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ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER SINGULAR EQUATIONS

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Abstract. The problem on the existence of a positive in the interval $]a,b[$ solution of the boundary value problem

$$u'' = f(t,u) + g(t,u)u'; \quad u(a+) = 0, \quad u(b-) = 0$$

is considered, where the functions f and $g\colon\ ]a,b[\times]0,+\infty[\ \to \mathbb{R}$ satisfy the local Carathéodory conditions. The possibility for the functions f and g to have singularities in the first argument (for $t=a$ and $t=b$) and in the phase variable (for $u=0$) is not excluded. Sufficient and, in some cases, necessary and sufficient conditions for the solvability of that problem are established.

Keywords: second order singular equation, two-point boundary value problem, solvability

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1. Statement of the problem and formulation of main results

The following notation is used throughout the paper.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0,+\infty[$.

$L(]a,b[;D)$, where $D\subset\mathbb{R}$, is the set of functions $p\colon\ ]a,b[\ \to D$ which are Lebesgue integrable on the segment $[a,b]$.

$L_{\rm loc}(]a,b[;D)$, where $D\subset\mathbb{R}$, is the set of functions $u\colon\ ]a,b[\ \to D$ which are Lebesgue integrable on each segment contained in $]a,b[$.

$C([a,b];D)$, where $D\subset\mathbb{R}$, is the set of continuous functions $u\colon\ [a,b]\to D$.

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$AC'([a,b];D)$, where $D \subset \mathbb{R}$, is the set of functions $u\colon [a,b] \to D$ which are absolutely continuous together with their first derivatives on $[a,b]$.

$AC'_{\rm loc}(I;D)$, where $I \subseteq]a,b[$, $D \subset \mathbb{R}$, is the set of functions $u\colon I \to D$ such that $u \in AC'(I_0;D)$ for every segment $I_0 \subset I$.

$\mathrm{Car}(]a,b[\times D;\mathbb{R})$, where $D \subset \mathbb{R}$, is the Carathéodory class, i.e., the set of functions $f\colon]a,b[\times D \to \mathbb{R}$ such that $f(t,\cdot)\colon D \to \mathbb{R}$ is continuous for almost all $t \in]a,b[$, $f(\cdot,x)\colon]a,b[\to \mathbb{R}$ is measurable for all $x \in D$, and

$$\sup\{|f(\cdot,x)|\colon x \in D_0)\} \in L(]a,b[;\mathbb{R}_+)$$

for any compact $D_0 \subset D$.

$\mathrm{Car}_{\rm loc}(]a,b[\times D;\mathbb{R})$, where $D \subset \mathbb{R}$, is the set of functions $f\colon]a,b[\times D \to \mathbb{R}$ whose restrictions to $[a+\varepsilon, b-\varepsilon]\times D$ belong to $\mathrm{Car}([a+\varepsilon,b-\varepsilon]\times D;\mathbb{R})$ for any $\varepsilon \in]0, \frac{1}{2}(b-a)[$.

$[p]_- = \frac{1}{2}(|p|-p)$.

$u(s+)$ and $u(s-)$ are one-sided limits of the function u at the point s from the right and from the left, respectively.

Consider the boundary value problem

$$u'' = f(t,u) + g(t,u)u', \tag{1.1}$$

$$u(a+) = 0, \quad u(b-) = 0, \tag{1.2}$$

where $f, g \in \mathrm{Car}_{\rm loc}(]a,b[\times]0,+\infty[;\mathbb{R})$. Under a solution of problem (1.1), (1.2) we understand a function $u \in AC'_{\rm loc}(]a,b[;]0,+\infty[)$ satisfying equation (1.1) almost everywhere in $]a,b[$ and boundary conditions (1.2).

The aim of the present paper is to investigate the problem of solvability of problem (1.1), (1.2) provided the functions f and g possess singularities both in the independent (for $t=a$ and $t=b$) and in the phase (for $u=0$) variable. Singular problems of such a type arise frequently in applications (cf., for example, [1], [3], [4], [6]–[8], [22]–[24]). The first essential step in their investigation was made by S. Taliaferro in his work [25] in which he established a necessary and sufficient condition for the solvability of problem (1.1), (1.2) with $g(t,x) \equiv 0$ and $f(t,x) = -h(t)/x^\lambda$, where $\lambda > 0$ and $h \in L_{\rm loc}(]a,b[;\mathbb{R}_+)$. Problem (1.1), (1.2) has been more often considered in the case when the function g does not depend on the second argument, and $f(t,x) \leqslant 0$ for $a<t<b$, $x>0$ (cf., for example, [1]–[18], [21]–[27] and references therein). In that case, equation (1.1) is in its turn easily reduced to a two-term equation of the type

$$u'' = f(t,u) \tag{1.1$_0$}$$

with nonpositive right-hand side. The restriction on the sign of the function f was overcome for the first time in [19], where criteria for the solvability of problem (1.1_0), (1.2) were established under the assumption that

$$\int_a^b (s-a)(b-s) f_r^*(s)\,\mathrm{d}s < +\infty \quad \text{for} \quad r > 1, \tag{1.3}$$

where $f_r^*(\cdot) = \max\{|f(\cdot,x)|\colon \frac{1}{r} \leqslant x \leqslant r\}$ for $r > 1$ (see also later works [12]–[14]). In [17], the equation of a more general type $u'' = H(t,u,u')$ was considered, but again under the assumption that the function H is nonpositive. So, in spite of a large number of publications, the question of the solvability of problem (1.1), (1.2) has not yet been studied throughly enough. Below we will give new sufficient, and in some cases, necessary and sufficient conditions for the solvability of problem (1.1), (1.2). Moreover, as it has been noted above, the possibility for both the functions f and g to have singularities in the first argument and in the phase variabe, is not excluded. Note also that Theorem 1.2 below enables one to establish criteria for the solvability of problem (1.1), (1.2) even in the case when condition (1.3) is not fulfilled.

Before we proceed to formulating the main results, we introduce the following definition.

Definition 1.1. The continuous function $\sigma\colon\,]a,b[\; \to \;]0,+\infty[$ is said to be a lower (upper) function of equation (1.1) if $\sigma \in AC'_{\mathrm{loc}}(]a,b[\setminus\{t_1,t_2,\dots,t_n\};]0,+\infty[)$, where $a < t_1 < t_2 < \ldots < t_n < b$, there exist finite limits $\sigma(a+)$, $\sigma(b-)$, $\sigma'(t_i+)$, $\sigma'(t_i-)$, $i=\overline{1,n}$,

$$\sigma'(t_i-) < \sigma'(t_i+) \quad (\sigma'(t_i-) > \sigma'(t_i+)), \quad i=\overline{1,n},$$

and almost everywhere in $]a,b[$ the inequality

$$\sigma''(t) \geqslant f(t,\sigma(t)) + g(t,\sigma(t))\sigma'(t) \quad (\sigma''(t) \leqslant f(t,\sigma(t)) + g(t,\sigma(t))\sigma'(t))$$

is fulfilled.

Definition 1.1 is a particular case of the definition of lower and upper functions introduced in [15] (see also [17] and [19]).

