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Bouncing solutions of an equation with attractive singularity

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For any $n, m \in \mathbb{N}$, we prove the existence of $2m\pi$-periodic solutions, with n bouncings in each period, for a second-order forced equation with attractive singularity by using the approach of successor map and Poincaré–Birkhoff twist theorem.

1. Introduction

In the paper [15], Lazer and Solimini initiated the study of the singular equation

$$x'' + \frac{1}{x^\alpha} = p(t), \tag{1.1}$$

with $\alpha > 0$ and p a continuous 2π-periodic function. By using upper and lower solutions, they proved that

$$\bar{p} := \frac{1}{2\pi}\int_0^{2\pi} p(t)\,\mathrm{d}t > 0$$

is a necessary and sufficient condition for the existence of a periodic solution. This result was generalized in [10] to a wider class of equations, including

$$x'' + g(x) = p(t), \tag{1.2}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$\lim_{x\to 0^+} g(x) = +\infty, \qquad \lim_{x\to +\infty} g(x) = 0. \tag{1.3}$$

Throughout this paper, we use the notation $\mathbb{R}^+ =]0, +\infty[$ and $\mathbb{R}_+ = [0, +\infty[$. In this situation, it is said that the nonlinearity has an attractive singularity in the origin. Nowadays, the dynamics of ordinary differential equations with attractive singularities is generally well understood, especially when the nonlinearity is monotone (see [4,16]). However, bouncing solutions in this context have not been studied in detail. It is known that there are solutions that collide with the singularity, and hence it is natural to look for a good definition of bouncing solutions (that is, solutions with impacts) and study the underlying dynamics. In [10], a definition of a 'generalized solution' was proposed, which allowed some kind of collision with the singularity. There is also a well-developed theory about singular boundary-value problems of Dirichlet type, known as generalized Emdem–Fowler equations in the related literature (see, for instance, [2, 11] and the references therein). Such solutions touch the singularity at the extremes of the interval of definition and our idea is to continue such solutions by assuming an elastic impact. To summarize, our aim in this paper is to consider equation (1.2) as a bouncing oscillator with elastic impacts and obtain the existence and multiplicity of periodic solutions and subharmonics of any order by applying a generalized Poincaré–Birkhoff Theorem to the so-called successor map, a concept which is used by Ortega in [17, 18] to study the boundedness of the solutions for a second-order equation without singularity.

The dynamics of impact oscillators has recently attracted the attention of many researchers (see [3,9,13,14,19,21] and the references therein), but, to our knowledge, the study of impact dynamics in singular equations has not been considered until now.

Let us formally state the concept of a bouncing solution.

DEFINITION 1.1. A continuous function $x : \mathbb{R} \to \mathbb{R}_+$ is called a *bouncing solution* of equation (1.2) if there exists a doubly infinity sequence $\{t_i\}_{i\in\mathbb{Z}}$ such that, for all $i \in \mathbb{Z}$, the following conditions hold:

(1) $x(t_i) = 0$;

(2) $x'(t_i^-) = -x'(t_i^+)$;

(3) $x \in C^2(]t_i, t_{i+1}[, \mathbb{R}^+)$ is a classical solution of (1.2).

Any of the t_i such that $x(t_i) = 0$ are called a *bouncing* of the solution x.

From the classical results in [10, 15], it is known that, when $\bar{p} < 0$, classical periodic solutions cannot exist; in fact, we will prove that every solution is of bouncing type.

As we will see, the situation is very different depending on the behaviour of the potential of the nonlinearity in the origin. In the context of singular systems, it is said that the singularity is *weak* if

$$\int_0^1 g(s)\,\mathrm{d}s < +\infty, \tag{1.4}$$

also known as a *weak force condition*. If, on the contrary,

$$\int_0^1 g(s)\,\mathrm{d}s = +\infty, \tag{1.5}$$

we speak about a *strong singularity* or a *strong force condition*. In general, the dynamics of any singular system experiments important changes depending if the singularity is weak or strong, as can be observed in the related literature. In our case, if the singularity is weak, the impact velocity will be finite. This will be the more difficult case and we devote almost all our attention to it. If, on the contrary, the singularity is strong, then the impact velocity becomes infinity and the problem is simpler; we comment this case briefly in the last section.

The main result of this paper is the following.

