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# Twist periodic solutions of repulsive singular equations☆

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**Abstract**

Motivated by the Lazer–Solimini equation and the Brillouin equation, which present a repulsive singularity at the origin, we will develop in this paper some criterion for the (positive) periodic solution to be of twist type. As an application of the criterion, we will give also some quantitative estimates to the region of parameters so that the equations have twist periodic solutions. These, together with the Moser Twist Theorem, imply that the equations have rich dynamics in the neighborhood of the twist periodic solutions.


*Keywords:* Twist; Periodic solution; Lyapunov stability; Singular equation

## 1. Introduction

This paper is devoted to the study of the existence and uniqueness of Lyapunov-stable periodic solutions of second-order ordinary differential equations with a repulsive singularity at the origin. Our attention will be mainly focused on two models. The first one is the Lazer–Solimini equation

$$x'' - \frac{1}{x^\alpha} = p(t) - s$$

☆ Dedicated to Professor Jean Mawhin on the occasion of his 60th birthday.

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with $p \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, $\int_0^{2\pi} p(t)\,dt = 0$ and $\alpha \geqslant 1$. This is perhaps the simplest example of equation with a singular restoring force and it models, for instance, the motion of a charged particle inside an electric field or the motion of a piston inside a cylinder filled with a perfect gas. The survey [6] provides a nice historical overview of the equation. In a paper which has became a classic [4], Lazer and Solimini proved that there is a positive periodic solution if and only if $s > 0$. This result was extended in [3] to equations with a linear damping term. In this context, it was proved in [13] that under some conditions there exists an asymptotically stable solution, but up to the knowledge of the authors there are no stability results available in the literature for the conservative case.

The second model we will analyze is the so-called Brillouin equation

$$x''(t) + \gamma(1 + \delta\cos t)x(t) = \frac{1}{x(t)}$$

with $\gamma, \delta$ positive numbers. This equation governs a focusing system for an electron beam immersed in a periodic magnetic field [1]. From the mathematical point of view, many efforts have been done in the improvement of the results about existence of periodic solution [2,17–20]. The study of stability of such a solution has been initiated very recently [14,15].

In general, we shall consider the periodic problem

$$\begin{aligned} &x'' + g(t,x) = 0, \\ &x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{aligned} \tag{1}$$

where $g(t,x)$ is $2\pi$-periodic in $t$ and of class $C^{0,4}$ in $(t,x)$. It is known that the stability of a solution depends closely on the third-order approximation of (1) with respect to this solution. This is a classical idea that can be traced back to Moser [11] in 1960s, but it has been renewed recently by Ortega [8–10]. These papers develop different criteria for existence of a periodic solution of twist type based on the third approximation. A solution is said to be of *twist type* if the first coefficient of Birkhoff $\beta$, also called the twist coefficient, is nonzero. If strong resonances are avoided, a solution of twist type is always Lyapunov-stable [11]. Poincaré–Birkhoff Theorem and K.A.M. theory imply the existence of subharmonic solutions with minimal period tending to infinity as well as quasiperiodic solutions in a neighborhood of the solution. In this paper, we will use a compact expression of the twist coefficient developed in [5] to obtain a new stability criterion with applications to the model equations (3) and (5).

## 2. Uniqueness and linear stability of periodic solution

Let us define

$$K(q) = \begin{cases} \dfrac{1}{q(2\pi)^{2/q}}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)}\right)^2 & \text{if } 1 \leqslant q < \infty, \\ \dfrac{2}{\pi} & \text{if } q = \infty. \end{cases} \tag{2}$$

In [12], it is proved that $K(q)$ is the best constant in the Sobolev inequality

$$C\|u\|_q^2 \leqslant \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0,2\pi).$$

Let us fix some notation: for a given $p \in [1,+\infty]$, we define $p^* = p/(p-1)$; also, $q^+(t) = \max\{q(t),0\}$ is the positive part of $q(t)$.

**Proposition 2.1.** *Let us assume that there are* $m$, $M > 0$ *such that any solution of* (1) *verifies* $m \leqslant x(t) \leqslant M$ *for all* $t$. *Besides, let us assume that there exists* $p \in [1,+\infty]$ *such that the following two conditions hold*:

(i) $\|\sup_{x\in[m,M]} g_x^+(t,x)\|_p < K(2p^*)$.
(ii) $\int_0^{2\pi} [\inf_{x\in[m,M]} g_x(t,x)]\,\mathrm{d}t > 0$.

