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NON-COLLISION PERIODIC SOLUTIONS OF FORCED DYNAMICAL SYSTEMS WITH WEAK SINGULARITIES

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Abstract. We prove the existence of periodic solutions in a second order differential system with a singular potential of attractive or repulsive type and forced periodically. The proof is based on a Krasnoselskii fixed point theorem for absolutely continuous operators on a Banach space, and this makes possible to avoid any kind of "strong force" condition.

1. **Introduction.** The purpose of this work is to study the existence of non-collision T-periodic solutions of periodically forced Lagrangian systems with a singular potential in the origin. Generally speaking, a potential $V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is said *singular* if

$$\lim_{x \to 0} V(x) = +\infty.$$

Then, we can consider the system

$$-u'' + \nabla V(u) = p(t) \tag{1.1}$$

with $p \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$. In the related literature, it is said that such a system has an *attractive singularity*, since near the origin the gradient is inward directed. Alternatively, it is said that the system

$$u'' + \nabla V(u) = p(t) \tag{1.2}$$

possesses a *repulsive singularity.* A non-collision T-periodic solution is just a function $u \in C^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n \setminus \{0\})$ which solves the system. Both situations have deserved the attention of many mathematicians and physicists because they include some important examples like gravitational or coulombian potentials. Usually, the method of proof is either of variational nature (see the monograph [2] which include a complete bibliography up to 1993 and more recently [1, 9, 11, 13, 14, 15, 16, 20]) or either is based in topological degree arguments [5, 7, 18]. Among those papers that use variational arguments, we will pay special attention to [8, 12] as touchstones of our results. Last section contains a brief comparative discussion.

In both lines of research previously mentioned, a common device in order to avoid collisions of the solution with the singularity is to assume some type of *strong singularity* condition, that roughly means that $V(x) \simeq \frac{1}{|x|^\alpha}$ with $\alpha \geq 2$ near the singularity. This type of condition was first introduced by Gordon [3, 4] and since then it has been assumed frequently in the related literature and many efforts have

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been done in order to avoid or weaken it (see for instance [9, 10, 13, 14]). Our intention is to apply a new technique in this context leading to existence results for potential with a very weak singularity. The proof is rather elementary and uses the known Krasnoselskii fixed point theorem for absolutely continuous operators on a Banach space that exhibit a cone compression or expansion of norm type [6, p.148]. This result has been employed very recently for the study of scalar equations with singular nonlinearities [17, 19]. Here, we intend to apply a similar technique to a whole system of differential equations. To this purpose, a quadratic growth of the potential in infinity is required. Concretely, our "model" potential is

$$V(x) = \frac{1}{|x|^\alpha} + k^2 \frac{|x|^2}{2}, \tag{1.3}$$

with α, k positive numbers. This is closely related with the situation studied in [8]. Our main result applied to such a potential reads as follows.

Theorem 1.1. *Let us assume that the potential V is given by* (1.3) *and that $p \in C(I\!R/T\mathbb{Z}, I\!R^n)$ verifies that there exists $v \in I\!R^n_+$ such that the scalar product $(v, p(t))$ as a function of t does not change sign. Then, equation* (1.1) *has at least a non-collision T-periodic solution for any $k > 0$, whereas equation* (1.2) *has at least a non-collision T-periodic solution for any $0 < k < \frac{\pi}{T}$.*

The paper is organized as follows. Section 2 exposes some notation and also the main tool to be used in the proofs of next section, namely the yet mentioned Krasnoselskii's fixed point theorem. In Section 3, some existence results are stated and proved. Section 4 is devoted to applications of such results. Finally, Section 5 contains some remarks and further extensions.

