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SOME REMARKS ON A NEUMANN BOUNDARY VALUE PROBLEM ARISING IN FLUID DYNAMICS

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Abstract

It is proved that the Neumann boundary value problem, which Mays and Norbury have recently connected with a certain fluid dynamics equation, has a positive solution for any positive value of a particular parameter. Uniform bounds for the solutions and symmetry on a given range of the parameter are also introduced. The proofs include Krasnoselskii's classical fixed-point theorem on cones of a Banach space and basic comparison techniques.

1. Introduction

In a recent paper by Mays and Norbury [3], the Neumann boundary value problem

$$\begin{aligned} Lu \equiv -u'' + q^2 u &= u^2(1 + \sin x), \\ u'(0) = 0 &= u'(\pi), \end{aligned} \tag{1.1}$$

was studied using analytical and numerical methods. This problem was considered as a simplified version of a fluid dynamics equation introduced by Benjamin [1]. The results in [3] are mostly of a numerical nature and show the existence of a solution if $q^2 \in (0, 10)$. It is important to obtain analytical results which could confirm and/or complement the numerical understanding of this problem [3]. This is the aim of this note. In Section 2 the existence of a solution for any value of the parameter $q > 0$ is rigorously proved. The proof relies on a fixed-point theorem for completely continuous Krasnoselskii operators and the positivity of the Green's function of the linear part of the problem, as has already been observed in [3]. In Section 3 uniform bounds for the solutions are deduced as well as symmetry for a certain range of values of q, by using basic comparison arguments. All these results confirm the numerical evidence from [3], although the range where symmetry appears is more conservative and uniqueness remains an open problem.

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2. Existence of solutions

The main result is the following.

THEOREM 2.1. *Problem* (1.1) *has a positive solution for any positive* q.

The proof is based on the following fixed-point theorem for cones in a Banach space [2, p. 148] and some arguments recently developed in [4].

THEOREM 2.2. *Let* $\mathscr{B}$ *be a Banach space and let* $\mathscr{P} \subset \mathscr{B}$ *be a cone in* $\mathscr{B}$. *Assume* Ω_1, Ω_2 *are open subsets of* $\mathscr{B}$ *with* $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ *and let* $A : \mathscr{P} \cap (\Omega_2/\overline{\Omega}_1) \to P$ *be a completely continuous operator such that one of the following conditions is satisfied*:

(1) $\|Au\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$;
(2) $\|Au\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$.

Then A *has at least one fixed point in* $\mathscr{P} \cap (\overline{\Omega}_2/\Omega_1)$.

As was observed in [3], the Green's function $k(x, y)$ of the operator L with Neumann conditions is a positive and continuous function on $[0, \pi] \times [0, \pi]$. Thus problem (1.1) can be written as the fixed-point problem

$$u(x) = \int_0^{\pi} k(x, y)u^2(y)(1 + \sin y)\, dy \equiv Au. \tag{2.1}$$

PROOF OF THEOREM 2.1. We follow along the lines of [4, Section 3]. If we denote

$$m = \min k(x, y), \qquad M = \max k(x, y), \qquad x, y \in [0, \pi],$$

then evidently $M > m > 0$. In order to apply Theorem 2.2, let us consider the Banach space $\mathscr{B} = C([0, \pi])$ with the L^{∞}-norm $\|\cdot\|_{\infty}$, and define the following cone in $\mathscr{B}$:

$$\mathscr{P}_0 = \left\{ u \in \mathscr{B} : \min_{x \in [0,\pi]} u(x) \geq \frac{m}{M} \|u\|_{\infty} \right\}.$$

Let us prove that $A\mathscr{P}_0 \subset \mathscr{P}_0$. For a given $u \in \mathscr{P}_0$, we have

$$\begin{aligned} \min_{x \in [0,\pi]} Au(x) &\geq \int_0^{\pi} m u^2(y)(1 + \sin y)\, dy \\ &\geq \frac{m}{M} \int_0^{\pi} k(x, y)u^2(y)(1 + \sin y)\, dy = \frac{m}{M} Au(x), \end{aligned}$$

for all $x \in [0, \pi]$, so in particular $\min_{x \in [0,\pi]} Au(x) \geq (m/M)\|Au\|_{\infty}$.