*Then, if there exists a solution of problem* (1), *it is unique and elliptic.*

**Proof.** Let us assume that $x_1(t), x_2(t)$ are two different solutions of problem (1). Then, $y(t) = x_1(t) - x_2(t)$ is a periodic solution of the Hill's equation $y'' + a(t)y = 0$, where

$$a(t) = \frac{g(t,x_1(t)) - g(t,x_2(t))}{x_1(t) - x_2(t)}.$$

An elemental application of the mean value theorem together with condition (i) proves that $\|a^+\|_p < K(2p^*)$. Now, we apply [16, Lemma 2.1] (see also inequality (2.10) and Remark 2.2 of the mentioned paper) to conclude that $y(t)$ cannot vanish in $[0,2\pi]$. On the other hand, condition (ii) implies that every solution of Hill's equation is oscillatory, and the contradiction is done.

In consequence, if there is a solution $x(t)$, it is unique. In this case, the linearized equation $y'' + g_x(t,x(t))y = 0$ is stable as a consequence of the main result of [22]. □

## 3. Some new stability criteria

In this section, we prove a new stability criterion based on the third-order approximation. The stability of a given solution $u(t)$ of problem (1) can be determined in most cases by studying the stability of the equilibrium $x \equiv 0$ for the third-order approximation

$$x'' + a(t)x + b(t)x^2 + c(t)x^3 + \cdots = 0,$$

where $a, b, c$ are the coefficient of the Taylor expansion of the function $g(t, u(t) + x)$ around 0 up to degree three. Ortega's results [8–10] cover the case $b \geqslant 0$ or $b \leqslant 0$ and $c \leqslant 0$. The case of coefficients with changes of sign was studied in [7,21,5]. In view of our models, here we are interested mainly in the case $c \geqslant 0$.

For a given solution $u(t)$ of problem (1), let us define the functions

$$a(t) = g_x(t,u(t)), \quad b(t) = \tfrac{1}{2}\, g_{xx}(t,u(t)), \quad cc(t) = \tfrac{1}{6}\, g_{xxx}(t,u(t))$$

and the constants $a_* = \inf_{t\in[0,2\pi]} a(t)$, $a^* = \sup_{t\in[0,2\pi]} a(t)$, $b_* = \inf_{t\in[0,2\pi]} |b(t)|$, $b_* = \sup_{t\in[0,2\pi]} |b(t)|$, $c_* = \inf_{t\in[0,2\pi]} c(t)$, $c^* = \sup_{t\in[0,2\pi]} c(t)$.

**Theorem 3.1.** *Let us assume that there exists a solution u of problem* (1) *such that*

(i) $0 < a_* \leqslant a^* < \frac{1}{16}$,
(ii) $c_* > 0$,
(iii) $10b_*^2 a_*^{3/2} > 9c^*(a^*)^{5/2}$.

*Then, the solution u(t) is of twist type.*

**Proof.** We recall some arguments from [5, Section 3.2]. By condition (i), the linearized equation $x'' + a(t)x = 0$ is stable as a consequence of the main result in [22]. Hence, the solution $u(t)$ is elliptic. Let $\Psi(t) = \Phi_1(t) + i\Phi_2(t)$ be the (complex) solution of $x'' + a(t)x = 0$ with the initial conditions $\Psi(0) = 1, \dot{\Psi}(0) = i$. We can write $\Psi(t) = r(t)e^{i\phi(t)}$. If the characteristic multipliers are $\lambda = e^{\pm i\theta}$, we know from [5, Propositions 3.1 and 3.2] that the twist character of $u(t)$ is determined by the sign of the twist coefficient

$$\beta^* = -\frac{3}{8}\int_{[0,2\pi]} c(t)r^4(t)\,dt + \int\int_{[0,2\pi]^2} b(t)b(s)r^3(t)r^3(s)\chi_2(|\phi(t)-\phi(s)|)\,dt\,ds,$$

where the kernel $\chi_2(\cdot)$ is

$$\chi_2(x) = \frac{3}{16}\frac{\cos(x-\theta/2)}{\sin(\theta/2)} + \frac{1}{16}\frac{\cos 3(x-\theta/2)}{\sin(3\theta/2)}, \quad x \in [0,\theta].$$