THEOREM 1.2. *Let us assume that p is a continuous 2π-periodic function with $\bar{p} < 0$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz-continuous function verifying (1.3) and (1.4). It is also assumed that there is $\epsilon > 0$ such that g is strictly decreasing in $]0, \epsilon[$. Then, for any $m \in \mathbb{N}$, there exist at least two $2m\pi$-periodic solutions with one bouncing in each period. Moreover, for any $n, m \in \mathbb{N}$, $n \geqslant 2$, there exists at least one $2m\pi$-periodic solution with n bouncings in each period.*

The paper is organized as follows. In § 2 we prove that every solution is of bouncing type and define the successor map. Section 3 is devoted to the study of some limiting properties of successor map. The main result is proved in § 4 via a generalized version of the Poincaré–Birkhoff Theorem. Finally, § 5 collects some remarks and comments.

2. Bouncing solutions and the successor map

Our first aim will be to prove that every classical solution of (1.2) can be extended to a bouncing solution defined on the whole real line if $\bar{p} < 0$. Moreover, if the singularity is weak, we will prove that this bouncing solution is unique. This will be done through several lemmas.

LEMMA 2.1. *Let us assume that p is a continuous 2π-periodic function with $\bar{p} < 0$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz-continuous function verifying (1.3). Then every classical solution x of (1.2) has a finite maximal interval of definition $]w_-, w_+[$ such that*

$$x(w_-) = x(w_+) = 0.$$

Proof. By contradiction, let us suppose that $w_+ = +\infty$. From the equation, we have that

$$x''(t) < p(t) \tag{2.1}$$

for all t. We write $\tilde{p} = p - \bar{p}$. Fixing a t_0 and integrating on $[t_0, t]$, we have

$$x'(t) < \bar{p}(t - t_0) + \int_{t_0}^{t} \tilde{p}(s)\,\mathrm{d}s + x'(t_0).$$

Then $\lim_{t\to+\infty} x'(t) = -\infty$ because $\bar{p} < 0$ and $\int_{t_0}^{t} \tilde{p}(s)\,\mathrm{d}s$ is bounded. But this is impossible because x is positive. A symmetric argument is valid for w_-, so the maximal interval of definition is finite.

By continuation of solutions, there exists a sequence $\{t_n\} \to w_+$ such that one of the following cases holds:

(a) $x(t_n) \to 0$;

(b) $x'(t_n) \to +\infty$;

(c) $x'(t_n) \to -\infty$;

(d) $x(t_n) \to +\infty$.

If (b) or (d) holds, then it would be possible to take a new sequence $\{\tau_n\}$ such that $\{x''(\tau_n)\} \to +\infty$, in contradiction with (2.1). On the other hands, if (c) holds, then there is a sequence $\{\tau_n\} \to w_+$ such that $\{x''(\tau_n)\} \to -\infty$, and from the equation we deduce that $x(\tau_n) \to 0$. In conclusion, (a) is verified. By (1.3), it is possible to take $\xi > 0$ such that $g(x) > p(t)$ for all t and $0 < x < \xi$. Then a classical solution of (1.2) has no minima lower that ξ, so, in consequence, if (a) holds, then $\lim_{t\to w_+} x(t) = 0$. Analogous reasonings prove that $\lim_{t\to w_-} x(t) = 0$, so the proof is concluded. □

The next lemma shows the most important difference between a weak and a strong singularity.

LEMMA 2.2. *In the conditions of lemma 2.1, every classical solution of (1.2) verifies that*

$$-\infty < x'(w_+) < 0, \qquad 0 < x'(w_-) < +\infty$$

if (1.4) holds, and

$$x'(w_+) = -\infty, \qquad x'(w_-) = +\infty$$

if (1.5) holds.

Proof. First, let us suppose that (1.4) holds. Let P be such that $|p(t)| \leqslant P$ for all t. As a consequence of lemma 2.1, it is clear that there exists $t_0 < w_+$ such that $x'(t_0) = 0$ and $x'(t) < 0$ for all $t_0 < t < w_+$. Then

$$x'' = p(t) - g(x) \geqslant -P - g(x)$$

for all t. Multiplying this inequality by x' and integrating in $[t_0, w_+)$, we get

$$\tfrac{1}{2}x'(w_+)^2 \leqslant Px(t_0) + \int_0^{x(t_0)} g(s)\,\mathrm{d}s < +\infty. \tag{2.2}$$

The proof for $-\infty < x'(w_-) < 0$ is identical. On the other hand, if (1.5) holds, an analogous argument starting from the inequality $x'' \leqslant P - g(x)$ works in order to prove the second assertion. □

From now on, we will concentrate our attention in the weak singularity case, and hence condition (1.4) is assumed in the following. The strong case will be briefly commented in the last section.