Theorem 1.1. *Let σ_1 and σ_2 be respectively lower and upper functions of equation* (1.1) *and let*

$$\sigma_1(t) \leqslant \sigma_2(t) \quad \text{for} \quad a < t < b, \tag{1.4}$$
$$\sigma_1(a+) = 0, \;\; \sigma_1(b-) = 0, \;\; \sigma_2(a+) \neq 0, \;\; \sigma_2(b-) \neq 0.$$

Assume, moreover, that for every $0 < \eta < \min\{\sigma_2(t)\colon\ a \leqslant t \leqslant b\}$ *there exist* $\gamma \in \,]a,b[$ *and functions* $p_\eta, q_\eta \in L_{\rm loc}(]a,b[;\mathbb{R}_+)$ *such that*

$$\int_a^b q_\eta(s)\,{\rm d}s < +\infty, \qquad \int_a^b (s-a)(b-s)p_\eta(s)\,{\rm d}s < +\infty \tag{1.5}$$

and

$$|f(t,x)| \leqslant p_\eta(t), \quad g(t,x)\,{\rm sgn}(\gamma - t) \geqslant -q_\eta(t)$$
$$\text{for } a < t < b, \ \sigma_{1\eta}(t) \leqslant x \leqslant \sigma_2(t),$$

where

$$\sigma_{1\eta}(\cdot) = \max\{\eta, \sigma_1(\cdot)\}. \tag{1.6}$$

Then the problem (1.1), (1.2) *has at least one solution* u *such that*

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t) \quad \text{for} \quad a < t < b. \tag{1.7}$$

Remark 1.1. Theorem 1.1 covers the case when both the functions f and g have nonintegrable singularities with respect to the independent variable. Note that singularities of the function g may be "sufficiently large". As an example, consider the problem

$$u'' = \frac{\lambda}{t(1-t)} - \frac{\lambda}{t(1-t)u^3} + \Big(1 + \frac{1-2t}{t^2(1-t)^2}\Big)\frac{u'}{u^2}, \tag{1.8}$$
$$u(0+) = 0, \quad u(1-) = 0,$$

where $\lambda > 0$. We can easily see that $\sigma_2(t) \equiv 1$ is an upper function of the equation, and $\sigma_1(t) = \varepsilon t(1-t)$ for $0 \leqslant t \leqslant 1$, where $0 < \varepsilon < \frac{\lambda}{2+\lambda}$ is a lower function. Putting now $\gamma = \frac{1}{2}$, by Theorem 1.1 problem (1.8) has at least one solution.

Corollary 1.1. *Let a function* f *be nondecreasing in the second argument and let there exist* $r > 0$ *such that*

$$f(t,r) \leqslant 0 \quad \text{for} \quad a < t < b, \tag{1.9}$$

with the strict inequality on the subset of $]a,b[$ *of a positive measure. Let, moreover, on the set* $]a,b[\,\times\,]0,+\infty[$ *the inequality*

$$|g(t,x)| \leqslant q^*(t) \tag{1.10}$$

hold, where $q^* \in L(]a,b[;\mathbb{R}_+)$. *Then the condition*

$$\int_a^b (s-a)(b-s)|f(s,x)|\,{\rm d}s < +\infty \quad \text{for} \quad 0 < x \leqslant r \tag{1.11}$$

is necessary and sufficient for the solvability of problem (1.1), (1.2).

Remark 1.2. In the case when the function g does not depend on the second argument, problem (1.1), (1.2) under the conditions of Corollary 1.1 is uniquely solvable. Note also that for $g(t,x) \equiv 0$ the above corollary implies Theorem 1.2 in [19] (see also [17], Theorem 4.3_1]).

Corollary 1.2. *Let a function f be nondecreasing in the second argument and there exist $r > 0$ such that conditions* (1.9) *are fulfilled. Let, moreover, there exist $c \in \,]a,b[$ such that the mapping $(t,x) \longmapsto g(t,x)\operatorname{sgn}(c-t)$ is nondecreasing in the second argument, and*

$$\int_a^b |g(s,x)|\,\mathrm{d}s < +\infty \quad \textit{for} \quad x > 0.$$

Then condition (1.11) *is necessary and sufficient for the solvability of problem* (1.1), (1.2).

As an example, consider the equation

$$u'' = h(t)\Big(\delta u^{\alpha} - \frac{1}{u^{\lambda}}\Big) + \frac{g(t)}{u^{\mu}}u' - \varphi(t), \tag{1.12}$$

where $\delta \geqslant 0$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $h, \varphi \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$, $h(t) \not\equiv 0$, $c \in \,]a,b[$, and

$$g(t)\operatorname{sgn}(c-t) \leqslant 0 \quad \text{for} \quad a < t < b, \qquad \int_a^b |g(s)|\,\mathrm{d}s < +\infty.$$

Then by Corollary 1.2, for the solvability of the problem (1.12), (1.2) it is necessary and sufficient to have

$$\int_a^b (s-a)(b-s)h(s)\,\mathrm{d}s < +\infty \quad \text{and} \quad \int_a^b (s-a)(b-s)\varphi(s)\,\mathrm{d}s < +\infty. \tag{1.13}$$

Consider now the equation

$$u'' = \frac{h(t)}{u}, \tag{1.14}$$

where $h \in L_{\mathrm{loc}}(]a,b[;\mathbb{R})$ can, in general, change its sign.

Corollary 1.3. *Let the function h admit the representation*

$$h(t) = p(t) - q(t), \quad p(t) \geqslant 0, \quad q(t) \geqslant 0 \quad \textit{for} \quad a < t < b,$$

where

$$\int_a^b \frac{p(s)}{(s-a)(b-s)}\,\mathrm{d}s < +\infty, \qquad \int_a^b (s-a)(b-s)q(s)\,\mathrm{d}s < +\infty$$

and

$$(1.15)\qquad \begin{aligned} &(b-t)\int_a^t (s-a)q(s)\,\mathrm{d}s + (t-a)\int_t^b (b-s)q(s)\,\mathrm{d}s \\ &\quad > (t-a)(b-t)\sqrt{b-a}\sqrt{\int_a^b (s-a)(b-s)q(s)\,\mathrm{d}s} \\ &\quad \times \exp\left[\frac{2}{b-a}\int_a^b \frac{p(s)}{(s-a)(b-s)}\,\mathrm{d}s\right] \quad \textit{for} \quad a<t<b. \end{aligned}$$

Then problem (1.14), (1.2) *has at least one solution.*

According to Corollary 1.3, for example the problem

$$u'' = \left(\frac{kt(1-t)}{\sqrt{|2t-1|}} - \lambda\right)\frac{1}{u}; \quad u(0+) = 0, \quad u(1-) = 0,$$

where $k > 0$ and $\lambda > \frac{2}{3}\mathrm{e}^{8k}$, has at least one solution.

Theorem 1.2. *Let σ_1 and σ_2 be respectively lower and upper functions of equation* (1.1) *satisfying conditions* (1.4). *Assume, moreover, that for every $0 < \eta < \min\{\sigma_2(t)\colon\ a \leqslant t \leqslant b\}$ there exist functions $p_\eta, q_\eta \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$ such that conditions* (1.5) *and*

$$f(t,x) \geqslant -p_\eta(t), \quad |g(t,x)| \leqslant q_\eta(t) \quad \textit{for} \quad a<t<b, \quad \sigma_{1\eta}(t) \leqslant x \leqslant \sigma_2(t)$$

are fulfilled, where $\sigma_{1\eta}$ is the function defined by (1.6). *Then problem* (1.1), (1.2) *has at least one solution.*

Remark 1.3. Theorem 1.2 covers also the case when condition (1.3) is not fulfilled for the function f. Indeed, consider the problem

$$(1.16)\qquad u'' = u - \frac{1}{u^2} + \frac{\lambda(1-2t)}{u^2}u' + \frac{\alpha}{t^2(1-t)^2}; \quad u(0+) = 0, \quad u(1-) = 0,$$

where $\alpha > 0$ and $\lambda \in \mathbb{R}$. It is easily seen that $\sigma_1(t) = \varepsilon t(1-t)$, where $0 < \varepsilon < \frac{1}{1+\alpha+|\lambda|}$, is a lower function and $\sigma_2(t) \equiv 1$ is an upper function. Consequently, by Theorem 1.2, problem (1.16) is solvable. It should also be noted that in the case $\lambda \leqslant 0$ and $\alpha < 0$, problem (1.16) by Corollary 1.2 has no solution.

From Theorem 1.2, for the equation

$$u'' = h(t)u^{\lambda} - \frac{p(t)}{u^{\mu}} + \varphi(t), \tag{1.17}$$

where $\lambda > 0$ and $\mu > 0$, we obtain the following

Corollary 1.4. *Let $r > 0$, $n > 0$, $p_0 > 0$, $p, h, \varphi \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$, and*

$$\int_a^b (s-a)(b-s)p(s)\,\mathrm{d}s < +\infty,$$

$$p_0 \leqslant p(t), \quad (\varphi(t) + h(t))[(t-a)(b-t)]^n \leqslant r \quad \text{for} \quad a < t < b.$$

Then problem (1.17), (1.2) *is uniquely solvable.*

As is readily seen from this corollary, the functions h and φ need not satisfy conditions (1.13).