In order to estimate $\beta^*$, we know that $0 < 2\pi(a_*)^{1/2} \leqslant \theta \leqslant 2\pi(a^*)^{1/2} < \pi/2$ from [5, Lemma 3.6] and $(a^*)^{-1/4} \leqslant r(t) \leqslant (a_*)^{-1/4}$ from [7, Lemma 4.2]. As it was noted in [5], function $\chi_2(x)$ is positive if $0 < \theta < \pi/2$. Hence,

$$\beta^* \geqslant -\frac{3\pi}{4}\frac{c^*}{a_*} + b_*^2(a^*)^{-5/2}\int\int_{[0,2\pi]^2}\frac{\chi_2(|\phi(t)-\phi(s)|)}{r^2(t)r^2(s)}\,dt\,ds.$$

Taking into account that $\phi' = 1/r^2$ and $\int_0^{2\pi} dt/r^2(t) = \theta$, the previous integral is

$$\int\int_{[0,2\pi]^2}\frac{\chi_2(|\phi(t)-\phi(s)|)}{r^2(t)r^2(s)}\,dt\,ds = \int\int_{[0,\theta]^2}\chi_2(|u-v|)\,du\,dv = \frac{5\theta}{12}.$$

Thus

$$\beta^* \geqslant -\frac{3\pi}{4}\frac{c^*}{a_*} + b_*^2(a^*)^{-5/2}\frac{5\theta}{12} \geqslant -\frac{3\pi}{4}\frac{c^*}{a_*} + \frac{5\pi}{6}b_*^2(a^*)^{-5/2}(a_*)^{1/2}.$$

Now, condition (iii) implies that $\beta^* > 0$ and the proof is done. □

Analogously, we have the following result

**Theorem 3.2.** *Let us assume that there exists a solution u of problem* (1) *such that*

(i) $0 < a_* \leqslant a^* < \frac{1}{16}$,
(ii) $c_* > 0$,
(iii′) $10(b^*)^2(a^*)^{3/2} < 9c_* a_*^{5/2}$.

*Then, the solution* $u(t)$ *is of twist type.*

**Proof.** The proof is analogous to that of Theorem 3.1 by using the estimates

$$\beta^* \leqslant -\frac{3\pi}{4}\frac{c_*}{a^*} + (b^*)^2(a_*)^{-5/2}\frac{5\theta}{12} \leqslant -\frac{3\pi}{4}\frac{c_*}{a^*} + \frac{5\pi}{6}(b^*)^2(a_*)^{-5/2}(a^*)^{1/2}.$$

By (iii′) it is proved that the twist coefficient is negative. □

## 4. The Lazer–Solimini equation

In this section, we apply previous results to the Lazer–Solimini equation

$$x'' - \frac{1}{x^\alpha} = p(t) - s \tag{3}$$

with $p \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, $\int_0^{2\pi} p(t)\,\mathrm{d}t = 0$ and $\alpha \geqslant 1$.

**Proposition 4.1.** *Let us assume that* $s > 0$ *and that there exist* $m, M > 0$ *such that any* $2\pi$*-periodic solution* $u(t)$ *of* (3) *verifies* $m \leqslant u(t) \leqslant M$ *for all* $t$. *Then, if* $m > (4\alpha)^{(1/\alpha+1)}$, (3) *has a unique* $2\pi$*-periodic solution* $u(t)$ *which is elliptic. Moreover, if the following conditions hold*

(i) $m > (16\alpha)^{(1/\alpha+1)}$,
(ii) $\frac{M}{m} < \left(\frac{5(\alpha+1)}{3(\alpha+2)}\right)^{2/7\alpha+11}$,

*then, the solution* $u(t)$ *is of twist type.*

**Proof.** In this case, $g(t,x) = -1/x^\alpha - p(t) + s$ and the result is a direct consequence of Proposition 2.1 with $p = +\infty$ and Theorem 3.1. □

The next step is to obtain a priori bounds for the periodic solutions of (3). Let us define

$$\varepsilon(\alpha, p) = \frac{1}{s^\alpha} - 2\pi^2 s - \pi\|p^-\|_1,$$

$$M(\alpha, p) = \frac{1}{s^\alpha} + 2\pi^2 s + \pi\|p^-\|_1,$$

for $\alpha \geqslant 1$ and

$$m(1,p) = \max\{\exp[-\ln s - \|p - s\|_1(2\pi s + \|p^-\|_1)], \varepsilon(1,p)\},$$

$$m(\alpha,p) = \max\{[s^{\alpha+1/\alpha} + (\alpha+1)\|p - s\|_1(2\pi s + \|p^-\|_1)]^{-\alpha-1}, \varepsilon(\alpha,p)\},$$

for $\alpha > 1$.

