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Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem

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Abstract

This paper is devoted to study the existence of periodic solutions of the second-order equation $x'' = f(t, x)$, where f is a Carathéodory function, by combining some new properties of Green's function together with Krasnoselskii fixed point theorem on compression and expansion of cones. As applications, we get new existence results for equations with jumping nonlinearities as well as equations with a repulsive or attractive singularity. In this latter case, our results cover equations with weak singularities and are compared with some recent results by I. Rachunková, M. Tvrdý and I. Vrkoč.
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1. Introduction

The aim of this paper is to provide new existence results for the periodic boundary value problem

$$x'' = f(t, x),$$
$$x(0) = x(T), \quad x'(0) = x'(T),$$

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where f is a function of L^1-Carathéodory type (that is, measurable in the first variable and continuous in the second one) and T-periodic in t. Two of the most common techniques to approach this problem are: (1) the obtention of a priori bounds for the possible solutions and then the application of topological degree arguments [21] and (2) the theory of upper and lower solutions [3]. These techniques can be interconnected and have proved to be very strong and fruitful and became very popular in this research area. However, any method of proof has some limitations and in fact, for practical purposes, serious difficulties arise frequently in the search for a priori bounds or upper and lower solutions.

In this paper, we choose another strategy of proof which rely essentially on a fixed point theorem due to Krasnoselskii for completely continuous operators on a Banach space that exhibit a cone compression and expansion of norm type [17, p. 148] (see Theorem 3.1). This result has been extensively employed in the related literature, specially to study several kinds of separated boundary value problems (see for instance in [7,8,13,15] and their references), while for the periodic problem it is more difficult to find references, and only a very recent paper [22] is known to the author. The reason for this contrast may be the fact that in order to apply this fixed-point theorem, it is necessary to perform a study of the sign of Green's function for the linear equation, and the periodic boundary conditions are difficult in this study in all sense. We overcome this problem by using a new L^p-maximum principle developed in [27]. This tool is combined with some ideas from [22]. One of the most interesting features of this technique of proof is that it provides "a posteriori" bounds of the solution (in particular, the solution has constant sign) which makes it possible to apply the general results to a variety of equations by truncation.

The paper is organized as follows: in Section 2 a detailed analysis of the sign of Green's function of the linearized equation is carried out as an intermediate step for the proofs of the main results of this paper given in Section 3. Finally, Section 4 is devoted to applications of these general results, making assumptions on the behavior of the nonlinearity near 0 and infinity, following some ideas from [22]. Many illustrating examples are considered. In particular, we provide new results for two problems that have attracted the interest of many researchers in the recent years: jumping nonlinearities (Example 3) and equations with a singularity in the origin (Section 4.2). Specially, in Section 4.2 it is shown that our method of proof can deal also with weak singularities. We obtain also new existence results for singular equations and compare them with related results obtained recently in [25,31].

Let us fix some notation to be used in the following: 'a.e.' means 'almost everywhere' and 'for a.e.' means 'for almost every'. Given $a \in L^1(0,T)$, we write $a \succ 0$ if $a \geqslant 0$ for a.e. $t \in [0,T]$ and it is positive in a set of positive measure. Similarly, $a \prec 0$ if $-a \succ 0$. We write $f \in Car([0,T] \times (a,b), \mathbb{R})$ (where $-\infty \leqslant a < b \leqslant +\infty$) if $f : [0,T] \times (a,b) \to \mathbb{R}$ is a L^1-Carathéodory function, that is, it is continuous in the second variable and for every $a < r < s < b$ there exists $h_{r,s} \in L^1(0,T)$ such that $|f(t,x)| \leqslant h_{r,s}(t)$ for all $x \in [r,s]$ and a.e. $t \in [0,T]$. The usual L^p-norm is denoted by $||.||_p$, whereas $||.||$ is used for the norm of the supremum.

2. On the sign of Green's function

Let us consider the linear equation

$$x'' + a(t)x = 0 \tag{1}$$

with periodic conditions

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{2}$$

In this section, we assume conditions under which the only solution of problem (1)–(2) is the trivial one. As a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + a(t)x = h(t)$$

admits a unique T-periodic solution that can be written as

$$x(t) = \int_0^T G(t,s)h(s)\,ds,$$

where $G(t,s)$ is the Green's function of problem (1)–(2). The following result follows from the classical theory of Green's functions (see for instance [11]).