2. Some auxiliary propositions

In this section, lemmas on a priori estimates and a lemma on the solvability of problem (1.1), (1.2) will be established in the case when $f, g \in \mathrm{Car}_{\mathrm{loc}}(]a,b[\times\mathbb{R};\mathbb{R})$. Everywhere in what follows, functions $h_1, h_2 \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$ will be assumed to satisfy the conditions

$$\int_a^b (s-a)(b-s)h_1(s)\,\mathrm{d}s < +\infty, \qquad \int_a^b h_2(s)\,\mathrm{d}s < +\infty. \tag{2.1}$$

Lemma 2.1. *Let $r_0 > 0$, and let $h_1, h_2 \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$ satisfy conditions* (2.1). *Then there exist $c_0 > 0$ and functions $H_1 \in C([a, \frac{a+b}{2}];\mathbb{R}_+)$, $H_2 \in C([\frac{a+b}{2}, b];\mathbb{R}_+)$ satisfying the conditions $H_1(a) = 0$, $H_2(b) = 0$ and such that for any $a_1 \in]a, \frac{a+b}{2}[$, $b_1 \in]\frac{a+b}{2}, b[$ and $u \in AC'([a_1,b_1];\mathbb{R})$ satisfying the inequalities*

$$\begin{aligned} &u''(t) \geqslant -h_1(t) - h_2(t)|u'(t)| && \text{for} \quad a_1 < t < b_1,\\ &|u(t)| \leqslant r_0 && \text{for} \quad a_1 < t < b_1, \end{aligned} \tag{2.2}$$

the following estimates hold:

$$(t-a_1)(b_1-t)|u'(t)| \leqslant c_0 \quad \text{for } a_1 < t < b_1, \tag{2.3}$$

$$\begin{aligned} &u(t) \leqslant u(a_1) + H_1(t) \quad \text{for } a_1 \leqslant t \leqslant \frac{a+b}{2},\\ &u(t) \leqslant u(b_1) + H_2(t) \quad \text{for } \frac{a+b}{2} \leqslant t \leqslant b_1. \end{aligned} \tag{2.4}$$

P r o o f. Let a function $u \in AC'([a_1, b_1]; \mathbb{R})$ satisfy the conditions of the lemma. Suppose

$$\psi_0(t) = -h_2(t) \operatorname{sgn} u'(t), \quad \psi_1(t) = u''(t) + h_2(t)|u'(t)| \quad \text{for } a_1 < t < b_1.$$

Clearly, u is a solution of the equation

$$u'' = \psi_0(t)u' + \psi_1(t) \tag{2.5}$$

and

$$\psi_1(t) \geqslant -h_1(t) \quad \text{for} \quad a_1 < t < b_1. \tag{2.6}$$

Let $t_0 \in \,]a_1, b_1[$ be an arbitrary point such that $u'(t_0) \neq 0$. Then either

$$u'(t_0) > 0 \tag{2.7}$$

or

$$u'(t_0) < 0. \tag{2.8}$$

Suppose that (2.7) ((2.8)) is satisfied. Put

$$\mu(t) = \int_t^{b_1} \exp\left[\int_t^s \psi_0(\xi)\,\mathrm{d}\xi\right] \mathrm{d}s \quad \text{for} \quad t_0 < t < b_1$$
$$\left(\mu(t) = \int_{a_1}^t \exp\left[\int_t^s \psi_0(\xi)\,\mathrm{d}\xi\right] \mathrm{d}s \quad \text{for} \quad a_1 < t < t_0\right).$$

Multiplying both sides of (2.5) by μ and integrating from t_0 to b_1 (from a_1 to t_0), we obtain

$$-\mu(t_0)u'(t_0) - u(b_1) + u(t_0) = \int_{t_0}^{b_1} \mu(s)\psi_1(s)\,\mathrm{d}s$$
$$\left(\mu(t_0)u'(t_0) + u(t_0) - u(a_1) = \int_{a_1}^{t_0} \mu(s)\psi_1(s)\,\mathrm{d}s\right).$$

Hence by (2.2), (2.6) and (2.7) ((2.8)) we get

$$\begin{gathered}\mu(t_0)|u'(t_0)| \leqslant 2r_0 + \int_{t_0}^{b_1} h_1(s)\mu(s)\,\mathrm{d}s \\ \left(\mu(t_0)|u'(t_0)| \leqslant 2r_0 + \int_{a_1}^{t_0} h_1(s)\mu(s)\,\mathrm{d}s\right).\end{gathered} \tag{2.9}$$

We can easily check that

$$\begin{aligned} h_0^{-1}(b_1-t) &\leqslant \mu(t) \leqslant h_0(b_1-t) \quad \text{for} \quad t_0<t<t_1 \\ \big(h_0^{-1}(t-a_1) &\leqslant \mu(t) \leqslant h_0(t-a_1) \quad \text{for} \quad a_1<t<t_0\big), \end{aligned}$$

where

$$h_0 = \exp\bigg[\int_a^b h_2(s)\,\mathrm{d}s\bigg]. \tag{2.10}$$

Taking this into account, from (2.9) we find that inequality (2.3), where

$$c_0 = \bigg[2r_0(b-a) + \int_a^b (s-a)(b-s)h_1(s)\,\mathrm{d}s\bigg] \exp\bigg[2\int_a^b h_2(s)\,\mathrm{d}s\bigg], \tag{2.11}$$

is fulfilled.

□

Let us now show that estimates (2.4) are satisfied, where

$$\begin{aligned} H_1(t) = \bigg[&\frac{4r_0}{b-a}(t-a) + \int_a^t (s-a)h_1(s)\,\mathrm{d}s \\ &+ (t-a)\int_t^{\frac{a+b}{2}} h_1(s)ds\bigg] h_0^2 \quad \text{for} \quad a<t\leqslant \frac{a+b}{2}, \\ H_2(t) = \bigg[&\frac{4r_0}{b-a}(b-t) + \int_t^b (b-s)h_1(s)\,\mathrm{d}s \\ &+ (b-t)\int_{\frac{a+b}{2}}^t h_1(s)\,\mathrm{d}s\bigg] h_0^2 \quad \text{for} \quad \frac{a+b}{2}\leqslant t<b. \end{aligned} \tag{2.12}$$

Here the number h_0 is defined by (2.10).

Let us denote by w_1 a solution of the boundary value problem

$$w'' = \psi_0(t)w' - h_1(t) \quad w(a_1)=0, \quad w\Big(\frac{a+b}{2}\Big) = 2r_0$$

and show that

$$\widetilde{w}(t) = u(t) - u(a_1) - w_1(t) \leqslant 0 \quad \text{for} \quad a_1<t_0<\frac{a+b}{2}. \tag{2.13}$$

Assume the contrary. Let (2.13) be violated. Then there exist $t_* \in [a_1, \frac{a+b}{2}[$ and $t^* \in\,]t_*, \frac{a+b}{2}]$ such that

$$\widetilde{w}(t) > 0 \quad \text{for} \quad t_*<t<t^*, \quad \widetilde{w}(t_*)=0 \quad \text{and} \quad \widetilde{w}(t^*)=0. \tag{2.14}$$

Taking into consideration (2.5) and (2.6), we obtain

$$\widetilde{w}''(t) \geqslant \psi_0(t)\widetilde{w}'(t) \quad \text{for} \quad t_* < t < t^*.$$

However, this contradicts condition (2.14). Thus (2.13) is valid.

We can directly verify that

$$w_1(t) \leqslant H_1(t) \quad \text{for} \quad a_1 \leqslant t \leqslant \frac{a+b}{2},$$

where H_1 is the function defined by (2.12). Consequently, the first of inequalities (2.4) is valid.

