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Existence and Stability of Periodic Solutions of a Duffing Equation by Using a New Maximum Principle

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Abstract. The purpose of this paper is to obtain new criteria for existence and asymptotic stability of periodic solutions of a Duffing equation $x'' + cx' + g(t,x) = 0$, taking advantage of a new maximum principle with L^p-conditions combined with known relations between upper and lower solutions, topological degree and stability.

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1. Introduction

In this paper, we consider the periodic problem for the Duffing equation

$$x'' + cx' + g(t,x) = 0,$$
$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi),$$

with $c > 0$ and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Caratheodory function, that is, it is measurable in the first variable and continuous in the second one.

The research presented here was mainly motivated by the reading of [4], where stability properties of periodic solutions of a Duffing equation are derived by interpreting in terms of lower and upper solutions the known relation between topological degree and stability developed by Ortega [5], [6], [7]. The approach exposed in [4] is quite useful when the nonlinearity admits the decomposition $g(t,x) = g(x)+h(t)$. Our final purpose is to deal with situations where the previous decomposition does not hold. We develop a generalization of a new maximum principle [10] with L^p-conditions, and this together with a combination of ideas of

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Ortega and some tricks from [4] lead to new results that are applied, as practical examples, to an equation modeling the motion of a pendulum with a moving support and to an equation with oscillatory-expansive nonlinearity.

The paper is divided in five sections: after this Introduction, in Section 2 we extend the maximum principle given in [10] to operators with a viscous damping term. Section 3 mimics some arguments from [5] in order to connect index of periodic solutions and asymptotic stability. Section 4 provides a connection between the existence of upper and lower solutions in the reversed order and asymptotic stability in the line of [4]. Finally, Section 5 presents two practical examples where our results apply.

2. A new maximum principle

Throughout the paper, for a given $f \in L^1(0, 2\pi)$, we write $f \succ 0$ if $f \geq 0$ for almost every t and it is positive in a subset of positive measure.

Let us call $W = \{u \in W^{2,1}(0, 2\pi) : u(0) = u(2\pi),\, u'(0) = u'(2\pi)\}$. For a given $a \in L^1(0, 2\pi)$ and $c \geq 0$, let us define the differential operator

$$\begin{aligned} L : W &\to L^1(0, 2\pi), \\ Lu &:= u'' + cu' + a(t)u. \end{aligned}$$

The operator L is said *inversely positive* if it is invertible and

$$Lu \succ 0 \Longrightarrow u > 0.$$

Classically, when an operator is inversely positive it is said that a maximum principle holds.

Let us recall some results concerning periodic and anti-periodic eigenvalues (see for instance [3]). Consider the eigenvalue problem

$$u'' + (\lambda + \phi(t))u = 0 \tag{2.1}$$

subject to the periodic boundary condition

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{2.2}$$

or to the anti-periodic boundary condition

$$u(0) = -u(2\pi), \quad u'(0) = -u'(2\pi). \tag{2.3}$$

It is well known that there exist two sequences $\{\underline{\lambda}_k(\phi) : k \in \mathbb{N}\}$ and $\{\overline{\lambda}_k(\phi) : k \in \mathbb{Z}^+\}$ such that

- they have the following order:

  $$-\infty < \overline{\lambda}_0(\phi) < \underline{\lambda}_1(\phi) \leq \overline{\lambda}_1(\phi) < \cdots < \underline{\lambda}_k(\phi) \leq \overline{\lambda}_k(\phi) < \cdots$$

  and $\underline{\lambda}_k(\phi) \to +\infty$, $\overline{\lambda}_k(\phi) \to +\infty$ as $k \to \infty$.
- λ is an eigenvalue of (2.1) – (2.2) if and only if $\lambda = \underline{\lambda}_k(\phi)$ or $\overline{\lambda}_k(\phi)$ for some even k.
- λ is an eigenvalue of (2.1) – (2.3) if and only if $\lambda = \underline{\lambda}_k(\phi)$ or $\overline{\lambda}_k(\phi)$ for some odd k.

Lemma 2.1. *Let us consider* $a \in L^1(0, 2\pi)$ *and* $c \in \mathbb{R}$ *such that* $\underline{\lambda}_1(a) + \frac{c^2}{4} > 0$. *For a given* $t_0 \in \mathbb{R}$, *let* u *be the solution of the initial value problem*

$$\begin{gathered} u'' + cu' + a(t)u = 0, \\ u(t_0) = 0, \quad u'(t_0) = 1. \end{gathered}$$

Then, $u(t) > 0$ *for all* $t \in]t_0, t_0 + 2\pi]$.