Theorem 2.1. *Let us assume that the distance between two consecutive zeroes of a nontrivial solution of Eq.* (1) *is greater than T. Then, Green's function $G(t,s)$ has constant sign.*

Proof. As G is a continuous function defined on $[0,T]\times[0,T]$, we only have to prove that it does not vanish in any point. By contradiction, let us suppose that there exists $(t_0,s_0)\in[0,T]\times[0,T]$ such that $G(t_0,s_0)=0$. First, let us assume that $(t_0,s_0)\in(0,T)\times[0,T]$. It is known that for a given $s_0\in(0,T)$, $G(t,s_0)$ as a function of t is a solution of (1) in the intervals $[0,s_0)$ and $(s_0,T]$ such that

$$G(0,s_0) = G(T,s_0),$$

$$\frac{\partial}{\partial t}G(0,s_0) = \frac{\partial}{\partial t}G(T,s_0).$$

Then, we can construct

$$x(t) = \begin{cases} G(t,s_0), & t\in[s_0,T] \\ G(t-T,s_0), & t\in[T,s_0+T] \end{cases}$$

This function is class C^1 and in consequence it is a solution of Eq. (1) in the whole interval $[s_0,s_0+T]$. Also, we have that $x(t_0)=0$ and because of hypothesis two consecutive zeroes should be separated by a distance greater than T, there should be a unique zero of x in $[s_0, s_0 + T]$. Moreover as $x(s_0) = x(s_0 + T)$, $x'(t_0) = 0$ and by uniqueness of solution of the initial value problem, $x(t) \equiv 0$ for all t, in contradiction with the elementary properties of Green's function.

Analogously, if $t_0 \in [0, s_0)$, we get a contradiction by following the same reasoning with:

$$x(t) = \begin{cases} G(t - T, s_0), & t \in [s_0 - T, 0], \\ G(t, s_0), & t \in [0, s_0]. \end{cases}$$

Finally, if $s_0 = 0$ or $s_0 = T$, then $G(t, s_0)$ is a solution of (1) in $[0, T]$ such that $G(0, s_0) = G(T, s_0)$, and the same argument as before leads to a contradiction. The proof is concluded since the case $t_0 = 0$ or $t_0 = T$ has been already studied if we take into account that $G(t, s)$ is a symmetric function. $\square$

In order to apply the previous result, we are going to study two different situations. The first one is the following:

Corollary 2.2. *If* $a(t) \prec 0$, *then* $G(t, s) < 0$ *for all* $(t, s) \in [0, T] \times [0, T]$.

Proof. If $a(t) \prec 0$, then it is easy to verify that any nontrivial solution of (1) has at most one zero. Hence by Theorem 2.1 Green's function $G(t, s)$ has constant sign. Let us prove that this sign is negative. The unique T-periodic solution of the equation

$$x'' + a(t)x = 1$$

is just $x(t) = \int_0^T G(t, s)\, ds$, but an integration of the equation over $[0, T]$ yields

$$\int_0^T a(t)x(t)\, dt = T > 0.$$

As by hypothesis $a(t) \prec 0$, $x(t) < 0$ for some t, and as a consequence $G(t, s) < 0$ for all (t, s). $\square$

If on the contrary $a(t) \succ 0$, then the solutions of (1) are oscillating, i.e., there are infinite zeroes, and in order to get the required distance between zeroes, we need to use a maximum principle given in [27] and based on the proof some eigenvalues arguments from [32]. The following best Sobolev constants will be used:

$$K(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \left(\frac{2}{2+q}\right)^{1-2/q} \left(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})} \right)^2 & \text{if } 1 \leqslant q < \infty, \\ \frac{4}{T} & \text{if } q = \infty, \end{cases} \tag{3}$$

where Γ is the Gamma function. For a given p, let us define

$$p^* = \tfrac{p}{p-1} \quad \text{if } 1 \leqslant p < \infty$$

$$p^* = 1 \text{ if } p = +\infty.$$

Corollary 2.3. *Assume that* $a(t) \succ 0$ *and* $a \in L^p(0,T)$ *for some* $1 \leqslant p \leqslant \infty$. *If*

$$||a||_p \leqslant K(2p^*), \tag{4}$$

then $G(t,s) > 0$ *for all* $(t,s) \in [0,T] \times [0,T]$.

Proof. We use some arguments from [27] and they are recapitulated for the convenience of the reader. If $\lambda_1(a)$ is the first antiperiodic eigenvalue, the following inequality is proved in [32]:

$$\lambda_1(a) \geqslant \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{||a||_p}{K(2p^*)}\right).$$

Therefore, under assumption (4), we have $\lambda_1(a) \geqslant 0$. From here, it is not hard to prove that the distance between two consecutive zeroes of a nontrivial solution of Eq. (1) is greater than T (see [27, Lemma 2.1]). Hence, by Theorem 2.1 Green's function has constant sign, and the positive sign of G is determined as in the previous result. $\square$

Remark 1. If $p = +\infty$, then hypothesis (4) is equivalent to $||a||_\infty \leqslant (\frac{\pi}{T})^2$, which is a known criterion for the maximum principle yet used in the related literature (see [3] and the references therein).