Analogously one can prove that

$$u(t) \leqslant u(b_1) + w_2(t), \quad w_2(t) \leqslant H_2(t) \quad \text{for} \quad \frac{a+b}{2} \leqslant t \leqslant b_1,$$

where w_2 is a solution of the boundary value problem

$$w'' = \psi_0(t)w' - h_1(t); \quad w\Big(\frac{a+b}{2}\Big) = 2r_0, \quad w(b_1) = 0.$$

□

Lemma 2.2. *Let $r_0 > 0$, $\alpha \in \,]a, \frac{a+b}{2}[$, $\beta \in \,]\frac{a+b}{2}, b[$, $\gamma \in \,]\alpha,\beta[$, and let $h_1, h_2 \in L_{\rm loc}(]a,b[;\mathbb{R}_+)$ satisfy conditions* (2.1)*. Then there exists a function $\varphi \in L(]a,b[;\mathbb{R}_+)$ such that φ is bounded in $]a,b[$, and for any $a_1 \in \,]a,\alpha[$, $b_1 \in \,]\beta,b[$, and a function $u \in AC'([a_1,b_1];\mathbb{R})$ satisfying condition* (2.2) *and inequalities*

$$u''(t)\,\mathrm{sgn}((\gamma - t)u'(t)) \geqslant -h_1(t) - h_2(t)|u'(t)| \quad \textit{for} \quad a_1 < t < b_1, \tag{2.15}$$

$$u''(t) \geqslant -h_1(t) - h_2(t)|u'(t)| \quad \textit{for} \quad \alpha < t < \beta, \tag{2.16}$$

the estimate

$$|u'(t)| \leqslant \varphi(t) \quad \textit{for} \quad a_1 < t < b_1 \tag{2.17}$$

holds.

P r o o f. Let $u \in AC'([a_1,b_1];\mathbb{R})$ satisfy the conditions of the lemma. By virtue of (2.16), (2.2) and Lemma 2.1, the estimate

$$|u'(\gamma)| \leqslant \frac{c_0}{(\gamma-\alpha)(\beta-\gamma)} \tag{2.18}$$

is fulfilled, where c_0 is the number given by (2.11). Let us show that

$$|u'(t)| \leqslant \varphi_0(t) \quad \text{for} \quad a_1 < t < \gamma, \tag{2.19}$$

where

$$\varphi_0(t) = \left(\frac{c_0}{(\gamma - \alpha)(\beta - \gamma)} + \int_t^\gamma h_1(s)\,\mathrm{d}s \right) \exp\left[\int_a^b h_2(s)\,\mathrm{d}s \right] \quad \text{for} \quad a < t < \gamma. \tag{2.20}$$

Assume the contrary. Let (2.19) be violated. Then by (2.18) and (2.20), there exist $t_* \in \,]a_1, \gamma[$ and $t^* \in \,]t_*, \gamma[$ such that

$$\begin{aligned} |u'(t)| &> \varphi_0(t) \quad \text{for} \quad t_* \leqslant t < t^*, \\ |u'(t^*)| &= \frac{c_0}{(\gamma - \alpha)(\beta - \gamma)}. \end{aligned} \tag{2.21}$$

For the sake of definiteness we assume that $u'(t) > 0$ for $t_* < t < t^*$. Then by virtue of the lemma on the differential inequality, from (2.15) we find

$$|u'(t_*)| \leqslant \left(|u'(t^*)| + \int_{t_*}^{t^*} \exp\left[\int_{t_*}^s h_2(\xi)\,\mathrm{d}\xi \right] h_1(s)\,\mathrm{d}s \right) \leqslant \varphi_0(t_*).$$

However, the last inequality contradicts condition (2.21). Consequently, estimate (2.19) is valid.

Analogously we can show that

$$|u'(t)| \leqslant \varphi_1(t) \quad \text{for} \quad \gamma \leqslant t < b_1, \tag{2.22}$$

where

$$\varphi_1(t) = \left(\frac{c_0}{(\gamma - \alpha)(\beta - \gamma)} + \int_\gamma^t h_1(s)\,\mathrm{d}s \right) \exp\left[\int_a^b h_2(s)\,\mathrm{d}s \right] \quad \text{for} \quad \gamma < t < b.$$

Suppose now

$$\varphi(t) = \begin{cases} \varphi_0(t) & \text{for } a < t \leqslant \gamma \\ \varphi_1(t) & \text{for } \gamma < t < b \end{cases}.$$

It is evident from (2.19) and (2.22) that estimate (2.17) is satisfied. It is not also difficult to see that the function φ is bounded in $]a, b[$, and $\varphi \in L(]a, b[; \mathbb{R}_+)$. $\square$

Lemma 2.3. *Let* $f, g \in \mathrm{Car}_{\mathrm{loc}}(]a,b[\times\mathbb{R};\mathbb{R})$, *and let* σ_1 *and* σ_2 *be lower and upper functions of equation* (1.1) *satisfying conditions* (1.4). *Moreover, let there exist functions* $p_0, q_0 \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$ *such that*

$$\int_a^b (s-a)(b-s)p_0(s)\,\mathrm{d}s < +\infty, \quad \int_a^b q_0(s)\,\mathrm{d}s < +\infty$$

and

$$f(t,x) \geqslant -p_0(t), \ |g(t,x)| \leqslant q_0(t) \ \text{for } a<t<b, \ \sigma_1(t) \leqslant x \leqslant \sigma_2(t), \tag{2.23}$$

$$(|f(t,x)| \leqslant p_0(t), \ g(t,x)\,\mathrm{sgn}(c-t) \geqslant -q_0(t) \ \text{for } a<t<b, \ \sigma_1(t) \leqslant x \leqslant \sigma_2(t)). \tag{2.24}$$

Then problem (1.1), (1.2) *has at least one solution* u *satisfying* (1.7).

P r o o f. Let conditions (2.23) be fulfilled. Choose sequences $(t_{ik})_{k=1}^{+\infty}$, $(s_{ik})_{k=1}^{+\infty}$, and $(c_{ik})_{k=1}^{+\infty}$ $(i=1,2)$ such that

$$a < t_{1k+1} < t_{1k} < \frac{a+b}{2} < t_{2k} < t_{2k+1} < b, \quad k=1,2,3,\dots,$$

$$s_{1k+1} \in]t_{1k+1}, t_{1k}[, \ s_{2k+1} \in]t_{2k}, t_{2k+1}[, \ t_{11} < s_{11} < \frac{a+b}{2} < s_{21} < t_{21},$$

$$c_{1k} = \sigma_1(t_{1k}), \quad c_{2k} = \sigma_1(t_{2k}), \quad k=1,2,3,\dots, \tag{2.25}$$

$$\lim_{k\to+\infty} t_{1k} = a, \quad \lim_{k\to+\infty} t_{2k} = b. \tag{2.26}$$

Let $r_0 = \max\{|\sigma_1(t)| + |\sigma_2(t)| \colon a \leqslant t \leqslant b\}$ and let c_0 be the number defined by (2.11), where $h_1(t) = p_0(t)$, $h_2(t) = q_0(t)$ for $a<t<b$,

$$\widetilde{\varphi}(t) = \begin{cases} \dfrac{c_0}{(t-t_{1k+1})(t_{2k+1}-t)} & \text{for } t \in]s_{1k+1}, s_{1k}[\cup]s_{2k}, s_{2k+1}[, \\[2ex] \dfrac{c_0}{(t-t_{11})(t_{21}-t)} & \text{for } t \in]s_{11}, s_{21}[, \end{cases} \tag{2.27}$$

$$\varrho(t) = \widetilde{\varphi}(t) + |\sigma_1'(t)| + |\sigma_2'(t)| + 1 \ \text{for } a<t<b, \tag{2.28}$$

$$F(t,x,y) = f(t,x) + g(t,x)y \ \text{for } a<t<b, \ x,y \in \mathbb{R},$$

$$\widetilde{F}(t,x,y) = \begin{cases} F(t,x,y) & \text{for } |y| \leqslant \varrho(t), \\ \Big(2 - \dfrac{|y|}{\varrho(t)}\Big)F(t,x,y) & \text{for } \varrho(t) < |y| \leqslant 2\varrho(t), \\ 0 & \text{for } |y| \geqslant 2\varrho(t), \end{cases} \tag{2.29}$$

$$q(t) = |g(t, \sigma_1(t))| + |g(t, \sigma_2(t))| \quad \text{for } a < t < b,$$

$$(2.30)\quad F_0(t,x,y) = \begin{cases} \widetilde{F}(t, \sigma_1(t), y) + \dfrac{\sigma_1(t) - x}{1 + |\sigma_1(t) - x|} q(t) & \text{for } x \leqslant \sigma_1(t), \\ \widetilde{F}(t, x, y) & \text{for } \sigma_1(t) < x < \sigma_2(t), \\ \widetilde{F}(t, \sigma_2(t), y) + \dfrac{x - \sigma_2(t)}{1 + |x - \sigma_2(t)|} q(t) & \text{for } x \geqslant \sigma_2(t). \end{cases}$$

We can easily see that $F_0 \in \mathrm{Car}(]t_{1k}, t_{2k}[\times \mathbb{R}^2; \mathbb{R})$ and for any natural k there exists a function $F_k^* \in L(]t_{1k}, t_{2k}[; \mathbb{R}_+)$ such that

$$|F_0(t,x,y)| \leqslant F_k^*(t) \quad \text{for} \quad t_{1k} < t < t_{2k}, \quad x, y \in \mathbb{R}.$$

Therefore the boundary value problem

$$(2.31)\qquad u'' = F_0(t, u, u'),$$

$$(2.32)\qquad u(t_{1k}) = c_{1k}, \quad u(t_{2k}) = c_{2k}$$

has at least one solution u_k (cf., for example, Lemma 2.1 in [17]).