2. Notation and preliminaries.

For $i = 1, 2$, let us define the linear operators

$$L_i[u] := (-1)^i u'' + k^2 u.$$

It is very known that for any $k > 0$, the operator L_1 with periodic conditions has a well-defined positive Green's function $G_1 : [0, T] \times [0, T] \to I\!R$ such that

$$M_1 = \sup_{t,s} G_1(t, s) = \frac{1 + e^{kT}}{2k(e^{kT} - 1)}, \quad m_1 = \inf_{t,s} G_1(t, s) = \frac{e^{kT/2}}{k(e^{kT} - 1)} \tag{2.4}$$

On the other hand, L_2 with periodic conditions has also a well-defined Green's function G_2 but it is positive only if $0 < k < \frac{\pi}{T}$, being

$$M_2 = \sup_{t,s} G_2(t, s) = \frac{1}{2k \sin\left(\frac{kT}{2}\right)}, \quad m_2 = \inf_{t,s} G_2(t, s) = \frac{1}{2k} \cot\left(\frac{kT}{2}\right). \tag{2.5}$$

The set of vectors of $I\!R^n$ with positive components is denoted by $I\!R^n_+$. Given $x, y \in I\!R^n$, the usual scalar product is denoted by (x, y). The usual Euclidean norm is denoted by $|x|$, whereas $|x|_1 = \sum_{i=1}^n |x_i|$ is the l_1-norm. More generally, for a fixed vector $v \in I\!R^n_+$ we have a well-defined v-norm $|x|_v = \sum_{i=1}^n v_i |x_i|$. Evidently, if $v = (1, \ldots, 1)$ we recover the l_1-norm. With this v-norm, the ball centered in zero with radius R is denoted by $B(R) = \{x \in I\!R^n : |x|_v \leq R\}$.

For a fixed $v \in I\!R^n_+$ the following cone will be useful

$$\mathcal{S}_v = \{x \in I\!R^n : (v, x) > 0\}.$$

We recall the following fixed point theorem on cones due to M.A. Krasnoselskii [6].

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ ant let $\mathcal{A} : \mathcal{P} \cap (\overline{\Omega}_1/\Omega_2) \to \mathcal{P}$ be a completely continuous operator such that one of the following conditions is satisfied*

1. $\|\mathcal{A}u\| \leq \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geq \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_2$
2. $\|\mathcal{A}u\| \geq \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leq \|u\|$, *if* $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, $\mathcal{A}$ has at least one fixed point in $\mathcal{P} \cap (\overline{\Omega}_2/\Omega_1)$.

3. **Main results.** From now on, we will assume that $V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ verifies

$$\lim_{x \to 0} V(x) = +\infty \tag{3.6}$$

and there exists a fixed $v \in \mathbb{R}^n_+$ such that

$$\lim_{x \to 0,\, x \in \mathbb{R}^n_+} (v, \nabla V(x)) = -\infty. \tag{3.7}$$

Our first result concerns attractive systems.

Theorem 3.1. *Let us assume that $V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies conditions (3.6)–(3.7). If there exist positive numbers k and R such that*

$$\text{i)} \quad \big(v, p(t) + k^2 x - \nabla V(x)\big) \geq 0, \qquad \forall t, \forall x \in \mathcal{S}_v \cap B\left(\frac{M_1}{m_1} R\right),$$

$$\text{ii)} \quad \big|p(t) + k^2 x - \nabla V(x)\big|_v \leq \frac{R}{m_1 T}, \qquad \forall t, \forall x \in \mathcal{S}_v \cap \left(B\left(\frac{M_1}{m_1} R\right) \setminus B(R)\right),$$

then equation (1.1) has at least a non-collision T-periodic solution.