Proof. The change of variables $y = e^{\frac{ct}{2}} u$ leads to the Hill's equation

$$y'' + \left(a(t) - \frac{c^2}{4}\right) y = 0.$$

From the definition of eigenvalue, note that $\underline{\lambda}_1(a(t) - \frac{c^2}{4}) = \underline{\lambda}_1(a) + \frac{c^2}{4}$. Then, the result is a direct consequence of [10, Lemma 2.1]. □

Remark 2.2. By the comparison theory of Sturm, under the hypotheses of the previous lemma the distance between two consecutive zeroes of a solution of $Lu = 0$ is always greater than 2π.

Lemma 2.3. *If* $a \succ 0$ *and* $\underline{\lambda}_1(a) + \frac{c^2}{4} > 0$, *then the operator* L *is inversely positive.*

Proof. First, we prove that the inverse L^{-1} exists. By the linearity of the operator, it is enough to prove that $Lu = 0$ implies $u = 0$. By Remark 2.2, if $u \neq 0$ then the function u should have constant sign and by integrating $Lu = 0$ over a period a contradiction is done.

Now, let us assume that $Lu \succ 0$ for some $u \in W$. By the assumptions on u, it can be extended to the whole real line by periodicity. First, we prove that u does not vanishes. On the contrary, let us suppose that $u(t_0) = 0$ for some $t_0 \in [0, 2\pi]$. Let v be the solution of

$$\begin{gathered} v'' - cv' + a(t)v = 0, \\ v(t_0) = 0, \quad v'(t_0) = 1. \end{gathered}$$

From Lemma 2.1, we know that $v(t) > 0$ for every $t \in]t_0, t_0 + 2\pi]$. Then, $vLu \succ 0$. Besides,

$$vLu = vu'' + cu'v - uv'' + cuv'.$$

Integrating in $[t_0, t_0 + 2\pi]$ by parts and using that $u(t_0) = u(t_0 + 2\pi) = 0$ we get

$$\int_{t_0}^{t_0+2\pi} vLu = v(t_0 + 2\pi)u'(t_0 + 2\pi) = v(t_0 + 2\pi)u'(t_0).$$

Therefore, $u'(t_0) > 0$. Hence, we have proved that any zero of u has positive derivative, but this is a contradiction since u is a periodic continuous function. Therefore, u has constant sign. Now, by integrating the equation in $[0, 2\pi]$ we get

$$\int_0^{2\pi} a(t)u = \int_0^{2\pi} Lu > 0,$$

and taking into account that $a \succ 0$, we conclude that $u > 0$ and the proof is completed. □

Let us recall a lower bound for $\underline{\lambda}_1(\phi)$ from [12]. Let us define the positive part of a function ϕ as $\phi^+(t) = \max\{\phi(t), 0\}$. If the L^p norm $\|\phi^+\|_p$ satisfies

$$\|\phi^+\|_p \le K(2p^*), \qquad (p^* = p/(p-1)),$$

then (see (13) in [12])

$$\underline{\lambda}_1(\phi) \ge \frac{1}{4}\left(1 - \frac{\|\phi^+\|_p}{K(2p^*)}\right). \tag{2.4}$$

Here $K(q)$ is the best Sobolev constant in the following inequality:

$$C\|u\|_q^2 \le \|u'\|_2^2 \qquad \text{for all } u \in H_0^1(0, 2\pi).$$

Explicitly (see Talenti [8]),

$$K(q) = \begin{cases} \frac{1}{q(2\pi)^{2/q}}\left(\frac{2}{2+q}\right)^{1-2/q}\left(\frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)}\right)^2, & \text{if} \quad 1 \le q < +\infty, \\ \frac{2}{\pi}, & \text{if} \quad q = +\infty. \end{cases} \tag{2.5}$$

Thus we have the following

Corollary 2.4. *Let us assume that* $\|a^+\|_p < \left(1 + \frac{c^2}{4}\right) K(2p^*)$ *for some* $1 \le p \le +\infty$*. Then, the conclusion of Lemma* 2.1 *holds.*