LEMMA 2.3. *In the conditions of theorem 1.2, the singular initial-value problem (IVP)*

$$\left.\begin{aligned} x'' + g(x) = p(t), \\ x(t_0) = 0, \qquad x'(t_0^+) = v_0 > 0 \end{aligned}\right\} \tag{2.3}$$

has a unique solution.

Proof. Existence of a solution is proved through a shooting argument. From [11, theorem 2] (see also remark 2), it is known that equation (1.2) has a solution with Dirichlet conditions

$$x(t_0) = 0, \qquad x(t_0 + \lambda) = \lambda^2.$$

If such a solution is denoted by x_λ, the proof of existence is performed in the first four steps.

STEP 1. Let $t_\lambda \in (t_0, t_0 + \lambda]$ be such that x_λ attains its maximum,

$$x_\lambda(t_\lambda) = \max\{x(t) : t \in [t_0, t_0 + \lambda]\}.$$

Then

$$\lim_{\lambda \to 0^+} x_\lambda(t_\lambda) = 0.$$

Proof of step 1. To prove this, we argue by contradiction. Let us assume that there exist $K > 0$ and $\{\lambda_n\}$ a sequence converging to zero such that $x_{\lambda_n}(t_{\lambda_n}) > K$ for any n. Observe that, as $x(t_0+\lambda) = \lambda^2$, this means that $t_{\lambda_n} < t_0 + \lambda$ and $x'(t_{\lambda_n}) = 0$. On the other hand, as $x(t_0) = 0$, there must be some $\tilde{t}_{\lambda_n} < t_{\lambda_n}$ (if there are many, we take the biggest one) such that $x(\tilde{t}_{\lambda_n}) = \frac{1}{2}K$. By a repeated application of the mean-value theorem on the interval $[\tilde{t}_{\lambda_n}, t_{\lambda_n}]$, we get a sequence t_n such that

$$\lim_{n \to +\infty} x''(t_n) = -\infty. \tag{2.4}$$

Besides, $x(t_n) \geqslant \frac{1}{2}K$ because $t_n \in [\tilde{t}_{\lambda_n}, t_{\lambda_n}]$. However, from the equation, the following uniform bound is obtained,

$$x''(t_n) = p(t_n) - g(x(t_n)) \geqslant -P - \max\{g(s) : s \geqslant \tfrac{1}{2}K\},$$

in contradiction with (2.4). □

STEP 2. $\lim_{\lambda \to 0^+} x'_\lambda(t_0) = 0$.

Proof of step 2. By the mean-value theorem, there exists $t_1 \in (t_0, t_0 + \lambda)$ such that $x'_\lambda(t_1) = \lambda$. If t_1 is the first one to the right of t_0 verifying this property, then $x'_\lambda(t) > 0$ for all $t \in (t_0, t_1)$, and then it is possible to multiply the equation by x'_λ and integrate over (t_0, t_1), resulting in

$$\tfrac{1}{2}\lambda^2 - \tfrac{1}{2}x'_\lambda(t_0)^2 > -P x_\lambda(t_1) - \int_0^{x_\lambda(t_1)} g(s)\,\mathrm{d}s.$$

Now, taking into account step 1 and (1.4), the right-hand side of this inequality tends to zero, and the result is done. □

STEP 3. $\lim_{\lambda \to +\infty} x'_\lambda(t_0) = +\infty$.

Proof of step 3. Again, by the mean-value theorem, there exists $t_1 \in (t_0, t_0 + \lambda)$ such that $x'_\lambda(t_1) = \lambda$. Now we use that $x''(t) < p(t)$ for all t. An integration over $[t_0, t_1]$ leads to

$$x'_\lambda(t_0) > x'_\lambda(t_1) - \int_{t_0}^{t_1} p(t)\,\mathrm{d}t = \lambda - \int_{t_0}^{t_1} p(t)\,\mathrm{d}t \to +\infty$$

as $\lambda \to +\infty$. □

STEP 4. Existence.

Proof of step 4. By the continuous dependence of the solution and its derivative with respect to the regular initial conditions $x_\lambda(t_0 + \lambda)$, $x'_\lambda(t_0 + \lambda)$, it turns out that $x'_\lambda(t)$ is a continuous function of λ, and hence $x'_\lambda(t_0)$ is also continuous in λ. For a given v_0, it is proved that there are $\lambda_1 < \lambda_2$ such that $x'_{\lambda_1}(t_0) < v_0 < x'_{\lambda_2}(t_0)$, so, by continuity, there is $\lambda \in]\lambda_1, \lambda_2[$ such that $x'_\lambda(t_0) = v_0$. Therefore, problem (2.3) has at least a solution. □

STEP 5. Uniqueness.