Now let us define the open balls

$$\Omega_1 = \left\{ u \in \mathscr{B} : \|u\|_\infty < \frac{1}{2\pi M} \right\} \quad \text{and} \quad \Omega_2 = \left\{ u \in \mathscr{B} : \|u\|_\infty < \frac{M^2}{\pi m^3} \right\}.$$

Clearly, $0 \in \Omega_1$. On the other hand, observe that the radius of Ω_1 is less than that of Ω_2, so $\overline{\Omega}_1 \subset \Omega_2$.

Now, if $u \in \mathscr{P}_0 \cap \partial\Omega_1$,

$$\|Au\|_\infty \leq 2\pi M \|u\|_\infty^2 = \|u\|_\infty,$$

whereas if $u \in \mathscr{P}_0 \cap \partial\Omega_2$,

$$\|Au\|_\infty \geq m \int_0^\pi u^2(y)(1 + \sin y)\, dy \geq m \int_0^\pi u^2(y)\, dy \geq \frac{m^3}{M^2}\pi \|u\|_\infty^2 = \|u\|_\infty.$$

Therefore (2.1), and in consequence problem (1.1), has a solution $u \in \mathscr{P}_0 \cap (\overline{\Omega}_2 / \Omega_1)$.

3. Uniform bounds and symmetry of the solutions

Note that from the proof of Theorem 2.1 the following bounds of the solution are deduced:

$$\frac{m}{2\pi M^2} \leq u(x) \leq \frac{M^2}{\pi m^3}.$$

However, these bounds are valid only for this particular solution; in principle there may exist other solutions outside these limits. Our following goal is to get uniform bounds for every solution of problem (1.1).

THEOREM 3.1. *There exist constants ϵ, C (only depending on q) such that any solution of problem* (1.1) *verifies*

$$\epsilon \leq u(x) \leq C, \quad x \in [0, \pi].$$

PROOF. First, it is important to consider that, as was observed in [3], every solution of (1.1) is positive. An integration of the equation gives

$$q^2 \|u\|_1 = \int_0^\pi u^2(1 + \sin x)\, dx \geq \|u\|_2^2,$$

and by the Cauchy-Schwartz inequality, $\|u\|_2 \leq q^2\sqrt{\pi}$. Moreover,

$$u'(x) = \int_0^x u''(s)\, ds = \int_0^x \left(q^2 u(s) - u^2(s)(1 + \sin s)\right) ds < q^2 \|u\|_1 \leq q^4 \pi,$$

$$-u'(x) = \int_x^\pi u''(s)\, ds = \int_x^\pi \left(q^2 u(s) - u^2(s)(1 + \sin s)\right) ds < q^2 \|u\|_1 \leq q^4 \pi,$$

so in consequence $\|u'\|_\infty < q^4\pi$.

On the other hand, any non-constant solution of (1.1) must have an inflexion point, that is, there exists $x_0 \in]0,\pi[$ such that $u''(x_0) = 0$. From this equation, it is easy to deduce that

$$q^2/2 < u(x_0) < q^2.$$

We can now deduce the upper bound C as follows:

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(s)\,ds < q^2 + \pi^2 q^4 =: C. \tag{3.1}$$

We still need to obtain the lower bound ϵ. It will be done by comparison of u with solution $\tilde{u}$ of the autonomous initial value problem

$$-\tilde{u}'' + q^2\tilde{u} = \tilde{u}^2,$$
$$\tilde{u}(0) = \epsilon, \quad \tilde{u}'(0) = 0.$$

By continuous dependence of the solution on the initial conditions it is easy to realise that if ϵ is small enough, $\tilde{u}$ is positive, increasing, convex and $\tilde{u} < q^4/4$, $x \in [0,\pi]$.

Evidently ϵ depends on q. By contradiction, let us assume that $u(x_m) = \min u(x) < \epsilon$. Without loss of generality, it can be assumed that $x_m < \pi$ (if $x_m = \pi$, we can continue the argument with $w(x) = u(\pi - x)$, which is also a solution of (1.1)). Let us define $z(x) = u(x) - \tilde{u}(x)$. Note that

$$u(x_m) < \epsilon \le \tilde{u}(x_m), \quad u'(x_m) = 0 \le \tilde{u}'(x_m),$$

so $z(x_m) < 0$, $z'(x_m) \le 0$. Evidently, z cannot be identically zero. We are going to prove that $z(x) < 0$ for all $x > x_m$. If this is not true, there exists $x_1 > x_m$ such that $z(x_1) < 0$, $z'(x_1) = 0$ and $z''(x_1) \ge 0$ ($z(x_1)$ would be a local minimum of z). Subtracting the equations,

$$-z''(x_1) = z(x_1)(u(x_1) + \tilde{u}(x_1) - q^2) + \sin(x_1)u^2(x_1) > 0,$$

because $u(x_1) \le \tilde{u}(x_1) < q^2/4$. This is a contradiction and hence it is proved that $z(x) < 0$ for all $x > x_m$.