**Lemma 4.1.** *Any $2\pi$-periodic solution $u(t)$ of (3) verifies*

$$(0 <) \quad m(\alpha,p) < u(t) < M(\alpha,p) \quad \textit{for all } t \in [0, 2\pi].$$

**Proof.** Since $u(t)$ is $2\pi$-periodic, there exists $t_0$ such that $u'(t_0) = 0$. Then, for a given $t \in [t_0, t_0 + 2\pi]$,

$$-u'(t) = -\int_{t_0}^{t} u''(\tau)\,\mathrm{d}\tau < \int_{t_0}^{t} (s - p(\tau))\,\mathrm{d}\tau \leqslant 2\pi s + \|p^-\|_1,$$

$$u'(t) = -\int_{t}^{t_0+2\pi} u''(\tau)\,\mathrm{d}\tau < \int_{t}^{t_0+2\pi} (s - p(\tau))\,\mathrm{d}\tau \leqslant 2\pi s + \|p^-\|_1.$$

Thus,

$$\|u'\|_\infty < 2\pi s + \|p^-\|_1. \tag{4}$$

Besides, a simple integration over a period gives

$$\int_0^{2\pi} \frac{1}{u^\alpha(t)}\,\mathrm{d}t = 2\pi s.$$

Hence, there is $t_1$ such that $u(t_1) = s^{-1/\alpha}$. For any $t \in [t_1 - \pi, t_1 + \pi]$,

$$|u(t) - u(t_1)| = \left| \int_{t_1}^{t} u'(\tau)\,\mathrm{d}\tau \right| \leqslant \pi\|u'\|_\infty < 2\pi^2 s + \pi\|p^-\|_1.$$

Therefore,

$$\frac{1}{s^\alpha} - 2\pi^2 s - \pi\|p^-\|_1 < u(t) < \frac{1}{s^\alpha} + 2\pi^2 s + \pi\|p^-\|_1.$$

Of course, the lower bound above can be negative, so we provide an alternative lower bound. Let us define $u(t_M) = \max\{u(t)\colon t \in [0, 2\pi]\}$ and $u(t_m) = \min\{u(t)\colon t \in [t_M, t_M + 2\pi]\}$. Then, multiplying the equation by $u'(t)$ and integrating,

$$-\int_{u(t_M)}^{u(t_m)} \frac{\mathrm{d}\tau}{\tau^\alpha} = \int_{t_M}^{t_m} (p(\tau) - s)u'(\tau)\,\mathrm{d}\tau.$$

We analyze the bounds only for $\alpha = 1$, because the case $\alpha > 1$ is analogous. By using (4),

$$\ln u(t_M) - \ln u(t_m) < \|p - s\|_1\,(2\pi s + \|p^-\|_1).$$

Then, taking into account that $u(t_M) > u(t_1)$,

$$\ln u(t_m) > \ln u(t_1) - \|p - s\|_1\,(2\pi s + \|p^-\|_1),$$

so finally

$$u(t_m) > \exp[-\ln s - \|p - s\|_1\,(2\pi s + \|p^-\|_1)]. \qquad \square$$

As a consequence of these bounds and Proposition 4.1 a variety of quantitative results about stability can be proved, for instance

**Corollary 4.1.** *For any fixed* $p \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, $\int_0^{2\pi} p(t)\,\mathrm{d}t = 0$ *and* $\alpha \geqslant 1$, *there are* $s_1 > s_0 > 0$ *such that if* $0 < s < s_1$, *Eq.* (3) *has a unique* $2\pi$*-periodic solution which is elliptic, whereas if* $0 < s < s_0$, *such a solution is of twist type.*

**Proof.** It is easy to verify that $m(\alpha, p) \to +\infty$ as $s \to 0^+$, so there is $s_1 > 0$ such that $m > (4\alpha)^{1/\alpha+1}$ if $0 < s < s_1$, so the first part of the result is proved by Proposition 4.1.