We will show that

$$(2.33)\qquad v(t) = \sigma_1(t) - u_k(t) \leqslant 0 \quad \text{for} \quad t_{1k} \leqslant t \leqslant t_{2k}.$$

Assume on the contrary that for some $\widetilde{t} \in]t_{ik}, t_{2k}[$ the inequality $v(\widetilde{t}) > 0$ holds. Since $v(t_{ik}) = 0$, $i = 1, 2$, there exist $t_* \in [t_*, t_{2k}[$ and $t^* \in]t_k, t_{2k}]$ such that $v \in AC'([t_*, t^*]; \mathbb{R})$,

$$(2.34)\qquad v(t) > v(t_*) > 0 \quad \text{for} \quad t_* < t < t^*, \quad v(t_*) = v(t^*),$$

$$(2.35)\qquad |v'(t)| < \frac{v(t)}{1 + v(t)} \quad \text{for} \quad t_* < t < t^*.$$

Then

$$(2.36)\qquad |u_k'(t)| < \varrho(t) \quad \text{for} \quad t_* < t < t^*.$$

In view of (2.29), (2.30) and (2.34)–(2.36), the inequality

$$v''(t) \geqslant F(t, \sigma_1(t), \sigma_1'(t)) - F(t, \sigma_1(t), u_k'(t)) + q(t)\frac{v(t)}{1 + v(t)}$$
$$= g(t, \sigma_1(t))v'(t) + q(t)\frac{v(t)}{1 + v(t)} \geqslant 0$$

is satisfied almost everywhere in $]t_*, t^*[$. Since $v \in AC'([t_*, t^*]; \mathbb{R})$, the above inequality contradicts conditions (2.34). Consequently, inequality (2.33) holds. Analogously we can see that $u_k(t) \leqslant \sigma_2(t)$ for $t_{1k} \leqslant t \leqslant t_{2k}$. Thus

$$\sigma_1(t) \leqslant u_k(t) \leqslant \sigma_2(t) \quad \text{for} \quad t_{1k} \leqslant t \leqslant t_{2k}. \tag{2.37}$$

Taking this into account, by (2.30) we have

$$u_k''(t) = \widetilde{F}(t, u_k(t), u_k'(t)) \quad \text{for} \quad t_{1k} \leqslant t \leqslant t_{2k}, \tag{2.38}$$

whence in view of condition (2.23) we conclude that the function u_k satisfies the conditions of Lemma 2.1 for $a_1 = t_{1k}$ and $b_1 = t_{2k}$. Hence,

$$(t - t_{1k})(t_{2k} - t)|u_k'(t)| \leqslant c_0 \quad \text{for} \quad t_{1k} < t < t_{2k} \tag{2.39}$$

and

$$\begin{aligned} u_k(t) &\leqslant c_{1k} + H_1(t) \quad \text{for } t_{1k} < t < \frac{a+b}{2}, \\ u_k(t) &\leqslant c_{2k} + H_2(t) \quad \text{for } \frac{a+b}{2} \leqslant t \leqslant t_{2k}, \end{aligned} \tag{2.40}$$

where H_1 and H_2 are the functions appearing in Lemma 2.1. Due to (2.27)–(2.29) and (2.39), from (2.38) we find

$$u_k''(t) = F(t, u_k(t), u_k'(t)) \quad \text{for} \quad s_{1k} < t < s_{2k}. \tag{2.41}$$

Now (2.37), (2.39) and (2.41) imply uniform boundedness and equicontinuity of sequences $(u_k)_{k=1}^{+\infty}$ and $(u_k')_{k=1}^{+\infty}$ in $]a, b[$ (i.e., on each compact contained in $]a, b[$). Therefore without loss of generality we assume that

$$\lim_{k \to +\infty} u_k(t) = u_0(t), \qquad \lim_{k \to +\infty} u_k'(t) = u_0'(t)$$

uniformly in $]a, b[$. Clearly, u_0 is a solution of equation (1.1). Moreover, from (1.4), (2.25), (2.37) and (2.40) we conclude that u_0 satisfies boundary conditions (1.2) and inequalities (1.7).

In the case when (2.24) is satisfied, the lemma can be proved analogously. The only difference is that there is no need to introduce sequences $(s_{ik})_{k=1}^{+\infty}$, $i = 1, 2$, but we must put $\widetilde{\varphi}(t) = \varphi(t)$ for $a < t < b$, where φ is the function appearing in Lemma 2.2 for the case $\alpha = \frac{a+\gamma}{2}$, $\beta = \frac{b+\gamma}{2}$, and $h_1(t) = p_0(t)$, $h_2(t) = q_0(t) + \widetilde{q}(t)$. Here $\widetilde{q}(t) = 0$ for $t \in]a, \alpha[\cup]\beta, b[$, and $\widetilde{q}(t) = \max\{|g(t, x)| \colon \sigma_* \leqslant x \leqslant \sigma^*\}$ for $\alpha < t < \beta$, where $\sigma_* = \min\{\sigma_1(t) \colon \alpha \leqslant t \leqslant \beta\}$, $\sigma^* = \max\{\sigma_2(t) \colon \alpha \leqslant t \leqslant \beta\}$. □

Finally, for the sake of convenience we give without proof some lemmas on properties of solutions of the linear equation

$$u'' = p(t)u + q(t)u', \tag{2.42}$$

where $p, q \in L_{\mathrm{loc}}(]a, b[; \mathbb{R})$.

Definition 2.1. Equation (2.42) is said to be oscillatory on the segment $[a, b]$ if every its nontrivial solution has at least one zero in the interval $]a, b[$.

Lemma 2.4. *Let*

$$p(t) \leqslant 0 \quad \text{for} \quad a < t < b, \tag{2.43}$$

$$\int_a^b |q(s)| \,\mathrm{d}s < +\infty, \quad \int_a^b (s-a)(b-s)|p(s)| \,\mathrm{d}s < +\infty,$$

and

$$\int_a^b (s-a)^2(b-s)^2|p(s)| \,\mathrm{d}s \geqslant (b-a)^3 \exp\left[8 \int_a^b |q(s)| \,\mathrm{d}s\right].$$

Then equation (2.42) *is oscillatory.*

Lemma 2.5. *Let* (2.43) *be fulfilled and*

$$\int_a^b |q(s)| \,\mathrm{d}s < +\infty, \quad \int_a^b (s-a)|p(s)| \,\mathrm{d}s < +\infty \quad \left(\int_a^b (b-s)|p(s)| \,\mathrm{d}s < +\infty\right),$$

$$\int_a^b (s-a)|p(s)| \,\mathrm{d}s < (b-a) \exp\left[-3 \int_a^b |q(s)| \,\mathrm{d}s\right]$$

$$\left(\int_a^b (b-s)|p(s)| \,\mathrm{d}s < (b-a) \exp\left[-3 \int_a^b |q(s)| \,\mathrm{d}s\right]\right).$$

Then equation (2.42) *has a solution* v_1 *(a solution* v_2*) satisfying the conditions*

$$v_1'(t) > 0 \quad \text{for} \quad a < t \leqslant b, \quad v_1(a+) = 1$$

$$\big(v_2'(t) < 0 \quad \text{for} \quad a \leqslant t < b, \quad v_2(b-) = 1\big).$$

The above two lemmas immediately follow from the results obtained in [20].

