocities (that is, an analogous to Lemma 3.6). To this purpose, some lemmas are needed. For the moment, let us assume that $\|a\|_\infty > 0$.

LEMMA 4.1. *The solution* $x(t; \tau, v)$ *of problem* (1.8) *with initial conditions* $x(\tau; \tau, v) = 0,\ x'(\tau; \tau, v) = v > 0$ *satisfies*

$$\begin{aligned}\left(r(\tau_1) + \frac{P}{M+1}\right)\exp(-(M+1)(\tau_2 - \tau_1)) &\leq r(\tau_2) + \frac{P}{M+1} \\ &\leq \left(r(\tau_1) + \frac{P}{M+1}\right)\exp((M+1)(\tau_2 - \tau_1))\end{aligned} \tag{4.1}$$

for $\tau_2 - \tau_1 \geq 0$*, where* $r(t) = ((x(t; \tau, v))^2 + (x'(t; \tau, v))^2)^{1/2}$.

Proof. This inequality is proved by using the Gronwall lemma as in (3.14). ◻

Let $n, m \in \mathbb{N}$ be such that $n > 2m(\sqrt{\|a\|_\infty})$. Then, there exists $\sigma > 0$, such that $n > 2m(\sqrt{\|a\|_\infty} + 2\sigma)$. Let us fix the positive numbers $T = \frac{n\pi}{\sqrt{\|a\|_\infty}}$ and

$$\delta = \frac{1}{2}\left|\frac{\pi}{\sqrt{\|a\|_\infty + 2\sigma}} - \frac{\pi}{\sqrt{\|a\|_\infty + \sigma}}\right|.$$

Note that σ (and in consequence δ) can be chosen arbitrarily small. By using the assumption (1.7), it is possible to take $d > 0$ (depending on σ) such that

$$\max_{0 \leq t \leq 2\pi} |a(t)x + g(t, x) - p(t)| \leq (\|a\|_\infty + \sigma)x \quad \text{for } x \geq d.$$

The following estimation is obtained by using the previous lemma.

LEMMA 4.2. *Let $x(t;\tau,v)$ be the solution of* (1.8) *with initial conditions*

$$x(\tau;\tau,v)=0, \quad x'(\tau;\tau,v)=v>0.$$

Then for d, δ, and $T>0$ as given before, there exists $v_\delta>0$ such that if $v\ge v_\delta$, then there exists $\tau_d^+>\tau$ such that $x(\tau_d^+;\tau,v)=d$ and $x(t;\tau,v)<d$ for $t\in(\tau,\tau_d^+)$, and moreover $|\tau_d^+-\tau|<\delta$. Besides, if there exists $\tau_d^->\tau_d^+$ such that $x(\tau_d^-;\tau,v)=d$, $x'(\tau_d^-;\tau,v)<0$, and $|\tau_d^--\tau|<T$, then there exists $\hat\tau>\tau_d^+$ such that $x(\hat\tau;\tau,v)=0$ and $|\hat\tau-\tau_d^-|<\delta$. Moreover, if δ is small enough, then

$$\frac{v_d^+}{2}\le v\le 2v_d^+. \tag{4.2}$$

Proof. Firstly, the global existence of $x(t;\tau,v)$ right to τ is assured from the assumptions. Suppose there is no time $t\in(\tau,\tau+1)$ such that $x(t;\tau,v)=d$, that is, $0<x(t;\tau,v)<d$ for $t\in(\tau,\tau+1)$. Then

$$x'(t;\tau,v)>\left(v+\frac{P}{M+1}\right)\exp(-(M+1))-d-\frac{P}{M+1},$$

so an integration gives

$$x(\tau+1;\tau,v)>\left(v+\frac{P}{M+1}\right)\exp(-(M+1))-d-\frac{P}{M+1}>d$$

if v is large enough. Thus we have proved the existence of τ_d^+. Moreover,

$$v_d^+=x'(\tau_d^+;\tau,v)>\left(v+\frac{P}{M+1}\right)\exp(-(M+1)(\tau_d^+-\tau))-d-\frac{P}{M+1}.$$