Proof of step 5. Let us assume that there are two different solutions, x, y, of problem (2.3). We define $\delta(t) = x(t) - y(t)$. Then $\delta(t_0) = \delta'(t_0) = 0$. If $\delta(t) = \delta'(t) = 0$ for some $t > t_0$, we automatically have that both solutions are the same by uniqueness of the regular IVP. In particular, δ cannot be identically zero in a given sub-interval of the common interval of the definition of x, y. Let us prove that there is $t_1 > t_0$ such that $\delta(t) \neq 0$ for all $t \in]t_0, t_1[$. Otherwise, there exists a decreasing sequence $\{t_n\}$ going to t_0 such that $\delta(t_n) = 0$ for each n. Now we use the decreasing character of g near 0. By continuity of the solutions, there exists N such that $x(t), y(t) < \epsilon$ for all $t \in]t_0, t_N[$. To fix ideas, let us assume that $\delta(t) > 0$ for all $t \in]t_{N+1}, t_N[$, with the remaining case being identical. Then, by the monotonicity of g,

$$\delta''(t) = g(y(t)) - g(x(t)) > 0, \quad t \in]t_{N+1}, t_N[.$$

In conclusion, δ is a positive convex function in $]t_{N+1}, t_N[$, vanishing in the extrema t_{N+1}, t_N, which is a contradiction.

We can assume, without loss of generality, that $\delta(t) > 0$ for $t \in]t_0, t_1[$ (otherwise, x, y can be interchanged). Besides, taking t_1 close enough to t_0, we get $x(t), y(t) < \epsilon$ for all $t \in]t_0, t_1[$, and, by using again the monotonicity of g, the function δ is convex in $]t_0, t_1[$. Taking into account that $\delta'(t_0) = 0$, we conclude that $\delta'(t) > 0$ for all $t \in]t_0, t_1[$.

Also, t_1 can be chosen such that $x'(t), y'(t) > \frac{1}{2}v_0$ for all $t \in]t_0, t_1[$, by continuity of the derivatives.

Let G be a primitive of g with $G(0) = 0$ (here it is used the weak condition (1.4)). As g is positive, G is increasing in its domain. From the equation, it is easy to see that x, y are solutions of

$$x'(t)^2 = v_0^2 - 2G(x(t)) + 2\int_{t_0}^t p(s)x'(s)\,\mathrm{d}s, \quad t \in]t_0, t_1[.$$

A simple subtraction gives

$$x'(t)^2 - y'(t)^2 = 2\int_{t_0}^t p(s)\delta'(s)\,\mathrm{d}s + 2[G(y(t)) - G(x(t))], \quad t \in]t_0, t_1[.$$

Since $\delta(t) \geqslant 0$, the last term is non-positive by the increasing character of G, so

$$\delta'(t)(x_1'(t) + x_2'(t)) \leqslant 2\int_{t_0}^t p(s)\delta'(s)\,\mathrm{d}s, \quad t \in]t_0, t_1[,$$

and hence

$$0 \leqslant \delta'(t) \leqslant \frac{2}{v_0}\int_{t_0}^t p(s)\delta'(s)\,\mathrm{d}s, \quad t \in]t_0, t_1[.$$

Finally, Gronwall's inequality implies that $\delta'(t) = 0$ for all $t \in]t_0, t_1[$, but $\delta(t_0) = 0$ so in conclusion $x(t) = y(t)$ for all $t \in]t_0, t_1]$ and the proof is finished. □

REMARK 2.4. Throughout this paper, the decreasing character of g near zero is used only to prove uniqueness in the previous IVP.

As a consequence of the previous results, every classical solution of equation (1.2) can be continued to a bouncing solution in a unique way.

For a given $\tau \in \mathbb{R}$ and $v \in \mathbb{R}^+$, let us denote by $x(t; \tau, v)$ the unique solution of (2.3) with $t_0 = \tau$ and $v_0 = v$. By lemma 2.1, this solution has a finite interval of definition and vanishes at some time $\hat{\tau} > \tau$. This is the time of the following impact. As the bouncing is elastic, the velocity after this impact is

$$\hat{v} = -x'(\hat{\tau}; \tau, v),$$

and, by lemma 2.2, this $\hat{v}$ is finite. Then the map

$$\mathcal{S} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+, \qquad \mathcal{S}(\tau, v) = (\hat{\tau}, \hat{v})$$

is well defined and one to one. Moreover, such a function is continuous, as is proved in the following result.