Lemma 2.6. *Let* $p(t) \geqslant 0$ *for* $a < t < b$, *and* $\int_a^b (s-a)(b-s)p(s) \,\mathrm{d}s < +\infty$. *Then Green's function* G *of the problem*

$$u'' = p(t)u; \quad u(0+) = 0, \quad u(b-) = 0$$

admits the estimates (cf., for example, [16]*).*

$$\frac{1}{\mu(b-a)}(t-a)(b-\tau) \leqslant -G(t,\tau) \leqslant \frac{\mu^2}{b-a}(t-a)(b-\tau)$$
$$\text{for} \quad a \leqslant t < \tau \leqslant b,$$
$$\frac{1}{\mu(b-a)}(\tau-a)(b-t) \leqslant -G(t,\tau) \leqslant \frac{\mu^2}{b-a}(\tau-a)(b-t)$$
$$\text{for} \quad a \leqslant \tau \leqslant t \leqslant b,$$

where $\mu = \exp\big[(b-a)^{-1}\int_a^b (s-a)(b-s)p(s)\,\mathrm{d}s\big]$.

3. Proof of main results

P r o o f of Theorem 1.1. Let

$$\sigma_* = \min\{\sigma_2(t)\colon\ a \leqslant t \leqslant b\}, \quad \sigma^* = \max\{\sigma_2(t)\colon\ a \leqslant t \leqslant b\}, \quad \varepsilon = \frac{\sigma_*}{2}.$$

Choose $a_0 \in\,]a,\gamma[$ and $b_0 \in\,]\gamma,b[$ such that $\sigma_1(t) < \varepsilon$ for $t \in\,]a,a_0[\,\cup\,]b_0,b[$. Denote by v_1 and v_2 respectively solutions of the boundary value problems

$$v'' = -q_\varepsilon(t)v' - p_\varepsilon(t); \quad v(a+) = \varepsilon, \quad v(a_0) = \sigma^* + 1,$$
$$v'' = q_\varepsilon(t)v' - p_\varepsilon(t); \quad v(b_0) = \sigma^* + 1, \quad v(b-) = \varepsilon.$$

Obviously, there exist $a_1 \in\,]a,a_0[$ and $b_1 \in\,]b_0,b[$ such that

$$\varepsilon < v_1(t) < \sigma_2(t), \quad v_1'(t) > 0 \quad \text{for} \quad a < t < a_1,$$
$$v_1(a_1) = \sigma_2(a_1), \quad v_1'(a_1) \geqslant \sigma_2'(a_1+),$$
$$\varepsilon < v_2(t) < \sigma_2(t), \quad v_2'(t) < 0 \quad \text{for} \quad b_1 < t < b,$$
$$v_2(b_1) = \sigma_2(b_1), \quad v_2'(b_1) \leqslant \sigma_2'(b_1-).$$

Taking this and the conditions of the theorem into account, we conclude that

$$v_1''(t) \leqslant -\big[f(t,v_1(t)) + g(t,v_1(t))v_1'(t)\big]_- \quad \text{for} \quad a < t < a_1,$$
$$v_2''(t) \leqslant -\big[f(t,v_2(t)) + g(t,v_2(t))v_2'(t)\big]_- \quad \text{for} b_1 < t < b.$$

Choose sequences $(t_{1k})_{k=1}^{+\infty}$ and $(t_{2k})_{k=1}^{+\infty}$ such that $t_{11} \in\,]a,a_1[$, $t_{21} \in\,]b_1,b[$, $t_{1k+1} < t_{1k}$, $t_{2k} < t_{2k+1}$, $k = 1,2,\ldots$,

$$\sigma_1(t) < \sigma_1(t_{1k}) \quad \text{for} \quad a < t < t_{1k}, \quad \sigma_1(t) < \sigma_1(t_{2k}) \quad \text{for} \quad t_{2k} < t < b,$$
$$\sigma_1(t_{1k+1}) < \sigma_1(t_{1k}), \quad \sigma_1'(t_{1k}) \geqslant 0, \quad \sigma_1(t_{2k+1}) < \sigma_2(t_{2k}), \quad \sigma_1'(t_{2k}) \leqslant 0,$$
$$\sigma_1(t_{1k}) = \sigma_1(t_{2k}), \quad \lim_{k\to+\infty} t_{1k} = a, \quad \lim_{k\to+\infty} t_{2k} = b.$$

Suppose

$$
\chi_k(x) = \begin{cases} \sigma_1(t_{1k}) & \text{for } x < \sigma_1(t_{1k}) \\ x & \text{for } x \geqslant \sigma_1(t_{1k}) \end{cases} \qquad k = 1, 2, \dots,
$$

$$
F_k(t, x, y) = \begin{cases} -\big[f(t, \chi_k(x)) + g(t, \chi_k(x))y\big]_- & \text{for } t \in \,]a, t_{1k}[\, \cup \,]t_{2k}, b[, \\ f(t, \chi_k(x)) + g(t, \chi_k(x))y & \text{for } t \in \,]t_{1k}, t_{2k}[, \end{cases}
$$

$$
\widetilde{\sigma}_{1k}(t) = \begin{cases} \sigma_1(t_{1k}) & \text{for } a \leqslant t \leqslant t_{1k}, \\ \sigma_1(t) & \text{for } t_{1k} < t < t_{2k}, \\ \sigma_1(t_{2k}) & \text{for } t_{2k} \leqslant t \leqslant b, \end{cases} \tag{3.1}
$$

$$
\widetilde{\sigma}_2(t) = \begin{cases} v_1(t) & \text{for } a \leqslant t \leqslant a_1, \\ \sigma_2(t) & \text{for } a_1 < t < b_1, \\ v_2(t) & \text{for } b_1 \leqslant t \leqslant b, \end{cases}
$$

and consider the boundary value problem

$$
u'' = F_k(t, u, u'), \tag{3.2$_k$}
$$

$$
u(a+) = \sigma_1(t_{1k}), \quad u(b-) = \sigma_1(t_{2k}). \tag{3.3$_k$}
$$

It is easy to see that $F_k \in \mathrm{Car}_{\mathrm{loc}}(]a, b[\times \mathbb{R}^2; \mathbb{R})$, and $\widetilde{\sigma}_{11}$ and $\widetilde{\sigma}_2$ are respectively lower and upper functions of equation (3.2_1). Therefore, by Lemma 2.3, problem (3.2_1), (3.3_1) has at least one solution u_1 satisfying

$$
\widetilde{\sigma}_{11}(t) \leqslant u_1(t) \leqslant \widetilde{\sigma}_2(t) \quad \text{for} \quad a \leqslant t \leqslant b.
$$

Further, $\widetilde{\sigma}_{12}$ is a lower and u_1 is an upper function of equation (3.2_2). Therefore, by Lemma 2.3, problem (3.2_2), (3.3_2) has a solution u_2 satisfying the condition

$$
\widetilde{\sigma}_{12}(t) \leqslant u_2(t) \leqslant u_1(t) \quad \text{for} \quad a \leqslant t \leqslant b.
$$

Continuing this process, we obtain a sequence of functions $(u_k)_{k=1}^{+\infty}$ satisfying equation (3.2_k), conditions (3.3_k), and

$$
\widetilde{\sigma}_{1k+1}(t) \leqslant u_{k+1}(t) \leqslant u_k(t) \quad \text{for} \quad a \leqslant t \leqslant b, \quad k = 1, 2, \dots. \tag{3.4}
$$

Applying (3.1), (3.4) and Lemma 2.2 (in the case, $r_0 = \sigma^*$, $\alpha = a_0$, $\beta = b_0$, $\eta = \min\{\widetilde{\sigma}_1(a_1), \widetilde{\sigma}_1(b_1)\}$, $h_1(t) = p_\eta(t)$, $h_2(t) = q_\eta(t) + q^*(t)$, where $q^*(t) = \max\{|g(t, x)| \colon \varepsilon_0 \leqslant x \leqslant \sigma^*\}$ and $\varepsilon_0 = \min\{\sigma_1(t) \colon a_0 \leqslant t \leqslant b_0\}$), we find that sequences $(u_k)_{k=1}^{+\infty}$

and $(u_k')_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous in $]a,b[$ (i.e., on each segment contained in $]a,b[$). Therefore without loss of generality we can assume that

$$\lim_{k\to+\infty} u_k(t) = u_0(t), \qquad \lim_{k\to+\infty} u_k'(t) = u_0'(t)$$

uniformly in $]a,b[$.