By the same reason, there is $s_2$ such that $m > (16\alpha)^{1/\alpha+1}$ if $0 < s < s_2$. Hence, (i) of Proposition 4.1 holds for $0 < s < s_2$. Besides,

$$\frac{M(\alpha, p)}{m(\alpha, p)} \leqslant \frac{M(\alpha, p)}{\varepsilon(\alpha, p)} \to 1 \quad \text{when } s \to 0^+.$$

Taking into account that $5(\alpha + 1)/3(\alpha + 2) > 1$ for any $\alpha \geqslant 1$, there exists $s_3 > 0$ such that (ii) of Proposition 4.1 holds for $0 < s < s_3$. Finally, we only have to take $s_0 = \min\{s_2, s_3\}$. □

It is important to remark that this is not a result of "small parameter" type, and in fact quantitative estimations of $s_0, s_1$ can be computed, as shown in the following example.

**Example 4.1.** The equation

$$x'' - \frac{1}{x} = \sin t - s,$$

verifies Corollary 4.1 with $s_0 = 0.0009$ and $s_1 = 0.097$.

## 5. The Brillouin equation

In this section we consider the Brillouin equation

$$x''(t) + \gamma(1 + \delta \cos t)x(t) = \frac{1}{x(t)} \tag{5}$$

with $\gamma, \delta$ positive constants and $0 < \delta \leqslant 1$. There are few stability results concerning this equation. In [14] it is proved the existence of a $2\pi$-periodic elliptic solution in an adequate parameter region $\gamma - \delta$. Later, it was proved in [15] that such a region verifies a property of "practical stability" in the sense that all the points in this region provide a Lyapunov-stable (twist) solution except possibly a set of zero measure. However, up to our knowledge there are no results providing a concrete region of stability in the related literature. The following result fills partially this gap.

**Theorem 5.1.** *Let us assume the following conditions:*

(a) $\gamma(1 + \delta) < \frac{1}{32}$,

(b) $800(1 + \delta)^9 < 81\mathrm{e}^{-16\gamma\delta}(1 - \delta + \mathrm{e}^{-4\gamma\delta})^5$.

*Then, Eq.* (5) *has a* $2\pi$*-periodic solution* $u(t)$ *of twist type, which verifies also*

$$\frac{1}{\sqrt{\gamma(1+\delta)}} < u(t) < \frac{1}{\sqrt{\gamma}}\,e^{\gamma\delta(1+\cos t)} \tag{6}$$

*for all* $t$.

**Proof.** For the existence and localization of the $2\pi$-periodic solution we use the upper and lower solution method as in [14,15]. It is known that

$$\beta(t) \equiv \frac{1}{\sqrt{\gamma(1+\delta)}} < \alpha(t) = \frac{1}{\sqrt{\gamma}}\,e^{\gamma\delta(1+\cos t)}$$

are upper and lower solutions, respectively, of Eq. (5). Moreover, condition (a) makes possible the construction of a convergent scheme of upper and lower solutions leading to $2\pi$-periodic solution $u(t)$ verifying (6). See [15, Theorem 2] for details.

Now, our aim is to apply Theorem 3.2 to prove the twist character of $u(t)$. In this case, $g(t,x)=\gamma(1+\delta\cos t)x-1/x$, so it is easy to see that coefficients $a,b,c$ as defined in Section 2 are

$$a(t)=\gamma(1+\delta\cos t)+\frac{1}{u^2}, \quad b(t)=-\frac{1}{u^3}, \quad c(t)=\frac{1}{u^4}.$$

By using (6), we can estimate

$$\gamma(1-\delta)+\gamma e^{-4\gamma\delta} \leqslant a_* < a^* \leqslant 2\gamma(1+\delta),$$

$$\gamma^{3/2}e^{-6\gamma\delta} \leqslant b_* < b^* \leqslant \gamma^{3/2}(1+\delta)^{3/2},$$

$$\gamma^2 e^{-8\gamma\delta} \leqslant c_* < c^* \leqslant \gamma^2(1+\delta)^2.$$

With these bounds, it is just a matter of elementary computations to verify that conditions (a) and (b) imply that assumptions of Theorem 3.2 hold. □

**Remark 5.1.** As a direct consequence, for any value $0 \leqslant \gamma < 1/32$, there exists $\delta_0(\gamma) > 0$ such that if $0 < \delta < \delta_0(\gamma)$, Eq. (5) has a $2\pi$-periodic solution $u(t)$ of twist type. The region of parameters fulfilling the assumptions of Theorem 5.1 is drawn in Fig. 1.