PROPOSITION 2.5. *In the conditions of theorem 1.2, the function* $\mathcal{S} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ *is continuous.*

Proof. We prove that $\mathcal{S}$ is a composition of two continuous functions. First, let us define the function $F_1 : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ in the following way. For a given $(\tau, v) \in \mathbb{R} \times \mathbb{R}^+$, by lemma 2.3, there exists a unique solution $x(t)$ such that $x(\tau) = 0$ and $x'(\tau) = v$. Then we define $F_1(\tau, v) = (t_M, x_M)$, where $x(t_M) = x_M$ is the maximum of x in its interval of definition as a classical solution of the equation (such a maximum exists as a consequence of lemma 2.1). Function F_1 is one to one: if two points have the same image, this would mean that there are two solutions with the same initial conditions $x(t_M) = x_M > 0$, $x'(t_M) = 0$, but this is impossible by uniqueness. Therefore, $F_1 : \mathbb{R} \times \mathbb{R}^+ \to \operatorname{Im} F_1$ is a bijective function. The inverse function $F_1^{-1} : \operatorname{Im} F_1 \to \mathbb{R} \times \mathbb{R}^+$ is continuous by the classical result of continuity with respect to initial conditions, since now the initial conditions $x(t_M) = x_M > 0, x'(t_M) = 0$ are regular. In consequence, F_1 is also continuous because it is the inverse of a continuous and bijective function with open domain (this is a direct consequence of the invariance of domain theorem (see, for instance, [7, proposition IV-7.4])).

On the other hand, let us define $F_2 : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ in the following way. For a given (t_M, x_M), there exists a unique solution $x(t)$ with the (regular) initial conditions $x(t_M) = x_M, x'(t_M) = 0$. By lemmas 2.1 and 2.2, such a solution has finite maximal interval of definition $]w_-, w_+[$ with $x(w_+) = 0$ and $-\infty < x'(w_+) < 0$. Then we define $F_2(t_M, x_M) = (w_+, -x'(w_+))$. Again, this function is continuous by the classical result of continuity with respect to (regular) initial conditions. To finish the proof, we only have to point out that $\mathcal{S} = F_2 \circ F_1$. $\square$

REMARK 2.6. This proposition is equivalent to the continuity of the solution of the singular IVP (2.3) with respect to initial conditions.

Following [1,17–19], function $\mathcal{S}$ is called the successor map for (1.2), although, in this context, 'impact map' would also be adequate. The cited references of Ortega in [17–19] contain a careful study of this function as well as interesting considerations over its relation with the dynamics of the equation (also see [20] for the consideration of a periodic solution). In our case, this study is quite different, mainly because of the singular character of the equation. We perform this study in the following section.

3. Twist property for successor map

Let us recall that $P = \|p(t)\|_\infty$. Due to condition (1.3), there exists $\xi > 0$ such that $g(\xi) = P$ and $g(x) > P$ for all $0 < x < \xi$. From now on, such numbers P, ξ are fixed and will be used in the proofs below. An initial lemma is the following.

LEMMA 3.1. *Under the assumptions of theorem 1.2, let us consider $x(t)$ as the classical solution of (1.2) with initial conditions $x(t_0) = x_0 > 0$, $x'(t_0) = 0$. If $]w_-(x_0), w_+(x_0)[$ is its maximal interval of definition, then*

$$\lim_{x_0 \to 0} [w_+ - w_-] = 0.$$

Proof. Looking at the equation, we realize that all solutions below ξ are concave. Hence, if $x_0 < \xi$, then x_0 is the absolute maximum of $x(t)$ on the whole interval of definition $]w_-, w_+[$. As a consequence, if we write

$$M(x_0) = \min\{g(x(t)) : t \in]w_-, w_+[\},$$

then $M(x_0) \to +\infty$ as $x_0 \to 0$. Now, from the equation, $x''(t) < P - M(x_0) < 0$ and, by integrating,

$$x'(w_+) - x'(w_-) < (P - M(x_0))(w_+ - w_-) < 0. \tag{3.1}$$

By an identical argument as in the proof of lemma 2.2 (see (2.2)), it is proved that

$$|x'(w_+)| < \sqrt{2\left(Px_0 + \int_0^{x_0} g(s)\,\mathrm{d}s\right)} \to 0 \quad \text{as } x_0 \to 0.$$

Analogously, it can be proved that $\lim_{x_0 \to 0} |x'(w_-)| = 0$, and the conclusion follows easily from (3.1). □

The following proposition is the main result of this section.