Clearly, u_0 is a solution of equation (1.1), and

$$0 < \sigma_1(t) \leqslant u_0(t) \leqslant u_k(t) \quad \text{for} \quad a < t < b, \quad k = 1, 2, \dots.$$

This with regard to (3.3_k) implies

$$\begin{aligned} 0 \leqslant \lim_{k\to a+} \inf u_0(t) \leqslant \lim_{k\to a+} \sup u_0(t) \leqslant \sigma_1(t_{1k}), \quad k = 1, 2, \dots, \\ 0 \leqslant \lim_{k\to b-} \inf u_0(t) \leqslant \lim_{k\to b-} \sup u_0(t) \leqslant \sigma_1(t_{2k}), \quad k = 1, 2, \dots. \end{aligned}$$

Consequently, $u_0(a+) = 0$ and $u_0(b-) = 0$. □

P r o o f of Corollary 1.1. First we prove the sufficiency. By Theorem 1.1, it suffices to show that there exist lower and upper functions σ_1 and σ_2 of equation (1.1) satisfying conditions (1.4).

Choose $\delta \in]0, r[$ so small that either of the equations

$$u'' = \frac{1}{\delta} f(t,r)u + q^*(t)u' \tag{3.5}$$

and

$$u'' = \frac{1}{\delta} f(t,r)u - q^*(t)u' \tag{3.6}$$

is oscillatory in the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, respectively. For this, by Lemma 2.4, it is sufficient to assume

$$\begin{aligned} \delta < \frac{8}{(b-a)^3} \exp\left[-8\int_a^b q^*(s)\,\mathrm{d}s\right] \\ \times \min\left\{\int_a^{\frac{a+b}{2}} (s-a)^2\Big(\frac{a+b}{2} - s\Big)^2 |f(s,r)|\,\mathrm{d}s, \int_{\frac{a+b}{2}}^b \Big(s - \frac{a+b}{2}\Big)^2 (b-s)^2 |f(s,r)|\,\mathrm{d}s\right\}. \end{aligned}$$

Denote by u_1 and u_2 the solutions of equations (3.5) and (3.6), respectively, satisfying the initial conditions

$$\begin{aligned} u_1(a+) = 0, \quad u_1'(a+) = 1, \\ u_2(b-) = 0, \quad u_2'(b-) = -1. \end{aligned}$$

(The condition $\int_a^b (s-a)(b-s)|f(s,r)|\,\mathrm{d}s < +\infty$ guarantees the existence of u_1 and u_2—cf., for example, [16], [17].) Then there exist $t_1 \in \,]a, \frac{a+b}{2}[$ and $t_2 \in \,]\frac{a+b}{2}, b[$ such that

$$u_1'(t) > 0 \quad \text{for} \quad a < t < t_1, \quad u_1'(t_1) = 0,$$
$$u_2'(t) < 0 \quad \text{for} \quad t_2 < t < b, \quad u_2'(t_2) = 0.$$

Let

$$\sigma_1(t) = \begin{cases} \dfrac{\delta}{u_1(t_1)} u_1(t) & \text{for } a \leqslant t \leqslant t_1 \\ \delta & \text{for } t_1 < t < t_2 \\ \dfrac{\delta}{u_2(t_2)} u_2(t) & \text{for } t_2 \leqslant t \leqslant b \end{cases}.$$

As is easily seen, $\sigma_1 \in AC'_{\mathrm{loc}}(]a,b[;\mathbb{R}_+)$, $\sigma_1(t) \leqslant \delta < r$ for $a \leqslant t \leqslant b$, and σ_1 is a lower function of equation (1.1).

Choose $\mu \in \,]0, r[$ so small that the equations

$$v'' = -\mu\Big[f\Big(t, \frac{1}{r}\Big)\Big]_- v - q^*(t)v'$$

and

$$v'' = -\mu\Big[f\Big(t, \frac{1}{r}\Big)\Big]_- v - q^*(t)v'$$

have respectively solutions v_1 and v_2 satisfying the conditions

$$v_1'(t) > 0 \quad \text{for} \quad a < t \leqslant \frac{a+b}{2}, \quad v_1(a+) = 1,$$
$$v_2'(t) < 0 \quad \text{for} \quad \frac{a+b}{2} \leqslant t < b, \quad v_2(b-) = 1.$$

For this, by Lemma 2.5, it is sufficient to assume that

$$\mu < \frac{b-1}{2\exp\big[3\int_a^b q^*(s)\,\mathrm{d}s\big]} \times \min\bigg\{\bigg(1 + \int_a^{\frac{a+b}{2}} (s-a)\Big[f\Big(s,\frac{1}{r}\Big)\Big]_- \mathrm{d}s\bigg)^{-1}, \bigg(1 + \int_{\frac{a+b}{2}}^b (b-s)\Big[f\Big(s,\frac{1}{r}\Big)\Big]_- \mathrm{d}s\bigg)^{-1}\bigg\}.$$

Let

$$\sigma_2(t) = \begin{cases} \lambda_1 v_1(t) & \text{for } a \leqslant t < \frac{a+b}{2}, \\ \lambda_2 v_2(t) & \text{for } \frac{a+b}{2} \leqslant t < b, \end{cases}$$

where

$$\lambda_i = \frac{1}{v_i\left(\frac{a+b}{2}\right)}\left(v_1\left(\frac{a+b}{2}\right) + v_2\left(\frac{a+b}{2}\right)\right)\left(r + \frac{1}{\mu}\right), \quad i = 1, 2.$$

Clearly, σ_2 is continuous, $\sigma_1 \in AC'_{\mathrm{loc}}\left(]a,b[\setminus\{\frac{a+b}{2}\}; \mathbb{R}_+\right)$,

$$r\sigma_2(t) > \mu\sigma_2(t) > 1, \quad \sigma_2(t) > r \quad \text{for} \quad a \leqslant t \leqslant b, \quad \sigma_2'\left(\frac{a+b}{2}-\right) > \sigma_2'\left(\frac{a+b}{2}+\right),$$

and σ_2 is an upper function of equation (1.1). Consequently, problem (1.1), (1.2) has at least one solution.

Now let us prove the necessity. Let u be a solution of problem (1.1), (1.2). Suppose $0 < x \leqslant r$ and choose $a_0 \in\,]a,b[$ such that

$$u(t) < x \quad \text{for} \quad a \leqslant t \leqslant a_0. \tag{3.7}$$

Since the function u is bounded in $[a, a_0]$, we have

$$\lim_{t\to a+} \inf (t-a)|u'(t)| = 0. \tag{3.8}$$

Multiplying both sides of equation (1.1) by

$$q(t) = \int_a^t \exp\left[\int_t^s g(\xi, u(\xi))\,\mathrm{d}\xi\right] \mathrm{d}s \quad \text{for} \quad a \leqslant t \leqslant b$$

and integrating from t to a_0, we obtain for $t \in\,]a,b[$ that

$$u'(t)q(t) = u'(a_0)q(a_0) + u(a_0) - u(t) - \int_a^{a_0} q(s)f(s,u(s))\,\mathrm{d}s. \tag{3.9}$$

Owing to (1.10), we can easily see that

$$\frac{1}{c}(t-a) \leqslant q(t) \leqslant c(t-a) \quad \text{for} \quad a < t < b,$$

where $c = \exp\left[\int_a^b q^*(s)\,\mathrm{d}s\right]$. Moreover, taking into account condition (3.7) and the fact that the function f is monotone, from (3.9) we easily find

$$\int_t^{a_0} (s-a)|f(s,x)|\,\mathrm{d}s \leqslant c\big(u(t) - u(a_0) - u'(a_0)q(a_0)\big) \\ + (t-a)|u'(t)| \quad \text{for} \quad a < t < a_0.$$

This, due to (1.2) and (3.8), implies

$$\int_a^{a_0} (s-a)|f(s,x)|\,\mathrm{d}s < +\infty. \tag{3.10}$$

Analogously we can show that $\int_{b_0}^b (b-s)|f(s,x)|\,\mathrm{d}s < +\infty$ for some $b_0 \in\,]a,b[$. Consequently, conditions (1.11) are satisfied. □