Corollary 2.5. *Let us assume that* $a \succ 0$ *and* $\|a\|_p < \left(1 + \frac{c^2}{4}\right) K(2p^*)$ *for some* $1 \le p \le +\infty$*. Then, L is inversely positive.*

3. Index and asymptotic stability of solutions

In this section, we consider the Duffing equation

$$x'' + cx' + g(t, x) = 0 \tag{3.1}$$

with $c > 0$ and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Caratheodory function such that the partial derivative g_x exists and it is also L^1-Caratheodory. For a given $1 \le p \le +\infty$, let us define the set

$$\Omega_{p,c} = \left\{a \in L^p(0, 2\pi) : a \succ 0, \|a\|_p < \left(1 + \frac{c^2}{4}\right) K(2p^*)\right\},$$

with p^*, K defined in the previous section. Let φ be a 2π-periodic solution of equation (3.1). Throughout this section, we assume that $a \in \Omega_{p,c}$ exists such that

$$g_x(t, \varphi(t)) < a(t) \qquad \text{for a.e. } t \in [0, 2\pi]. \tag{3.2}$$

For such an a, the operator L as defined in (2) is inversely positive. By defining the operators

$$\begin{aligned} F &: C[0, 2\pi] \to L^1(0, 2\pi), & H &: L^1(0, 2\pi) \to W, \\ Gx &= a(\cdot)x - g(\cdot, x), & H &= L^{-1}F, \end{aligned}$$

the solution φ can be seen as a fixed point of the operator H.

Next, we recall that an isolated 2π-periodic solution has a well-defined index $\gamma(\varphi)$ and $|\gamma(\varphi)| \le 1$ (see for instance [7, Chapter 2] for details).

The main result of this section resembles [5, Theorem 1.1] and [4, Lemma 2.5], see also [7, Chapter 3].

Theorem 3.1. *Let φ be an isolated 2π-periodic solution of equation* (3.1) *verifying* (3.2)*. Then, $\gamma(\varphi) = 1$ (resp. $\gamma(\varphi) = -1$) if and only if φ is asymptotically stable (resp. unstable).*

The proof is identical to that of the cited results, with the help in our case of the following Lemma.

Lemma 3.2. *Let us assume that $a \in L^1(0, 2\pi)$ verifies $\|a^+\|_p < \left(1 + \frac{c^2}{4}\right) K(2p^*)$ for some $1 \le p \le +\infty$. Then, the linear equation*

$$u'' + cu' + a(t)u = 0$$

does not admit negative characteristic multipliers.

Proof. The proof is analogous to the one in [7, Lemma 8], by using now Corollary 2.4 and Remark 2.2. □

It is interesting to note that this last lemma prevents the existence of subharmonics, as it is noted in [7, Lemma 8].

4. Upper and lower solutions on the reversed order

Let us begin with the formal definition of upper and lower solution.

Definition 4.1. A lower solution of equation (3.1) is a function $\alpha \in W$ such that $\alpha'' + c\alpha' + g(t, \alpha) \ge 0$ for a.e. t. An upper solution β is defined just reversing this inequality. If a lower (upper) solution is not a solution is called a *strict* lower (upper) solution.

There is a wide bibliography concerning upper and lower solutions, see for instance the survey [1]. If a couple of lower and upper solutions verifies the order $\alpha < \beta$, it is known [2] that an unstable periodic solution exists between them (see also [4]). We focused our attention in the case of upper and lower solutions on the reversed order $\alpha > \beta$. In the simple example $x'' + x = \sin t$, constant upper and lower solutions on the reversed order $\alpha = \pi/2 > \beta = 3\pi/2$ appear, however all solutions are unbounded. This proves that additional assumptions are required to assure even the existence of a periodic solution. Our main result is the following.

Theorem 4.2. *Let $\alpha > \beta$ be a couple of strict lower and upper solutions of equation* (3.1) *such that for a given $1 \le p \le +\infty$ there exists $a \in \Omega_{p,c}$ verifying*

$$g_x(t, x) \le a(t) \quad \text{for a.e. } t, \ \forall x \in [\beta(t), \alpha(t)]. \tag{4.1}$$

Then, (3.1) *has at least an asymptotically stable 2π-periodic solution $\beta < x < \alpha$ provided that the number of 2π-periodic solutions between β and α is finite.*

Remark 4.3. The proof is analogous to that of Theorem 1.2 in [4], by using now Theorem 3.1. If $p = +\infty$ we obtain basically Theorem 1.2 from [4].