Hence

$$\begin{aligned} d&=\int_\tau^{\tau_d^+}x'(s;\tau,v)\mathrm{d}s\\ &\ge\left[\left(v+\frac{P}{M+1}\right)\exp(-(M+1)(\tau_d^+-\tau))-d-\frac{P}{M+1}\right](\tau_d^+-\tau)\end{aligned}$$

and in consequence for a given δ we get $|\tau_d^+-\tau|<\delta$ by taking v large enough. The discussion for $\hat\tau$ is similar. Finally, if δ is small enough, then

$$\left(v_d^++\frac{P}{M+1}\right)\exp(-(M+1)\delta)-\frac{P}{M+1}\ge\frac{v_d^+}{2}$$

and

$$\left(v_d^++d+\frac{P}{M+1}\right)\exp(-(M+1)\delta)-\frac{P}{M+1}\le 2v_d^+.$$

Now, the estimation (4.2) follows easily from (4.1). $\square$

Define now

$$h(t,x)=\begin{cases}\dfrac{a(t)x+g(t,x)-p(t)}{x}, & x\ge d;\\[2mm] \dfrac{a(t)d+g(t,d)-p(t)}{d}, & x<d.\end{cases}$$

Then $h(t,x)$ is continuous and 2π-periodic with respect to t and verifies $|h(t,x)| \le \|a\|_\infty + \sigma$ for $x \ge 0$ and for all t. Let $x(t;\tau,v)$ be the solution of the equation $x''+h(t,x)x=0$ satisfying initial conditions $x(\tau;\tau,v)=0$, $x'(\tau;\tau,v)=v>0$. On the other hand, let $y_0(t;\tau,v)$ be the solution of the equation $x''+(\|a\|_\infty+\sigma)x=0$ satisfying the same initial conditions as $x(t;\tau,v)$. By using Sturm comparison theorem,

$$\hat{\tau}(h)-\tau \ge \frac{\pi}{\sqrt{\|a\|_\infty+\sigma}},$$

where $\hat{\tau}(h)$ is the next zero of $x(t;\tau,v)$ right to τ. Moreover, we have the following lemma.

LEMMA 4.3. *Let $x(t;\tau,v)$ be the solution of the equation $x''+h(t,x)x=0$ satisfying the initial conditions*

$$x(\tau;\tau,v)=0, \quad x'(\tau;\tau,v)=v>0.$$

Then, there is $v_\delta>0$ such that if $v \ge v_\delta$, then there exist τ_d^+, τ_d^- such that $x(\tau_d^+;\tau,v)=d$, $x'(\tau_d^+;\tau,v)=v_d^+>0$, $x(t;\tau,v)<d$ for $t\in(\tau,\tau_d^+)$ and $x(\tau_d^-;\tau,v)=d$, $x'(\tau_d^-;\tau,v)=v_d^-<0$, $x(t;\tau,v)>0$ for $t\in(\tau,\tau_d^-)$, respectively. Moreover,

$$\tau_d^- - \tau_d^+ > \frac{\pi}{\sqrt{\|a\|_\infty+2\sigma}}.$$

Proof. Recall that $|h(t,x)x| \le (\|a\|_\infty+\sigma)x$ for $x \ge 0$ and for all t, so the conclusion of Lemmas 4.1 and 4.2 are valid for $x(t;\tau,v)$ if $v>0$ is sufficiently large, thus we have $\tau_d^+ - \tau < \delta$. Note that $h(t,x)x = a(t)x + g(t,x) - p(t)$ for $x \ge d$ and for all t. This implies, under the assumption of Theorem 1.8, that there exists $\tau_d^- > \tau_d^+$ such that $x(\tau_d^-;\tau,v)=d$, $x'(\tau_d^-;\tau,v)=v_d^-<0$, and $x(t;\tau,v)>0$ for $t\in(\tau,\tau_d^-)$. By contradiction, if

$$\tau_d^- - \tau_d^+ \le \frac{\pi}{\sqrt{\|a\|_\infty+2\sigma}},$$

then $\tau_d^- - \tau < \tau_d^- - \tau_d^+ + \delta < T$, and Lemma 4.2 implies that the zero $\hat{\tau}(h)$ right to τ exists and $\hat{\tau}(h) - \tau_d^- < \delta$. Hence,

$$\tau_d^- - \tau_d^+ > \hat{\tau}(h) - \tau - 2\delta \ge \frac{\pi}{\sqrt{\|a\|_\infty+\sigma}} - 2\delta = \frac{\pi}{\sqrt{\|a\|_\infty+2\sigma}}.$$