PROPOSITION 3.2. *Under the assumptions of theorem 1.2, the following limits hold uniformly for $\tau \in [0, T]$,*

$$\lim_{v \to 0^+} [\hat{\tau}(\tau, v) - \tau] = 0, \tag{3.2}$$

$$\lim_{v \to +\infty} [\hat{\tau}(\tau, v) - \tau] = +\infty, \tag{3.3}$$

where $\hat{\tau}(\tau, v)$ is the first component of $\mathcal{S}(\tau, v)$.

Proof. A first consideration is that, if the limit is proved, uniformity is trivial since the interval is compact and the successor function $\mathcal{S}$ is continuous.

By definition, $x(t;\tau,v)$ as a classical solution has the maximal interval of definition $]\tau,\hat{\tau}[$. Let us take $t_0 \in\,]\tau,\hat{\tau}[$ such that $x(t_0;\tau,v)$ is the maximum of this function in the interval $]\tau,\hat{\tau}[$.

In view of lemma 3.1, in order to prove the first limit, it is sufficient to prove that

$$\lim_{v\to 0^+} x(t_0;\tau,v) = 0. \tag{3.4}$$

By contradiction, let us assume that there exist $K > 0$ and a sequence $\{v_n\}$ of positive numbers converging to zero such that $x(t_0;\tau,v_n) > K$. Then, for any n, there is $\tilde{t}_0 < t_0$ verifying $x(\tilde{t}_0;\tau,v_n) = \min\{K,\xi\}$. Let us recall that $\xi > 0$ is such that $g(\xi) = P$ and $g(x) > P$ for all $0 < x < \xi$. Moreover, if $\tilde{t}_0$ is the smallest one, then $x(t;\tau,v_n) < \xi$ for all $t \in\,]\tau,\tilde{t}_0[$ and, in consequence, $x''(t) < 0$ in this interval. A double integration of the preceding inequality leads to

$$x(\tilde{t}_0;\tau,v_n) = \min\{K,\xi\} < v_n(\tilde{t}_0 - \tau).$$

Then, passing to the limit shows that $v_n \to 0$ implies $\tilde{t}_0 \to +\infty$, and, of course, $t_0 \to +\infty$. Now, the desired contradiction comes out if we observe that, by integrating the equation over $[\tau,t_0]$, we get

$$0 < \int_\tau^{t_0} p(s)\,\mathrm{d}s + v_n,$$

but the right-hand side tends to $-\infty$ as $v_n \to 0$ because $\bar{p} < 0$ and $t_0 \to +\infty$. Therefore, equation (3.4) is true and hence (3.2) is proved.

To prove equation (3.3), let us define t_0 as above. By arguing one more time as in lemma 2.2, we have that

$$|v| < \sqrt{2\biggl(Px(t_0;\tau,v) + \int_0^{x(t_0;\tau,v)} g(s)\,\mathrm{d}s\biggr)},$$

so, in consequence,

$$\lim_{v\to+\infty} x(t_0;\tau,v) = +\infty.$$

Then, for v big enough, there exists $\tilde{t}_0 < t_0$ (we choose the biggest one) such that $x(\tilde{t}_0;\tau,v) = \xi$. Then, if $C = \max\{g(s) : s \geqslant \xi\}$, from the equation, $x''(t;\tau,v) > -P - C$ for all $t \in\,]\tilde{t}_0,t_0[$, so

$$x'(t;\tau,v) < (P+C)(t_0 - t)$$

for all $t \in\,]\tilde{t}_0,t_0[$. From here, a new integration on $]\tilde{t}_0,t_0[$ gives

$$x(t_0;\tau,v) < \tfrac{1}{2}(P+C)(t_0-\tilde{t}_0)^2 + \xi.$$

Now, going to the limit shows that $v \to +\infty$ implies $t_0 \to +\infty$, but, of course, $t_0 < \hat{\tau}(\tau,v)$, so (3.3) is proved. □

Remark 3.3. By using similar tricks, it is also possible to prove that

$$\lim_{v\to 0+} \hat{v}(\tau,v) = 0, \qquad \lim_{v\to+\infty} \hat{v}(\tau,v) = +\infty$$

uniformly in τ, where $\hat{\tau}(\tau, v)$ is the second component of $\mathcal{S}(\tau, v)$. Although this property will not be employed in the proof of the main result, it is interesting for a better understanding of the dynamics of the equation.