Proof. Let us take the Banach space $\mathcal{B}C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ with the norm

$$\|u\| = \max_t |u(t)|_v .$$

Writing equation (1.1) as

$$-u'' + k^2 u = p(t) + k^2 u - \nabla V(u),$$

a T-periodic solution is equivalent to a fixed point of the completely continuous operator

$$\mathcal{A}u := \int_0^T G_1(t,s) \left[p(s) + k^2 u(s) - \nabla V(u(s))\right] ds$$

Let us consider the cone

$$\mathcal{P} = \left\{ u \in C(\mathbb{R}/T\mathbb{Z}, \mathcal{S}_v) : \quad \min_t (v, u(t)) > \frac{m_1}{M_1} \|u\| \right\}$$

and the open sets

$$\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}, \qquad \Omega_2 = \left\{u \in \mathcal{B} : \|u\| < \frac{M_1}{m_1} R\right\},$$

with $r > 0$ to be fixed later. As a first step, let us prove that

$$\mathcal{A}\left(\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)\right) \subset \mathcal{P}.$$

In fact, if $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, by assumption *i)* together with the sign of G_1 we get

$$\begin{aligned}
(v, \mathcal{A}u(t)) &= \sum_{i=1}^n \int_0^T v_i G_1(t,s)\left[p_i(s) + k^2 u_i(s) - \frac{\partial V(u(s))}{\partial x_i}\right] ds > \\
&= \int_0^T G_1(t,s)\big(v, p(s) + k^2 u(s) - \nabla V(u(s))\big)\, ds > \\
&> m_1 \int_0^T \big(v, p(s) + k^2 u(s) - \nabla V(u(s))\big)\, ds \geq \\
&\geq \frac{m_1}{M_1} \max_t \int_0^T \big(v, G_1(t,s)\left[p(s) + k^2 u(s) - \nabla V(u(s))\right]\big)\, ds = \\
&= \frac{m_1}{M_1}\|\mathcal{A}(u)\|.
\end{aligned}$$

Let us take $u \in \partial\Omega_2 \cap \mathcal{P}$. Then, $\|u\| = \frac{M_1}{m_1}R$ so in consequence

$$\min_t (v, u(t)) > \frac{m_1}{M_1}\|u\| = R.$$

Therefore, $u(t) \in \mathcal{S}_v \cap \left(B\left(\frac{M_1}{m_1}R\right) \setminus B(R)\right)$, so condition *ii)* implies

$$\begin{aligned}
\|\mathcal{A}u\| &\leq M_1 \sum_{i=1}^n v_i \int_0^T \left|p_i(s) + k^2 u_i(s) - \frac{\partial V(u(s))}{\partial x_i}\right| ds = \\
&= M_1 \int_0^T \left|p(s) + k^2 u(s) - \nabla V(u(s)\right|_v ds \leq \frac{M_1}{m_1}R = \|u\|.
\end{aligned}$$

In order to apply Theorem 2.1, it remains to prove that $\mathcal{A}$ is expansive for small solutions. This is possible due to the singular character of the potential by taking r small enough. Indeed, assumption (3.7) guarantees that there exists $r < R$ such that

$$\big(v, p(t) + k^2 u(t) - \nabla V(u(t))\big) \geq \frac{M_1}{T m_1^2}(v, u(t))$$

for all $t \in [0,T]$, $u \in \partial\Omega_1 \cap \mathcal{P}$. Now, it is possible to estimate in the following way

$$\begin{aligned}
\|\mathcal{A}u\| &= \max_t \sum_{i=1}^n v_i \left|\int_0^T G_1(t,s)\left[p_i(s) + k^2 u_i(s) - \frac{\partial V(u(s))}{\partial x_i}\right] ds\right| \geq \\
&\geq \left|\int_0^T G_1(t,s) \sum_{i=1}^n v_i \left[p_i(s) + k^2 u_i(s) - \frac{\partial V(u(s))}{\partial x_i}\right] ds\right| \geq \\
&\geq m_1 \left|\int_0^T \big(v, p(s) + k^2 u(s) - \nabla V(u(s))\big)\, ds\right| \geq \\
&\geq \tfrac{M_1}{T m_1} \int_0^T (v, u(s))\, ds \geq \tfrac{M_1}{m_1} \min_t (v, u(t)) > \|u\|.
\end{aligned}$$

Then, the proof is done through a direct application of Theorem 2.1.

An analogous result holds for repulsive systems.