The following result provides a way to compute the maximum $M := \max_{t,s} G(t,s)$ and minimum $m := \min_{t,s} G(t,s)$ of a Green's function under the previous assumptions. Let us call u, v the solutions of the linear equation (1) with initial conditions $u(0) = 0,\ u'(0) = 1,\ v(T) = 0,\ v'(T) = -1$.

Proposition 2.0.1. (1) *Under the conditions of Corollary* 2.2,

$$m = \frac{v(0)}{2 + v'(0) - u'(T)}, \qquad M = \frac{\min_t (u(t) + v(t))}{2 + v'(0) - u'(T)}.$$

(2) *Under the conditions of Corollary* 2.3,

$$m = \frac{v(0)}{2 + v'(0) - u'(T)}, \qquad M = \frac{\max_t (u(t) + v(t))}{2 + v'(0) - u'(T)}.$$

Proof. We will make use of the basic theory of Green's functions again. In both cases, Green's function is of the form

$$G(t,s) = \alpha u(t) + \beta v(t) - \frac{1}{v(0)}[u(t)v(s)\mathscr{H}(s-t) + u(s)v(t)\mathscr{H}(t-s)], \tag{5}$$

where $\mathscr{H}$ is the Heavyside function (1 if the argument is nonnegative and zero otherwise) and α, β must be determined by imposing the boundary conditions.

First, by imposing that $G(0,0) = G(0,T)$ we get $\alpha = \beta$. Moreover, by imposing the jump condition on the derivative

$$\frac{\partial}{\partial t}G(0^+,0) - \frac{\partial}{\partial t}G(T^-,0) = 1$$

it is deduced that

$$\alpha = \frac{1}{2 + v'(0) - u'(T)}.$$

Note that α and $a(t)$ have the same sign.

On the other hand, $G(t,s_0)$ as a function of t is a solution of (1) with a positive jump in its derivative when $t = s_0$. Besides, $G(t,s_0)$ is concave in $[0,s_0)$ and $(s_0,T]$. Then, it is easy to conclude that

$$m = \min_{t\in[0,T]} G(t,t).$$

Let us call

$$h(t) := G(t,t) = \alpha(u(t) + v(t)) - \frac{1}{v(0)}u(t)v(t).$$

Then, by using that the wronskian $W(u,v)(t) = u'(t)v(t) - u(t)v'(t)$ is constant and is equal to $v(0)$ for all t, it is proved that

$$h''(t) = \alpha(u''(t) + v''(t)) = -a(t)\alpha(u(t) + v(t)) < 0.$$

As $h(0) = h(T)$, we conclude that

$$m = G(0,0) = \alpha v(0) = \frac{v(0)}{2 + v'(0) - u'(T)}.$$

We emphasize that this reasoning is valid in both cases included in this proposition.

Concerning the maximum, we only have to look at formula (5) to realize that it will be reached when $s = T$ (remember that $u(t)v(t)$ is positive for all $t\in[0,T]$). In consequence,

$$M = \max_{t\in[0,T]} G(t,T) = \max_{t\in[0,T]} [\alpha(u(t)+v(t))]$$

and a consideration over the sign of α in each case leads to the conclusion. □

As a direct application, we can compute the maximum and the minimum of the Green's function when $a(t) \equiv -k^2$, obtaining

$$m_k = -\frac{1+e^{kT}}{2k(e^{kT}-1)}, \quad M_k = -\frac{e^{kT/2}}{k(e^{kT}-1)}$$

and when $a(t) \equiv k^2 < (\frac{\pi}{T})^2$, obtaining

$$\tilde{m}_k = \frac{1}{2k}\cot\left(\frac{kT}{2}\right), \quad \tilde{M}_k = \frac{1}{2k\sin(\frac{kT}{2})}. \tag{6}$$

These explicit values will be employed in Section 4.

3. Main results

Let us consider the periodic boundary value problem

$$\begin{aligned} &x'' = f(t,x),\\ &x(0) = x(T), \quad x'(0) = x'(T), \end{aligned} \tag{7}$$

where $f \in Car([0,T]\times\mathbb{R},\mathbb{R})$.

