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Stable odd solutions of some periodic equations modeling satellite motion ☆

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Abstract

A new stability criterion is proved for second-order differential equations with symmetries in terms of the coefficients of the expansion of the nonlinearity up to the third order. Such a criterion provides solutions of twist type, which are Lyapunov-stable solutions with interesting dynamical properties. This result is connected with the existence of upper and lower solutions of a Dirichlet problem and applied to a known equation which model the planar oscillations of a satellite in an elliptic orbit, giving an explicit region of parameters for which there exists a Lyapunov-stable solution.

Keywords: Twist; Upper and lower solutions; Satellite equation; Lyapunov stability

1. Introduction

In this paper we are going to study the existence and stability of solutions of the periodic boundary value problem

$$\begin{aligned}&(mx')' + f(t,x) = 0,\\ &x(0) = x(T), \qquad x'(0) = x'(T),\end{aligned} \tag{1}$$

where $f \in C^{0,4}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}, \mathbb{R})$, $m \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ holds the symmetry conditions

$$f(-t,x) = -f(t,x), \qquad m(t) = m(-t) \tag{2}$$

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for all t. This type of symmetries appears with some frequency in nature. For instance, the planar oscillations of a satellite in an elliptic orbit is modeled by the equation

$$\big(1+e\cos(t)\big)x''-2e\sin(t)x'+\lambda\sin x=4e\sin(t),$$

where $e\in(0,1)$ is the eccentricity of the ellipse, $\lambda>0$ is the inertial parameter of the satellite, t is the true anomaly of the position of the satellite on the orbit, and x is the doubled angle between the radius vector to the mass center and one of its axis of inertia. Note that it can be written as

$$\Big(\big(1+e\cos(t)\big)^2x'\Big)'+\lambda\big(1+e\cos(t)\big)\sin x=4e\big(1+e\cos(t)\big)\sin t,$$

so it is of the form (1).

This equation was introduced by Beletskii in 1959 (see [3] and also [4] and references therein) and has been used, for instance, to describe the oscillations of Hyperion [20], a natural satellite of Saturn.

There exist a wide number of articles devoted to this study. Concerning existence of periodic solutions, some steps were performed by Kill [9], Torzhevskii [19], and Shlapak [17], but a major advance was done by Petrhysyn and Yu [16] by using Galerkin type finite-dimensional approximations. Finally, the problem of existence was solved by Hai [7] without additional restrictions over the parameters e,λ. The proof is of variational nature, obtaining the solution as a minimum of the action functional on certain ball centered on zero. In the second paper [8], the second solution is proved to exist by using the mountain pass theorem.

Concerning the stability of the solutions, the number of references is considerably fewer. Some numerical calculations [2,23] strongly suggest stability in some regions of parameters. In fact, some explicit criteria for linear stability were derived in [21]. The main object of this paper is to rigorously prove Lyapunov stability in a given region of parameters (λ,e). More concretely, the existence of a 2π-periodic solution of twist type is proved. A periodic solution φ of a general second-order periodic solution is said to be 4-elementary if it is linearly stable with Floquet's multipliers which are not roots of the unity up to the fourth order. Then, the solution φ is called of twist type [13,14] if it is 4-elementary and the first Birkhoff's coefficient of the associated Poincaré map is different to zero. This fact has many important consequences from the dynamical point of view. Moser's invariant curve theorem implies that a periodic solution of twist type is always Lyapunov stable [10,18]. Moreover, Poincaré–Birkhoff's fixed point theorem and KAM theory [1,18] imply the existence of subharmonics of arbitrary order, quasi-periodic solutions, and a chaotic behavior in the surroundings of the twist solutions. This is in accordance with the numerical results [2,23].

This paper can be seen as the natural continuation of [12], where a simpler equation modeling of the motion of a satellite in a circular orbit is studied [22].

To conclude this section, we describe briefly the structure of the paper. In Section 2, a general stability criterion based on the results in [11] is provided. In Section 3, this criterion is applied to equations with symmetries and connected with upper and lower solutions of the Dirichlet problem. Finally, Section 4 is devoted to a detailed study of the satellite equation. In particular, it is proved that the solution obtained in [7] is unstable, and a region of parameters for which there exists a solution of twist type is explicitly described. This region is drawn in Fig. 1.