As a consequence, $u(x) < q^2/4$ for all $x > x_m$. Now, in order to finish the reasoning we only have to point out that there must be an inflexion point $u(x_0)$ with $x_m < x_0 < \pi$, and as was observed before, $q^2/2 < u(x_0) < q^2$, leading to a contradiction. The consequence is that $u(x_m) \ge \epsilon$, and the proof is finished.

Note that constant C is explicitly defined in (3.1). This information can be used to prove the symmetry of the solutions (that is, $u(x) = u(\pi - x)$) on a certain range of values of q.

THEOREM 3.2. *Let us suppose that q is a positive constant such that*

$$3q^2 + 4\pi^2 q^4 \leq 1. \tag{3.2}$$

Then any solution of problem (1.1) *is symmetric.*

PROOF. Let u_1 be a solution, then it is easy to verify that $u_2 = u_1(\pi - x)$ is also a solution. Our purpose is to prove that $u_1 \equiv u_2$ under condition (3.2). Let us define $z = u_1 - u_2$. Then z is a solution of the problem

$$\begin{aligned} z'' + \alpha(x)z &= 0, \\ z'(0) = 0 &= z'(\pi), \end{aligned} \tag{3.3}$$

where $\alpha(x) = (1 + \sin x)(u_1 + u_2) - q^2$. Observe that by Theorem 3.1,

$$u_i(x) < C = q^2 + \pi^2 q^4, \quad x \in [0, \pi], \ i = 1, 2.$$

Therefore, using condition (3.2),

$$\alpha(x) < 1, \quad x \in [0, \pi]. \tag{3.4}$$

Let us prove that z is identically zero. Let us suppose that z is not the trivial solution of (3.3). Let us change to polar coordinates, $z = r \cos\theta$, $z' = -r \sin\theta$. By deriving z and z' we get respectively

$$\begin{aligned} r' \cos\theta - r \sin(\theta)\theta' &= -r \sin\theta, \\ -r' \sin\theta - r \cos(\theta)\theta' &= -\alpha(x) r \cos\theta. \end{aligned}$$

Multiplying the first equation by $\sin\theta$, the second one by $\cos\theta$ and adding, we obtain the equation

$$\theta' = \alpha(x)\cos^2\theta + \sin^2\theta. \tag{3.5}$$

Now, an integration in the interval $[0, x]$ and (3.4) give

$$\theta(x) - \theta(0) = \int_0^x (\alpha(s)\cos^2\theta + \sin^2\theta)\, ds < \int_0^x (\cos^2\theta + \sin^2\theta)\, ds = x, \tag{3.6}$$

for all $x \in (0, \pi]$.

On the other hand, note that $z(x) = -z(\pi - x)$, and therefore $z(\pi/2) = 0$. By the Sturm comparison theorem (compare with $z'' + z = 0$), this is the unique zero of z in the interval $[0, \pi]$. Besides, $z(0)z(\pi) < 0$ because z is not the trivial solution. We can assume without loss of generality that $z(0) > 0$ (if $z(0) < 0$ we work with $-z$). Then $\theta(0) = 0$ since $z'(0) = 0$. Moreover, $z(\pi/2) = 0$ and $z'(\pi/2) < 0$ (remember that z is not the trivial solution and $z(\pi/2)$ is the unique zero), so $\theta(\pi/2) = \pi/2$. But by (3.6), $\pi/2 = \theta(\pi/2) - \theta(0) < \pi/2$. This is a contradiction. The conclusion is that $z \equiv 0$ and therefore the proof is finished.

A numerical computation of condition (3.2) provides $q \in]0, 0.354446]$. As a final remark, the uniqueness of a positive solution on a given range of values of the parameter q is strongly suggested by numerical calculations. The analytical proof remains an open problem.

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