Finally, we state a technical lemma, which will be useful in the next section.

LEMMA 3.4. *Under the assumptions of theorem 1.2, for any $a_2 > a_1 > 0$, there exist $0 < b_1 < a_1 < a_2 < b_2$ such that, if $a_1 \leqslant v \leqslant a_2$, then $b_1 \leqslant \hat{v} \leqslant b_2$.*

The proof is trivial because the successor map is a continuous function.

4. Infinity of $2m\pi$-periodic bouncing solutions

To prove the existence of 2π-periodic and subharmonic solutions, we need the following generalized Poincaré–Birkhoff twist theorem, which is basically the version of Franks [8] and Ding [6] (also see [5]).

Let A and B be two annulus

$$A := \boldsymbol{S}^1 \times [a_1, a_2], \qquad B := \boldsymbol{S}^1 \times [b_1, b_2],$$

with $0 < b_1 < a_1 < a_2 < b_2 < +\infty$, where $\boldsymbol{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Let the map $f : A \to B$ possess a lift $\tilde{f} : \mathbb{R} \times [a_1, a_2] \to \mathbb{R} \times [b_1, b_2]$ with the form

$$\theta' = \theta + h(\theta, \rho), \qquad \rho' = g(\theta, \rho),$$

where h, g are continuous and 2π-periodic in θ. We say that $\tilde{f}$ satisfies the boundary twist condition if

$$h(\theta, a_1) \cdot h(\theta, a_2) < 0 \quad \text{for } \theta \in [0, 2\pi].$$

THEOREM 4.1 (twist theorem). *Assume that $f : A \to B$ is an area-preserving homeomorphism that possesses a lift $\tilde{f}$ satisfying the boundary twist condition and the area of the components of the complement of $f(A)$ in B are the same as the area of the corresponding components of the complement of A in B. Then f has at least two geometrically distinct fixed points (ρ_i, θ_i) $(i = 1, 2)$ satisfying $h(\theta_i, \rho_i) = 0$ for $i = 1, 2$.*

Proof. The proof basically combines the proofs from Franks [8] and Ding [6]. In [8], Franks showed that, by using a result from Oxtoby and Ulam, one can extend f to an area-preserving homeomorphism $F : B \to B$ such that F is the identity on the boundary of B (see the proof and the remark of theorem 4.2 in [8]). Then we can further assume that F is an area-preserving homeomorphism of $D := \{(\theta, \rho) : \rho \leqslant b_2\}$ to its image such that $O \in F(D \setminus B)$. Now we meet all the assumptions of the argument in [6]. According to the argument in [6], we can prove that F, and then f, has at least two fixed points in A. Moreover, the fixed points (θ_i, ρ_i) satisfy $h(\theta_i, \rho_i) = 0$ for $i = 1, 2$ (see [6, 12] for details). $\square$

REMARK 4.2. If A and B are two annulus in $\mathbb{R}^2$ and $f : A \to B$ is a homeomorphism, then we can define a lift $\tilde{f} : \mathbb{R} \times [a_1, a_2] \to \mathbb{R} \times [b_1, b_2]$ of f using polar coordinates. The definition of $\tilde{f}$ is $(\theta_0 + 2k\pi, \rho) \mapsto f(\theta_0, \rho) + (2k\pi, 0) + (2\tilde{k}\pi, 0)$, where $\theta_0 = \theta - 2k\pi \in [0, 2\pi)$ and $\tilde{k} \in \mathbb{Z}$ is a fixed constant.

Now we apply the above twist theorem to the successor map $\mathcal{S}$. From the discussion in § 2, we know that

$$\mathcal{S} : (\tau, v) \mapsto (\hat{\tau}, \hat{v})$$

is well defined, one to one and continuous in the domain $\mathbb{R} \times \mathbb{R}^+$. Moreover, it satisfies

$$\mathcal{S}(\tau + 2\pi, v) = \mathcal{S}(\tau, v) + (2\pi, 0).$$

Thus we can interpret τ and v as polar coordinates. It is easy to show that, for any $n, m \in N$, a fixed point of the map $\mathcal{S}^n(\tau, v) - (2m\pi, 0)$ corresponds a $2m\pi$-periodic bouncing solution of the equation with n bouncings in each period. We have the following lemma.