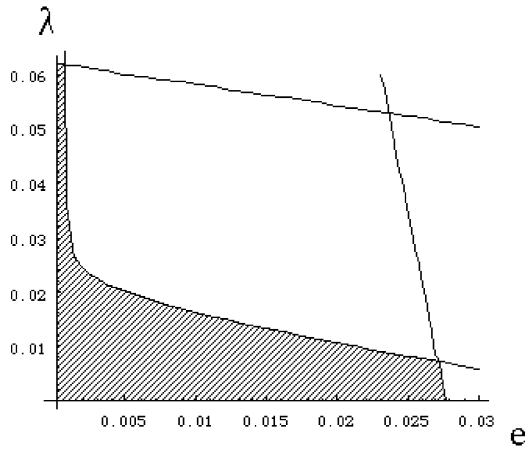

Fig. 1. Region of stability for the satellite equation.

2. Stability criterion

From now on, let us denote $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ the positive and negative part of a given function f. Let $\varphi(t)$ be a solution of problem (1). After a translation to the origin and a Taylor expansion, the equation can be written as

$$\big(m(t)x'\big)' + a(t)x + b(t)x^2 + c(t)x^3 + R(t, x) = 0, \tag{3}$$

where a, b, c are T-periodic and R denote the remaining terms. For the particular case $m(t) = 1$, the main result of [11] provides a stability criterion that it is reformulated in the following for reader's convenience.

Theorem 1 [11, Theorem 2.2]. *Assume that there exist positive numbers σ, γ such that*

$$\sigma^2 \leqslant a(t) \leqslant \gamma^2 \leqslant \left(\frac{\pi}{3T}\right)^2.$$

Then the equilibrium $x \equiv 0$ of (3) *is of twist type if the following condition holds*:

$$\sigma^6 \int_0^T c^-(t)\,dt - \sigma^2\gamma^4 \int_0^T c^+(t)\,dt > 2\gamma^5 \int_0^T b^+(t)\,dt \int_0^T b^-(t)\,dt.$$

For a general $m \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$, let us denote

$$\mu = \frac{1}{T} \int_0^T \frac{ds}{m(s)}.$$

Then, we can prove the following result.

Theorem 2. *Assume that there exist positive numbers σ, γ such that*

$$\sigma^2 \leqslant m(t)a(t) \leqslant \gamma^2 \leqslant \left(\frac{\pi}{3\mu T}\right)^2. \tag{4}$$

Then, the equilibrium $x \equiv 0$ of (3) *is of twist type if the following condition holds*:

$$\sigma^6 \int_0^T c^-(t)\,dt - \sigma^2\gamma^4 \int_0^T c^+(t)\,dt > 2\gamma^5 \int_0^T b^+(t)\,dt \int_0^T b^-(t)\,dt. \tag{5}$$

Proof. After the change on the independent variable

$$t \to \tau(t) = \int_0^t \frac{ds}{\mu m(s)},$$

Eq. (3) reads

$$y'' + \mu^2 m\big(t(\tau)\big)a\big(t(\tau)\big)y + \mu^2 m\big(t(\tau)\big)b(t)y^2 + \mu^2 m\big(t(\tau)\big)c(t)y^3 + R(t, y) = 0,$$

where $y(\tau) = x(t(\tau))$. Now, the result follows from the previous result and the identity

$$\int_0^T g\big(t(\tau)\big)m\big(t(\tau)\big)\mu^2\,d\tau = \mu \int_0^T g(s)\,ds,$$

which holds for every $g \in C(\mathbb{R}/T\mathbb{Z})$. □

Note that in this section it is not necessary to assume condition (2).

3. Odd periodic solutions and upper and lower solutions

From now on, let us assume that the symmetry condition (2) is satisfied.

In the rest of the paper, we shall often use the following diffeomorphism:

$$t \to \tau(t) = \int_0^t \frac{ds}{\mu m(s)}. \tag{6}$$

From condition (2), it is easy to check that $\tau(t) = \tau(-t)$ for all $t \in \mathbb{R}$ and $\tau(T/2) = T/2$, $\tau(T) = T$. Besides, diffeomorphism (6) transforms Eq. (1) into a Newtonian equation of the form

$$y'' + g(\tau, y) = 0$$

with g verifying $g(-\tau, -y) = -g(\tau, y)$.

We begin the section with the following proposition. Taking into account (2), the proof is immediate.

Proposition 1. *The odd extension of a solution of the Dirichlet problem*

$$(mx')' + f(t,x) = 0, \qquad x(0) = x(T/2) = 0 \tag{7}$$

is an odd solution of problem (1).