Let us define the sets of functions

$$\Lambda^- := \{a \in L^1(0,T)\colon\ a \prec 0\},$$

$$\Lambda^+ := \{a \in L^1(0,T)\colon\ a \succ 0, ||a||_p \leqslant K(2p^*) \text{ for some } 1 \leqslant p \leqslant +\infty\}.$$

From the study of Section 2, it is known that if $a \in \Lambda^- \cup \Lambda^+$, then the periodic problem for equation $x'' + a(t)x = 0$ has a Green's function $G(t,s)$ with a definite sign. Following some ideas from [22], along this section, we will exploit this fact together with the following fixed-point theorem for a completely continuous operator in a Banach space [17, p. 148].

Theorem 3.1. *Let $\mathscr{B}$ a Banach space, and let $\mathscr{P} \subset \mathscr{B}$ be a cone in $\mathscr{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathscr{B}$ with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $\mathscr{A} : \mathscr{P} \cap (\bar{\Omega}_1/\Omega_2) \to \mathscr{P}$ be a completely continuous operator such that one of the following conditions is satisfied*:

1. $||\mathscr{A}u|| \leqslant ||u||$, *if* $u \in \mathscr{P} \cap \partial\Omega_1$, *and* $||\mathscr{A}u|| \geqslant ||u||$, *if* $u \in \mathscr{P} \cap \partial\Omega_2$.

2. $||\mathscr{A}u|| \geqslant ||u||$, *if* $u \in \mathscr{P} \cap \partial\Omega_1$, *and* $||\mathscr{A}u|| \leqslant ||u||$, *if* $u \in \mathscr{P} \cap \partial\Omega_2$.

Then, $\mathscr{A}$ *has at least one fixed point in* $\mathscr{P} \cap (\bar{\Omega}_2/\Omega_1)$.

As in Section 2, we denote

$$M := \max_{t,s} G(t,s), \quad m := \min_{t,s} G(t,s).$$

Theorem 3.2. *Let us assume that there exist* $a \in \Lambda^+$ *and* $0 < r < R$ *such that*

$$f(t,x) + a(t)x \geqslant 0 \quad \forall x \in \left[\frac{m}{M}r, \frac{M}{m}R\right], \quad \text{a.e. } t. \tag{8}$$

Then, if one of the following conditions holds

(i)

$$f(t,x) + a(t)x \geqslant \frac{M}{Tm^2}x \quad \forall x \in \left[\frac{m}{M}r, r\right], \text{ a.e. } t,$$
$$f(t,x) + a(t)x \leqslant \frac{1}{TM}x \quad \forall x \in \left[R, \frac{M}{m}R\right], \text{ a.e. } t.$$

(ii)

$$f(t,x) + a(t)x \leqslant \frac{1}{TM}x \quad \forall x \in \left[\frac{m}{M}r, r\right], \text{ a.e. } t,$$
$$f(t,x) + a(t)x \geqslant \frac{M}{Tm^2}x \quad \forall x \in \left[R, \frac{M}{m}R\right], \quad \text{a.e. } t,$$

problem (7) *has a positive solution.*

Proof. From Section 2, it is known that $M > m > 0$. Let us write the equation as

$$x'' + a(t)x = f(t,x) + a(t)x$$

and define the open sets

$$\Omega_1 := \{x \in C(0,T):\ ||x|| < r\},$$
$$\Omega_2 := \left\{x \in C(0,T):\ ||x|| < \frac{M}{m}R\right\}.$$

By defining the open cone

$$\mathscr{P} := \left\{x \in C(0,T):\ \min_t x > \frac{m}{M}||x||\right\}$$

it is easy to prove that if $x \in \mathscr{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$, then

$$\frac{m}{M} r \leqslant x(t) \leqslant \frac{M}{m} R, \quad \forall t.$$

Let us define the completely continuous operator

$$\mathscr{A}x = \int_0^T G(t,s)[f(s,x(s)) + a(s)x(s)]\, ds. \tag{9}$$

Clearly, a solution of problem (7) is just a fixed point of this operator.

As a consequence of (8), we have that if $x \in \mathscr{P} \cap (\Omega_2 \backslash \Omega_1)$, then

$$\mathscr{A}x \geqslant m \int_0^T G(t,s) f(s,x(s))\, ds > \frac{m}{M} \int_0^T \max_t G(t,s) f(s,x(s))\, ds = \frac{m}{M} ||\mathscr{A}x||,$$

that is, $\mathscr{A}(\mathscr{P} \cap (\Omega_2 \backslash \Omega_1)) \subset \mathscr{P}$.