**Remark 5.2.** In the line of [7], a more general stability criterion based on the existence of upper and lower solutions in the reversed order can be written down. We have preferred this exposition through practical examples by reason of clearness.

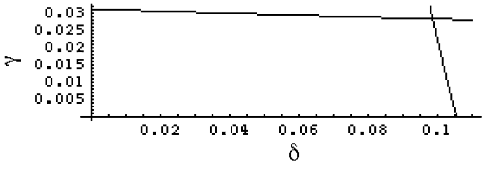


Fig. 1. Region of stability for the Brillouin equation.

## References

[1] V. Bevc, J.L. Palmer, C. Süsskind, On the design of the transition region of axi-symmetric magnetically focusing beam valves, J. Br. Inst. Radio Eng. 18 (1958) 696–708.
[2] T. Ding, A boundary value problem for the periodic Brillouin focusing system, Acta Sci. Natur. Univ. Pekinensis 11 (1965) 31–38 (in Chinese).
[3] P. Habets, L. Sanchez, Periodic solutions of some Liénard equations with singularities, Proc. Am. Math. Soc. 109 (1990) 1035–1044.
[4] A.C. Lazer, S. Solimini, On periodic solutions of nonlinear differential equations with singularities, Proc. Am. Math. Soc. 99 (1987) 109–114.
[5] J. Lei, X. Li, P. Yan, M. Zhang, Twist character of the least amplitude periodic solution of the forced pendulum, SIAM J. Math. Anal., to appear.
[6] J. Mawhin, Topological Degree and Boundary Value Problems for Nonlinear Differential Equations, Lectures Notes in Mathematics, Vol. 1537, Springer, Berlin, 1991, pp. 74–143.
[7] D. Nunez, The method of lower and upper solutions and the stability of periodic oscillations, Nonlinear Anal. 51 (2002) 1207–1222.
[8] R. Ortega, The twist coefficient of periodic solutions of a time-dependent Newton's equation, J. Dyn. Differential Equations 4 (1992) 651–665.
[9] R. Ortega, The stability of equilibrium of a nonlinear Hill's equation, SIAM J. Math. Anal. 25 (1994) 1393–1401.
[10] R. Ortega, Periodic solutions of a newtonian equation: stability by the third approximation, J. Differential Equations 128 (1996) 491–518.
[11] C. Siegel, J. Moser, Lectures on Celestial Mechanics, Springer, Berlin, 1971.
[12] G. Talenti, Best constant in Sobolev inequality, Ann. Math. Pura Appl. 110 (4) (1976) 353–372.
[13] P.J. Torres, Bounded solutions in singular equations of repulsive type, Nonlinear Anal. 32 (1998) 117–125.
[14] P.J. Torres, Existence and uniqueness of elliptic periodic solutions of the Brillouin electron beam focusing system, Math. Meth. Appl. Sci. 23 (2000) 1139–1143.
[15] P.J. Torres, Twist solutions of a Hill's equations with singular term, Adv. Nonlinear Stud. 2 (2002) 279–287.
[16] P.J. Torres, M. Zhang, A monotone iterative scheme for a nonlinear second order equation based on a generalized anti-maximum principle, Math. Nach. 251 (2003) 101–107.
[17] P. Yan, M. Zhang, Higher order nonresonance for differential equations with singularities, Math. Meth. Appl. Sci. 26 (2003) 1067–1074.
[18] Y. Ye, X. Wang, Nonlinear differential equations in electron beam focusing theory, Acta Math. Appl. Sinica 1 (1978) 13–41 (in Chinese).
[19] M. Zhang, Periodic solutions of Liénard equations with singular forces of repulsive type, J. Math. Anal. Appl. 203 (1996) 254–269.
[20] M. Zhang, A relationship between the periodic and the Dirichlet BVPs of singular differential equations, Proc. Roy. Soc. Edinburgh Sec. A 128 (1998) 1099–1114.
[21] M. Zhang, The best bound on the rotations in the stability of periodic solutions of a Newtonian equation, J. London Math. Soc. 67 (2003) 137–148.
[22] M. Zhang, W. Li, A Lyapunov-type stability criterion using $L^\alpha$ norms, Proc. Am. Math. Soc. 130 (2002) 3325–3333.