A common device in order to solve a Dirichlet problem is the method of upper and lower solutions. This technique has the advantage that provide bounds on the solution, which will be useful in order to apply the result of Section 2. The classical definition of upper and lower solution for the Dirichlet problem (7) is the following

Definition 1. A function $\alpha \in C^2(]0,2\pi[) \cap C([0,2\pi])$ is a lower solution of (7) if

$$(m\alpha')' + f(t,\alpha) \geqslant 0, \quad t \in]0,2\pi[,$$
$$\alpha(0) \leqslant 0, \qquad \alpha(T/2) \leqslant 0.$$

A function $\beta \in C^2(]0,2\pi[) \cap C([0,2\pi])$ is an upper solution of (7) if

$$(m\beta')' + f(t,\beta) \leqslant 0, \quad t \in]0,2\pi[,$$
$$\beta(0) \geqslant 0, \qquad \beta(T/2) \geqslant 0.$$

It is well known (see, for instance, [6]) that a couple $\alpha < \beta$ provides a solution of (7) between them. Hence, we have the following consequence.

Proposition 2. *If α, β is a couple of lower and upper solutions of* (7) *such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0,T/2]$, problem* (1) *has an odd solution φ such that*

$$\alpha(t) \leqslant \varphi(t) \leqslant \beta(t) \tag{8}$$

for all $t \in [0,T/2]$.

Proof. Performing the change (6), equation $(mx')' + f(t,x) = 0$ is transformed into the Newtonian equation

$$y'' + \mu^2 m\big(t(\tau)\big) f\big(t(\tau), y\big) = 0,$$

where $y(\tau) = x(t(\tau))$. Evidently, α, β are also upper and lower solutions of the Dirichlet problem or this equation, and we can apply the classical result (see, for instance, [6, Theorem 1.4]). $\square$

Following the notation of [11], we define the so-called auxiliary functions

$$A^* = \max_{t\in[0,T/2]} m(t) U\big(\partial_x f(t,\cdot)\big), \qquad A_* = \min_{t\in[0,T/2]} m(t) L\big(\partial_x f(t,\cdot)\big),$$
$$B_+(t) = U\left(\frac{1}{2}\big[\partial_x^2 f(t,\cdot)\big]^+\right), \qquad B_-(t) = U\left(\frac{1}{2}\big[\partial_x^2 f(t,\cdot)\big]^-\right),$$
$$C_+(t) = U\left(\frac{1}{6}\big[\partial_x^3 f(t,\cdot)\big]^+\right), \qquad C_-(t) = L\left(\frac{1}{6}\big[\partial_x^3 f(t,\cdot)\big]^-\right),$$

where the operators $L, U : C([0,T/2] \times \mathbb{R}) \to C([0,T/2])$ are defined by

$$L(f)(t) = \inf\{f(t,\xi):\ \alpha(t) \leqslant \xi \leqslant \beta(t)\},$$
$$U(f)(t) = \sup\{f(t,\xi):\ \alpha(t) \leqslant \xi \leqslant \beta(t)\}.$$

Now, we can state and prove the main result of this section.

Theorem 3. *Let α, β be a couple of lower and upper solutions of* (7) *such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0, T/2]$. In addition, let us suppose that*

(i) $$0 < A_* \leqslant A^* \leqslant \left(\frac{\pi}{3\mu T}\right)^2,$$

(ii) $$A_*^3 \int_0^{T/2} C_-(t)\,dt - A_*(A^*)^2 \int_0^{T/2} C_+(t)\,dt > 4(A^*)^{5/2}\left(\int_0^{T/2} B^+(t)\,dt\right)^2.$$

Then, problem (1) *has an odd solution φ verifying* (8) *which is of twist type.*

Proof. The existence of φ is given by Proposition 2. A computation of the coefficients of the expansion (3) gives

$$a(t) = \partial_x f(t,\varphi(t)), \qquad b(t) = \frac{1}{2}\partial_x^2 f(t,\varphi(t)), \qquad c(t) = \frac{1}{6}\partial_x^3 f(t,\varphi(t)).$$

Note that because of the property (2) and the oddness of φ, we have that $b(t)$ is odd and $c(t)$ is even. In any case, functions b^+, b^-, c^+, c^- are even, so in consequence,

$$\int_0^T b^+(t)\,dt = \int_0^T b^-(t)\,dt = 2\int_0^{T/2} b^+(t)\,dt,$$
$$\int_0^T c^+(t)\,dt = 2\int_0^{T/2} c^+(t)\,dt, \qquad \int_0^T c^-(t)\,dt = 2\int_0^{T/2} c^-(t)\,dt.$$