Theorem 3.2. *Let us assume that* $V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *satisfies conditions* (3.6)–(3.7). *If there exist* $0 < k < \frac{\pi}{T}$ *and* $C, R > 0$ *such that*

i) $\big(v, p(t) + k^2 x - \nabla V(x)\big) \geq 0, \qquad \forall t, \forall x \in \mathcal{S}_v \cap B\left(\frac{M_2}{m_2}R\right),$

ii) $\left|p(t) + k^2 x - \nabla V(x)\right|_v \leq \frac{R}{m_2 T}, \qquad \forall t, \forall x \in \mathcal{S}_v \cap \left(B\left(\frac{M_2}{m_2}R\right) \setminus B(R)\right),$

then equation (1.2) *has at least a non-collision* T*-periodic solution.*

Proof. The proof is identical by using now $G_2(t,s)$. The additional assumption $k < \frac{\pi}{T}$ is needed to have G_2 positive.

4. **Applications.** Let us recall the "model" potential proposed in the Introduction

$$V(x) = \frac{1}{|x|^\alpha} + k^2 \frac{|x|^2}{2},$$

with $\alpha > 0$. It is easy to verify that condition (3.6) holds. Besides, (3.7) holds for any $v \in \mathbb{R}^n_+$. If it is assumed that $(v, p(t)) \geq 0$ for all t, hypotheses of Theorems 3.1 and 3.2 hold by taking R large enough. On the other hand, if $(v, p(t)) \leq 0$ for all t, the symmetry of V enables the change of variables $y = -x$, so proof of Theorem 1.1 is completed. In general, we have the following Corollary.

Corollary 4.1. *Let us consider the potential* $V(x) = k^2 \frac{|x|^2}{2} + W(x)$ *with* $W \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *verifying* (3.6) *and* (3.7) *and such that*

$$\lim_{|x| \to +\infty} \nabla W(x) = 0.$$

If there exists $v \in \mathbb{R}^n_+$ *such that* $(v, p(t)) > 0$ *for all* t*, then equation* (1.1) *has at least a non-collision* T*-periodic solution for all* $k > 0$*, whereas equation* (1.2) *has at least a non-collision* T*-periodic solution for all* $0 < k < \frac{\pi}{T}$*. Besides, if* $W(x) = W(-x)$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$*, the previous statement is valid for any* $p \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ *such that* $(v, p(t)) \neq 0$ *for all* t*.*

Note that the previous result includes singularities weaker that any power, like for instance $W(x) = -\ln|x|$.

5. **Remarks and extensions.** We have formulated the results in its present form by reasons of clearness and simplicity, but they are still true for a time-dependent potential $V(t,x)$ if the inequalities involved in the hypotheses hold uniformly in t. In this sense, it is interesting to compare the obtained results with those of [8]. Clearly, the main advantage of our method is the possibility to handle with very weak singularities, but the counterpart is that quadratic growth of the potential in infinity is necessarily required, acting as a "trap" for the solution. Hence, classical examples like gravitational potential are not covered.

Other interesting reference is the paper of Solimini [12]. Again, the set of hypotheses required in both papers are independent. Our results allow a linear growth of the gradient field at infinite, whereas in [12] it is needed a vanishing gradient field at infinite (see assumption (F_3)). Also, we deal with repulsive or attractive singularities, while [12] is restricted only to the first case. As a counterpart, Solimini assumes mild conditions over the average of the external forcing $p(t)$, whereas we need some uniform bounds. As a consequence, the solution of Solimini can rotate around the origin, while our solution remains in the cone $\mathcal{S}_v$.

Finally, note that proofs do not rely in the variational nature of the problem, so it is not hard to get similar results for dissipative systems by adding a viscous damping term of the form $c_i u_i$. This is an important advantage if compared with papers of variational type like the previously mentioned [8, 12]. Finally, the inclusion of a more general term like $\frac{d}{dt}\nabla F(u)$ does not seem easy and remains as an open problem.

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