Remark 4.4. About the local finiteness of the solution set, there is an interesting discussion in [4, Remark 3.2]. It is proved that a sufficient condition is the analytic character of the nonlinearity in the second variable together with the presence of strict upper and lower solutions. The argument in [4] was done for L^∞-Caratheodory nonlinearities, but it is still valid without changes in our setting. In the next section, we propose two examples that verify these assumptions.

5. Some examples

5.1. A pendulum with oscillating support

As a first application of our results, let us consider a pendulum attached to a moving cart as shown in Figure 5.1. If the motion of the cart is described by a function $p(t) \in W^{2,1}(0, 2\pi)$, the angle $x(t)$ of the pendulum verifies the following differential equation

$$x'' + cx' + \frac{g}{l} \sin x + \frac{p''(t)}{l} \cos x = 0, \tag{5.1}$$

where $l > 0$ is the length of the pendulum and $c > 0$ is a viscous friction coefficient. Then, the following result holds.

Corollary 5.1. *Let us assume that* $\frac{g}{l} + \frac{p''(t)}{l} \in \Omega_{p,c}$ *for some* $1 \leq p \leq +\infty$. *Then, there exists an asymptotically stable* 2π*-periodic solution* x *of equation* (5.1) *verifying* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Proof. It is a direct consequence of Theorem 4.2, since $\alpha = \frac{\pi}{2} > \beta = -\frac{\pi}{2}$ are constant strict lower and upper solutions of equation (5.1). Then, assumption (4.1) is easily checked with $a(t) = \frac{g}{l} + \frac{p''(t)}{l}$. □

5.2. Oscillating-expansive nonlinearities

Let us consider the equation

$$x'' + cx' + a(t)x^\nu \sin(\log x) = h(t), \tag{5.2}$$

with a, $h \in L^1$ and $c > 0$. When $a \equiv 1$, it was studied in [4], obtaining an infinite number of asymptotically stable periodic solution. Now, we can generalize this result.

Corollary 5.2. *Let* $a \in L^1(0, 2\pi)$ *be such that* $a(t) \geq a_0 > 0$ *for a.e.* t. *Then, there exists an infinite number of asymptotically stable* 2π*-periodic solutions of equation* (5.2).

Proof. It is possible to construct a sequence $\beta_n > \alpha_n$ of strict upper and lower solutions on the reversed order going uniformly to $+\infty$ in the same way as in [4]. Moreover, the nonlinearity $g(t, x) = a(t)x^\nu \sin(\log x) - h(x)$ is strictly increasing in x between each pair β_n, α_n. If $\beta_n > \alpha_n > R$,

$$g_x(t, x) = a(t)x^{\nu-1} [\nu \sin(\log x) + \cos(\log x)] < \frac{a(t)}{(1 + \nu)R^{1-\nu}} \equiv a^*(t).$$

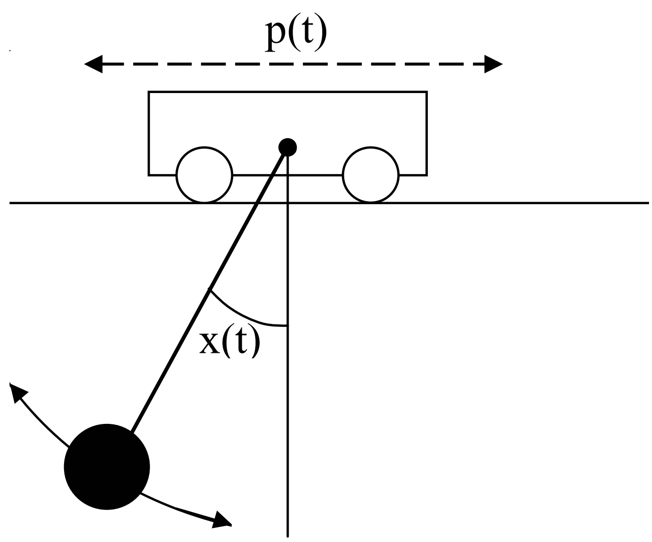

FIGURE 5.1. Pendulum with a moving suspension point

Then, taking R large enough, we get $a^* \in \Omega_{1,c}$ and in consequence Theorem 4.2 can be applied in each interval $[\beta_n, \alpha_n]$ with n large enough. □

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