Therefore, condition (5) of Theorem 2 reads

$$\sigma^6 \int_0^{T/2} c^-(t)\,dt - \sigma^2\gamma^4 \int_0^{T/2} c^+(t)\,dt > 4\gamma^5\left(\int_0^{T/2} b^+(t)\,dt\right)^2.$$

Now, using (i) we can take $\sigma^2 = A_*$, $\gamma^2 = A^*$, and then the result is an easy corollary of Theorem 2. □

4. Stability of solutions of a satellite equation

In this section we are going to focus our attention on the equation

$$\big(1 + e\cos(t)\big)x'' - 2e\sin(t)x' + \lambda\sin x = 4e\sin(t), \tag{9}$$

yet commented in Section 1, specially in the question of stability. First of all, note that if (9) is multiplied by $p(t) = 1 + e\cos(t)$, it can be written as

$$\big(p^2(t)x'\big)' + \lambda p(t)\sin x = 4ep(t)\sin t, \tag{10}$$

which is of the form (1) with $m(t) = p^2(t)$ and $f(t,x) = \lambda p(t)\sin x - 4ep(t)\sin t$. These functions verify the symmetry condition (2). From now on we will work with this last formulation.

Our first result proves that the solution found in [7] is unstable.

Theorem 4. *Let $e \in (0,1)$ and λ any real number. Then Eq.* (10) *has a 2π-periodic solution which is unstable.*

Proof. Let H be the Hilbert space of absolutely continuous 2π-periodic functions u such that $u' \in L^2(0,2\pi)$ with the inner product

$$(u,v) = u(0)v(0) + \int_0^{2\pi} p^2(t)u'v'\,dt,$$

where $p(t) = 1 + e\cos(t)$. The method of proof in [7] is to find a minimum in a certain ball of the action functional $\Phi : H \to \mathbb{R}$ defined as

$$\Phi(x) = \int_0^{2\pi} \left[p^2(t)\frac{x'^2}{2} + \lambda p(t)\cos x + 4ep(t)\sin(t)x \right] dt. \tag{11}$$

Such a minimum is a 2π-periodic solution of (10). At this point it is important to mention that Dancer and Ortega have proved that the minimizers of the action functional of a periodic Newtonian equation with analytic potential are always unstable (see [5,15]).

Performing the change (6) yet used in Section 3 (remember that $\mu = (1/T)\int_0^T \frac{ds}{m(s)}$), Eq. (10) is transformed in the Newtonian equation

$$y'' + \mu^2\lambda p^3\big(t(\tau)\big)\sin y = 4e\mu^2 p^3\big(t(\tau)\big)\sin\big(t(\tau)\big),$$

where $y(\tau) = x(t(\tau))$, whose action functional is

$$\Psi(y) = \int_0^{2\pi} \left[\frac{y'^2}{2} + \mu^2\lambda p^3\big(t(\tau)\big)\cos y + 4e\mu^2 p^3\big(t(\tau)\big)\sin\big(t(\tau)\big)x \right] d\tau.$$

The relation between Φ and Ψ is simply

$$\Phi(x) = \frac{1}{\mu}\Psi(y).$$

As $\mu > 0$, a minimum of Φ is also a minimum of Ψ (note that the operator $x \to x \circ \tau^{-1}$ is a homeomorphism with the usual topologies, so the image of a ball is a ball), and an application of [15, Theorem 3.4] concludes the proof. □

From the point of view of the application in consideration, it is desirable to obtain the existence of stable solutions, so we will look for a region of parameters of Lyapunov stability.

Let us define

$$M(\lambda, e) := \frac{(8e + \lambda\pi)\pi}{(1-e^2)^{3/2}}.$$

Our main result is as follows.

Theorem 5. *Let us assume that*

(H$_1$) $M(\lambda, e) \leqslant \dfrac{\pi}{2}$,

(H$_2$) $\dfrac{\lambda}{(1-e)^3} \leqslant \dfrac{1}{36}$,

(H$_3$) $0 < \lambda < \dfrac{(1-e)^{18}}{36\pi^2(1+e)^{15}} \dfrac{\cos^8 M(\lambda, e)}{\sin^4 M(\lambda, e)}$.

