SOLVING A PARADOX ABOUT DISCONTINUOUS DAMPING

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This note is intended to clarify some aspects of the scalar discontinuous O.D.E.

$$\ddot{x} + \epsilon \, \mathrm{sgn}(\dot{x}) + x = 1.$$

In a recent note [5], Professor Tuck pointed out an apparent paradox: starting from rest with zero displacement, the solution approaches the equilibrium $x = 1$ until a certain time $n_0\pi$ where n_0 is the unique integer satisfying the condition

$$\frac{1}{\epsilon} - 1 \leq 2n_0 \leq \frac{1}{\epsilon} + 1.$$

After this time the continuation of the solution does not seem possible.

Essentially, this paradox was raised at the beginning of this century by Painlevé [4], Lecornu [3] and other mathematicians, initiating a scientific controversy about the convenience or validity of Coulomb's law of friction.

The history of this topic is long and goes back to Charles Coulomb (1736-1806), who developed a large number of experiments in order to find the properties of the friction between surfaces without lubrication (dry friction). It turns out that the simplest way to model this phenomenon is by a term proportional to the sign of the derivative. In fact, our equation models the motion of a mass attached to a linear spring and submitted to a constant external force equal to 1, while the friction term can be attributed to dry friction between the mass and the floor (see Figure 1).

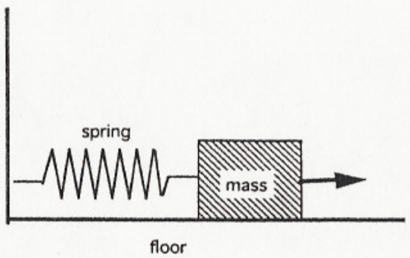


Figure 1: A spring-mass system with dry friction.

Nowadays, questions about non-existence or non-unicity of solutions arising in this context are generally well understood. The key idea is to choose the adequate notion of solution. Indeed, if we look for solutions in the Carathéodory sense (that is, functions of class C^1 with absolutely continuous derivative, satisfying the equation for almost all time), there is no way to continue once the solution reaches the interval $[1-\epsilon, 1+\epsilon]$ with zero derivative, as occurs in

the example of [5]. Hence a common device is to relax the original equation to the differential inclusion

$$\ddot{x} + x - 1 \in -\epsilon \, \mathrm{Sgn}(\dot{x})$$

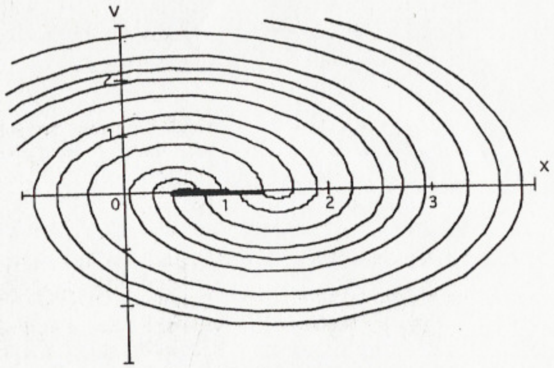
where Sgn(.) is the multivalued function

$$\mathrm{Sgn}(x) = \begin{cases} \mathrm{sgn}(x) & \text{if } x \neq 0 \\ [-1,1] & \text{if } x = 0 \end{cases}$$

With this reformulation, $[1-\epsilon, 1+\epsilon]$ is a whole interval of equilibria, also called stationary solutions, and any other solution reaches this interval at a finite time with zero derivative (as in [5]) and does not leave this point in the future. Thus there is unicity of solutions at the right but not at the left. This fact is in accordance with the phase portrait that can be easily drawn with the aid of a computer (see Figure 2).

In conclusion, a convexification of the damping term enables us to get an adequate mathematical setup for the problem.

As a final remark, there is a vast literature concerning discontinuous differential equations, differential inclusions, non-smooth mechanical problems and the relation between these topics, see for instance [1,2] and their references.



References

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