This contradiction completes the proof of Lemma 4.3. ∎

Finally, let us consider $x(t;\tau,v)$ the solution of (1.8) with initial conditions

$$x(\tau;\tau,v)=0, \quad x'(\tau;\tau,v)=v>0.$$

Let $\hat{\tau}$ be the first zero right to τ. If v is large enough, then there exist τ_d^-, $\tau_d^+ \in (\tau,\hat{\tau})$ such that $x(\tau_d^\pm;\tau,v)=d$, $v_d^+ = x'(\tau_d^+;\tau,v)>0$, $v_d^- = x'(\tau_d^-;\tau,v)<0$, and $x(t;\tau,v)<d$ for $t\in(\tau,\tau_d^+)\cup(\tau_d^-,\hat{\tau})$. Moreover, $|\tau_d^+ - \tau|<\delta$ and v_d^+ is arbitrarily large by using the estimation (4.2). On the other hand, let $x_h(t)$ be the solution of the equation $x''+h(t,x)x=0$ satisfying $x_h(\tau_d^+)=d$, $x_h'(\tau_d^+)=v_d^+>0$. If τ_h is such that $x_h(\tau_h)=0$, $x_h(t)>0$ for $t\in(t_h,\tau_d^+)$, then the estimation (4.2) implies that the initial velocity $v_h = x_h'(\tau_h)$ is arbitrarily large. Taking into account that $h(t,x)x = a(t)x+g(t,x)-p(t)$ for $x \ge d$ and for all t, Lemma 4.2 implies that the time in which the solution $x(t;\tau,v)$ of the equation $x''+a(t)x+g(t,x)=p(t)$ moves from (d_σ, v_d^+) to (d_σ, v_d^-) is larger than $\frac{\pi}{\sqrt{\|a\|_\infty + 2\sigma}}$. In consequence, if v large enough (more explicitly, $v \geq v_\delta$), then we have

$$\hat{\tau} - \tau \geq \frac{\pi}{\sqrt{\|a\|_\infty + 2\sigma}}. \tag{4.3}$$

Looking for the estimation of $\Pi_1(\mathcal{S}^n(\tau, v)) - \tau$, note that by using the argument leading to (3.15), it results that for a given $v_\delta > 0$ there is $v_{n,m}^+(\delta) > 0$ such that

$$\text{if } v > v_{n,m}^+(\delta) \quad \text{and} \quad \Pi_1(\mathcal{S}^n(\tau, v)) - \tau \leq 2m\pi, \text{ then } \Pi_2(\mathcal{S}^i(\tau, v)) > v_\delta \tag{4.4}$$

for $i = 0, 1, \ldots, n-1$. Hence, following the arguments of section 3, we can prove that the conclusions of Lemma 3.6 are true for the successor map of the problem (1.8) under the assumptions of Theorem 1.8. Note that if $a(t) \equiv 0$, then T is not well defined, but it is easy to prove, by using similar arguments as before, that $\Pi_1(\mathcal{S}(\tau, v)) - \tau \geq 2m\pi$ for v sufficiently large. Now, Theorem 1.8 can be proved by mimicking the arguments of sections 2 and 3 with minor modifications.

The property that the successor map $\mathcal{S}$ is well defined is not easy to check. For example, consider the equation $x'' - x = -1$. It has a singular point $(1, 0)$ in $x - x'$ phase plane and the solution $x(t; \tau, 1)$ starting from $x(\tau; \tau, 1) = 0,\ x'(\tau; \tau, 1) = 1$ will tend to $(1, 0)$ in $x - x'$ phase plane as $t \to +\infty$. Thus we can construct an equation by modifying the above equation such that the new equation is asymptotically linear and the successor map $\mathcal{S}$ of this equation is well defined for v sufficiently small and sufficiently large but $\mathcal{S}$ is not well defined for $v = 1$. In spite of that, in the following we show that $a(t)x + g(t, x) \geq 0$ is a sufficient condition to have $\mathcal{S}$ well defined. Actually, let us note that

$$\begin{aligned} x'(t; \tau, v) &= v - \int_\tau^t (a(s)x + g(s, x))\mathrm{d}s + \int_\tau^t p(s)\mathrm{d}s \\ &\leq v + \int_\tau^t p(s)\mathrm{d}s \to -\infty \quad \text{as } t \to +\infty. \end{aligned}$$

Thus, for any fixed $v > 0$, there exists a $\hat{\tau} > \tau$ such that $x(\hat{\tau}; \tau, v) = 0$ which implies that $\mathcal{S}$ is well defined for (τ, v). Hence Corollary 1.9 is proved.

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