Then, Eq. (10) *has a* 2π*-periodic solution which is of twist type.*

Proof. It is clear that $\beta(t) \equiv 0$ is an upper solution of the Dirichlet problem

$$\begin{aligned} &\big(p^2(t)x'\big)' + \lambda p(t)\sin x = 4ep(t)\sin t, \\ &x(0) = x(\pi) = 0. \end{aligned} \tag{12}$$

On the other hand, the unique solution of the linear problem

$$\begin{aligned} &\big(p^2(t)\alpha'\big)' = p(t)(\lambda + 4e\sin t), \\ &\alpha(0) = \alpha(\pi) = 0 \end{aligned}$$

is a lower solution of problem (12). From a trivial manipulation of the previous equation, one realizes that each local extrema of $\alpha(t)$ has to be a minimum, and in consequence $\alpha(t) < 0$ for all $t \in \,]0, \pi[$.

Therefore, $\alpha < \beta$ is a couple of ordered lower and upper solutions of problem (12). By Proposition 2, Eq. (10) has an odd 2π-periodic solution φ such that

$$\alpha(t) \leqslant \varphi \leqslant 0$$

for all $t \in \,]0, \pi[$.

Having in mind to apply Theorem 3, our next step will be to obtain an estimation of $\|\alpha\|_\infty$. By direct integration on $[0, \pi]$, it results in

$$\big\|\big(p^2(t)\alpha'\big)'\big\|_1 = 8e + \lambda\pi.$$

It is clear that there exists $t_0 \in \,]0, \pi[$ such that $\alpha'(t_0) = 0$. Then,

$$p^2(t)\alpha' = \int_{t_0}^{t} \big(p^2(s)\alpha'\big)'\,ds < \big\|\big(p^2(t)\alpha'\big)'\big\|_1 = 8e + \lambda\pi.$$

In consequence,

$$0 < -\alpha(t) = \int_t^{\pi} \alpha'(s)\,ds < (8e + \lambda\pi) \int_0^{\pi} \frac{ds}{p^2(s)} = \frac{(8e + \lambda\pi)\pi}{(1-e^2)^{3/2}} = M(\lambda, e)$$

for all $t \in]0, \pi[$. In conclusion,

$$\|\alpha\|_\infty < M(\lambda, e).$$

The next step is to compute the auxiliary functions of Theorem 3. In this case,

$$f(t, x) = \lambda p(t) \sin x - 4ep(t) \sin t.$$

Hence,

$$A^* = \max_t \bigl[p^2(t) U\bigl(\lambda p(t) \cos x\bigr)\bigr] \leqslant \lambda(1+e)^3,$$

and by hypothesis (H_1),

$$A_* = \min_t \bigl[p^2(t) L\bigl(\lambda p(t) \cos x\bigr)\bigr] > \lambda(1-e)^3 \cos M(\lambda, e).$$

Note that μ can be explicitly obtained as

$$\mu = \frac{1}{2\pi} \int_0^{2\pi} \frac{ds}{(1 + e\cos s)^2} = \frac{1}{(1-e^2)^{3/2}}.$$

Therefore, by using (H_1) and (H_2) we get

$$0 < A_* < A^* \leqslant \lambda(1+e)^3 \leqslant \frac{(1+e)^3(1-e)^3}{36} = \frac{(1-e^2)^3}{36} = \frac{1}{36\mu^2},$$

so condition (i) of Theorem 3 is verified. On the other hand,

$$B_+(t) = U\left(\frac{1}{2}\bigl[-\lambda p(t) \sin x\bigr]^+\right) = \frac{1}{2}\lambda p(t) \sin M(\lambda, e),$$

$$C_+(t) = U\left(\frac{1}{6}\bigl[-\lambda p(t) \cos x\bigr]^+\right) = 0,$$

$$C_-(t) = L\left(\frac{1}{6}\bigl[-\lambda p(t) \cos x\bigr]^-\right) = \frac{\lambda}{6} p(t) \cos M(\lambda, e).$$

In consequence, condition (ii) of Theorem 3 is reduced to

$$A_*^3 \int_0^{T/2} C_-(t)\,dt > 4(A^*)^{5/2} \left(\int_0^{T/2} B^+(t)\,dt\right)^2,$$

and by using hypothesis (H_3) it is not hard to verify that this inequality holds. Then, the proof is finished by using Theorem 3. □

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