P r o o f of Corollary 1.2. First we prove the sufficiency. To this end, by Theorem 1.1 it suffices to show that there exist lower and upper functions σ_1 and σ_2 of equation (1.1) satisfying conditions (1.4). According to Corollary 1.1, the equation

$$u'' = f(t,u)$$

has solutions u_1 and u_2 satisfying the conditions

$$\begin{aligned} &u_1(t) > 0 \quad \text{for} \quad a < t < c, \quad u_1(a+) = 0, \quad u_1(c-) = 0, \\ &u_2(t) > 0 \quad \text{for} \quad c < t < b, \quad u_2(c+) = 0, \quad u_2(b-) = 0. \end{aligned}$$

It is evident that there exist $t_1 \in \,]a,c[$ and $t_2 \in \,]c,b[$ such that

$$\begin{aligned} &u_1'(t) > 0 \quad \text{for} \quad a < t < t_1, \quad u_1'(t_1) = 0, \\ &u_2'(t) < 0 \quad \text{for} \quad t_2 < t < b, \quad u_2'(t_2) = 0. \end{aligned}$$

Set $\delta = u_2(t_2)/u_1(t_1)$ and

$$\sigma_1(t) = \begin{cases} \min\{1,\delta\} \cdot u_1(t) & \text{for} \quad a \leqslant t < t_1, \\ \min\{1,\delta\} \cdot u_1(t_1) & \text{for} \quad t_1 \leqslant t \leqslant t_2, \\ \min\left\{1, \dfrac{1}{\delta}\right\} \cdot u_2(t) & \text{for} \quad t_2 < t \leqslant b. \end{cases}$$

It is not difficult to see that σ_1 is a lower function of equation (1.1).

Choose $\nu > 0$ such that

$$\nu \int_a^c f(s,r) q_1(s)\,\mathrm{d}s = \int_c^b f(s,r) q_2(s)\,\mathrm{d}s,$$

where

$$q_1(t) = \int_a^t \exp\left[\int_c^s g(\xi,r)\,\mathrm{d}\xi\right] \mathrm{d}s, \;\; q_2(t) = \int_t^b \exp\left[\int_c^s g(\xi,r)\,\mathrm{d}\xi\right] \mathrm{d}s \;\text{ for }\; a < t < b.$$

Suppose

$$h(t) = \begin{cases} \max\{1,\nu\} \cdot f(t,r) & \text{for} \quad a < t < c \\ \max\left\{1, \dfrac{1}{\nu}\right\} \cdot f(t,r) & \text{for} \quad c < t < b \end{cases}$$

and consider the equation

$$u'' = h(t) + g(t,r)u'.$$

We can readily see that the above equation has a solution σ_2 satisfying the conditions

$$\sigma_2(t) > 1 + \max\{\sigma_1(t)\colon\ a \leqslant t \leqslant b\}, \quad \sigma_2'(t)\operatorname{sgn}(c-t) \geqslant 0 \quad \text{for} \quad a < t < b.$$

Taking now into account the fact that the functions f and $g\operatorname{sgn}(c-t)$ are monotone, we conclude that σ_2 is an upper function of equation (1.1) and conditions (1.4) hold. Hence problem (1.1), (1.2) has at least one solution.

Let us now prove the necessity. Let u be a solution of problem (1.1), (1.2). Clearly, (3.8) holds. Suppose $x \in\ ]0, r]$ and choose $a_0 \in\ ]a, c[$ such that

$$u(t) < x, \quad u'(t) > 0 \quad \text{for} \quad a < t < a_0. \tag{3.11}$$

Multiplying both sides of equation (1.1) by $t-a$ and integrating from t to a_0, we get

$$\begin{aligned}&(t-a)u'(t) - (a_0-a)u'(a_0) + u(a_0) - u(t)\\ &\quad = -\int_t^{a_0} (s-a)f(s,u(s))\,\mathrm{d}s + \int_t^{a_0} (s-a)g(s,u(s))u'(s)\,\mathrm{d}s \quad \text{for} \quad a < t < a_0.\end{aligned}$$

If we now take into consideration the fact that the functions f and g are monotone, then due to conditions (3.11) from the last equality we obtain

$$\int_t^{a_0} (s-a)|f(s,x)|\,\mathrm{d}s \leqslant (t-a)u'(t) - (a_0-a)u'(a_0) + u(a_0) - u(t) \quad \text{for} \quad a < t < a_0,$$

whence according to (1.2) and (3.8) we conclude that (3.10) is satisfied. Analogously we can see that $\int_{b_0}^b (s-a)|f(s,x)|\,\mathrm{d}s < +\infty$ for some $b_0 \in\ ]c, b[$. Consequently, (1.11) is valid. □

P r o o f of Corollary 1.3. Denote by v a solution of the problem

$$v'' = \frac{p(t)}{(t-a)^2(b-t)^2}\,v - \lambda q(t); \quad v(a+) = 0, \quad v(b-) = 0, \tag{3.12}$$

where

$$\lambda = \left[\int_a^b (s-a)(b-s)q(s)\,\mathrm{d}s\right]^{-\frac{1}{2}} \exp\left[\frac{-1}{b-a}\int_a^b \frac{p(s)\,\mathrm{d}s}{(s-a)(b-s)}\right]\sqrt{b-a}. \tag{3.13}$$

By Green's formula (cf., for example, [16], [17]),

$$v(t) = \lambda \int_a^b G_0(t,\tau)q(\tau)\,\mathrm{d}\tau \quad \text{for} \ \ a \leqslant t \leqslant b,$$

where G_0 is Green's function of the problem

$$v'' = \frac{p(t)}{(t-a)^2(b-t)^2}\, v; \quad v(a+) = 0, \quad v(b-) = 0.$$

Applying Lemma 2.6, we can easily find

$$v(t) = \frac{\lambda \varrho_0^2}{b-a} \int_a^b (s-a)(b-s)q(s)\,\mathrm{d}s \quad \text{for } a \leqslant t \leqslant b, \tag{3.14}$$

$$v(t) \geqslant \frac{\lambda \varrho_0^{-1}}{b-a} \biggl[(b-t) \int_a^t (s-a)q(s)\,\mathrm{d}s + (t-a) \int_t^b (b-s)q(s)\,\mathrm{d}s\biggr] \quad \text{for } a \leqslant t \leqslant b, \tag{3.15}$$

where $\varrho_0 = \exp\bigl[(b-a)^{-1} \int_a^b [(s-a)(b-s)]^{-1} p(s)\,\mathrm{d}s\bigr]$.

By (1.15) and (3.13), from (3.14) and (3.15) we obtain

$$(t-a)(b-t) \leqslant v(t) \leqslant \frac{1}{\lambda} \quad \text{for } a \leqslant t \leqslant b.$$

Owing to this, (3.12) results in

$$v''(t) \geqslant \frac{p(t) - q(t)}{v(t)} \quad \text{for } a < t < b.$$

Hence v is a lower function of equation (1.14).

Let w be a solution of the problem

$$w'' = -\frac{q(t)}{w}; \quad w(a+) = 1 + \max\{v(t)\colon\ a \leqslant t \leqslant b\} = w(b-).$$

Then it is clear that w is an upper function of equation (1.14). Hence, by Theorem 1.1, problem (1.14), (1.2) has at least one solution. □

Theorem 1.2 is proved similarly to Theorem 1.1.

P r o o f of Corollary 1.4. Suppose $\delta = \frac{1}{4}(b-a)^2$ and $k = 1 + \frac{n}{\mu}$, and choose $\varepsilon > 0$ such that

$$\varepsilon\delta^k < 1, \quad \varepsilon\delta^{k-1} < \frac{1}{2k}, \quad \varepsilon^{\mu}(r + \delta^n)\delta^{\mu} < p_0.$$

Then it can be readily verified that $\sigma_1(t) = \varepsilon[(t-a)(b-t)]^k$ for $a \leqslant t \leqslant b$ is a lower function of equation (1.17), and $\sigma_1(t) < 1$ for $a \leqslant t \leqslant b$.

Assume now that σ_2 is a solution of the problem

$$u'' = -\frac{p(t)}{u^\mu}; \quad u(a+) = 1, \quad u(b-) = 1.$$

Evidently, $\sigma_2(t) > 1$ for $a \leqslant t \leqslant b$.

Thus σ_1 and σ_2 are respectively lower and upper functions of equation (1.17) satisfying conditions (1.4). Hence by Theorem 1.2, problem (1.17), (1.2) is solvable. The uniqueness follows from the fact that the right-hand side of equation (1.17) is a nondecreasing function. □

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