LEMMA 4.3. *$\mathcal{S}$ is an area-preserving map with the area element $v\mathrm{d}v\mathrm{d}\tau$. Moreover, $\mathcal{S}$ is an exact simplectic map in its domain, that is, $\mathcal{S}$ is an area-preserving homotopic to the inclusion and, for any closed path γ in its domain,*

$$\int_\gamma \tfrac{1}{2} v^2 \,\mathrm{d}\tau = \int_{\mathcal{S}\circ\gamma} \tfrac{1}{2} v^2 \,\mathrm{d}\tau.$$

The proof of this lemma is the same as in [12] and the proof of proposition 2.3 in [17]. As a consequence of this lemma, we know that, for any annulus $A \subset B \subset \mathbb{R} \times \mathbb{R}^+$ with $\mathcal{S}(A) \subset B$, the area of the components of the complement of $\mathcal{S}(A)$ in B are the same as the area of the corresponding components of the complement of A in B.

Proof of theorem 1.2. In the previous section, we show that $\mathcal{S}$ has the property of

$$\begin{aligned} \Pi_1(\mathcal{S}(\tau, v)) - \tau &\to 0 && \text{as } v \to 0, \\ \Pi_1(\mathcal{S}(\tau, v)) - \tau &\to +\infty && \text{as } v \to +\infty \end{aligned}$$

uniformly in $\tau \in [0, 2\pi]$, where Π_1 is the projection for first factor. Thus, for any $n, m \in \mathbb{N}$, we can choose $v_-^{(n)} < v_+^{(n)}$ such that

$$\Pi_1(\mathcal{S}^n(\tau, v_-^{(n)})) - \tau < 2m\pi \quad \text{for } \tau \in [0, 2\pi]$$

and

$$\Pi_1(\mathcal{S}^n(\tau, v_+^{(n)})) - \tau > 2m\pi \quad \text{for } \tau \in [0, 2\pi].$$

Denote by $f = \mathcal{S}^n$, $(\tau', v') = \mathcal{S}^n(\tau, v)$, $a_1 = v_-^{(n)}$, $a_2 = v_+^{(n)}$. From lemma 3.4, we know that there are $b_1 < a_1 < a_2 < b_2$ such that

$$f(A := \{(\tau, v) : a_1 \leqslant v \leqslant a_2\}) \subset B := \{(\tau, v) : b_1 \leqslant v \leqslant b_2\}.$$

Thus the lift $\tilde{f}$ has the form

$$\tau' = \tau + h(\tau, v) - 2m\pi, \qquad v' = g(\tau, v),$$

with h, g continuous and $\tilde{f}$ a boundary twist in A. Therefore, using the twist theorem, we have that $\mathcal{S}(\tau, v) - (2m\pi, 0)$ has two fixed points. Thus, taking $n = 1$, it is proved that there are two $2m\pi$-periodic solutions with one bouncing in each period. However, if the number of bouncings n is greater or equal than 2, the two fixed points provided by the twist theorem may correspond to the same bouncing solution, so we can only assure the existence of one $2m\pi$-periodic solution if $n \geqslant 2$. $\square$

5. Further remarks

(1) By using the results of § 2, some interesting properties about bouncing solutions can be proved without difficulty. For instance, if, in theorem 1.2, m remains fixed and we let the number of bouncings n go to $+\infty$, then the corresponding periodic solution tends uniformly to 0.

(2) If $\bar{p} = 0$, our guess is that the dynamics is similar, but it remains as open problem.

(3) As it was observed in lemma 2.2, if the singularity is strong, then the impact velocity is infinity. In this case, the following general result can be formulated.

THEOREM 5.1. *Let us suppose that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz-continuous function verifying (1.3) and (1.5). Then, for any doubly infinity sequence $\{t_i\}_{i\in\mathbb{Z}}$, there exists a bouncing solution x of (1.2) such that $x(t_i) = 0$ for all $i \in \mathbb{Z}$.*

Proof. After a time rescaling, the main result of [11] can be applied in each interval $]t_i, t_{i+1}[$, obtaining a classical solution $x \in C^2(]t_i, t_{i+1}[, \mathbb{R}^+)$ of (1.2) such that $x(t_i) = 0 = x(t_{i+1})$. The elasticity condition in the bouncings holds automatically because of (1.5) and lemma 2.2, so the result is done. $\square$

Note that, in this case, no conditions are imposed on the mean value of p, so coexistence of periodic bouncing and non-bouncing solutions occurs if $\bar{p} > 0$. This type of coexistence result is an open problem in the weak case.

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