Let us suppose that condition (i) holds, being totally analogous the proof for condition (ii). If $x \in \partial\Omega_1 \cap \mathscr{P}$, then $||x|| = r$ and $\frac{m}{M} r \leqslant x(t) \leqslant r$ for all t. Therefore, by using (i),

$$\mathscr{A}x(t) \geqslant m \int_0^T f(t,x)\, ds \geqslant \frac{M}{Tm} \int_0^T x(s)\, ds \geqslant r = ||x||.$$

Similarly, if $x \in \partial\Omega_2 \cap \mathscr{P}$, then $||x|| = \frac{M}{m} R$ and $R \leqslant x(t) \leqslant \frac{M}{m} R$ for all t. As a consequence,

$$\mathscr{A}x(t) \leqslant M \int_0^T f(t,x)\, ds \leqslant M \frac{1}{TM} \int_0^T x(s)\, ds \leqslant R = ||x||.$$

Now, from Theorem 3.1 there exists $x \in \mathscr{P} \cap (\bar{\Omega}_2 / \Omega_1)$ which is a solution of problem (7). Therefore,

$$\frac{m}{M} r \leqslant x(t) \leqslant \frac{M}{m} R,$$

so in particular such a solution is positive. □

Remark 2. It is not hard to verify that the previous result is true for the equation $x'' = f(t,x,x')$, with f satisfying the required inequalities uniformly in x'. This remark can be extended to the whole paper. However, we have chosen the given presentation for the sake of brevity.

As a direct consequence of the previous result, the following corollary is straightforward by means of the change of variables $y = -x$.

Corollary 3.3. *Let us assume that there exist* $a\in\Lambda^+$ *and* $0<r<R$ *such that*

$$f(t,x)+a(t)x\leqslant 0 \quad \forall x\in\left[-\frac{M}{m}R,-\frac{m}{M}r\right],\ a.e.\ t. \tag{10}$$

Then, if one of the following conditions holds

(i)

$$f(t,x)+a(t)x\leqslant\frac{M}{Tm^2}x \quad \forall x\in\left[-r,-\frac{m}{M}r\right],\ \text{a.e. } t,$$

$$f(t,x)+a(t)x\geqslant\frac{1}{TM}x \quad \forall x\in\left[-\frac{M}{m}R,-R\right],\ \text{a.e. } t.$$

(ii)

$$f(t,x)+a(t)x\geqslant\frac{1}{TM}x \quad \forall x\in\left[-r,-\frac{m}{M}r\right],\quad \text{a.e. } t,$$

$$f(t,x)+a(t)x\leqslant\frac{M}{Tm^2}x \quad \forall x\in\left[-\frac{M}{m}R,-R\right],\quad \text{a.e. } t,$$

problem (7) *has a negative solution.*

In the following results, the knowledge of the sign of Green's function when $a\in\Lambda^-$ is used in order to get to a result similar to that in Theorem 3.2.

Theorem 3.4. *Let us assume that there exist* $a\in\Lambda^-$ *and* $0<r<R$ *such that*

$$f(t,x)+a(t)x\leqslant 0 \quad \forall x\in\left[\frac{M}{m}r,\frac{m}{M}R\right],\quad \text{a.e. } t. \tag{11}$$

Then, if one of the following conditions holds

(i)

$$f(t,x)+a(t)x\leqslant\frac{m}{TM^2}x \quad \forall x\in\left[\frac{M}{m}r,r\right],\ \text{a.e. } t,$$

$$f(t,x)+a(t)x\geqslant\frac{1}{Tm}x \quad \forall x\in\left[R,\frac{m}{M}R\right],\ \text{a.e. } t.$$

(ii)

$$f(t,x)+a(t)x\geqslant\frac{1}{Tm}x \quad \forall x\in\left[\frac{M}{m}r,r\right],\quad \text{a.e. } t,$$

$$f(t,x)+a(t)x\geqslant\frac{m}{TM^2}x \quad \forall x\in\left[R,\frac{m}{M}R\right],\quad \text{a.e. } t,$$

problem (7) *has a positive solution.*

Proof. The proof follows essentially the lines of Theorem 3.2. Now, we have that $m<M<0$. Let us define the open sets

$$\Omega_1 := \{x \in C(0,T): \ ||x|| < r\},$$

$$\Omega_2 := \left\{x \in C(0,T): \ ||x|| < \frac{m}{M} R\right\}$$

and the open cone

$$\mathscr{P} := \left\{x \in C(0,T): \ \min_t \ x > \frac{M}{m} ||x||\right\}.$$

If $x \in \mathscr{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$, then

$$\frac{M}{m} r \leqslant x(t) \leqslant \frac{m}{M} R, \quad \forall t.$$

Now, by defining the operator $\mathscr{A}$ as in (9), the proof continues as in Theorem 3.2. □

Corollary 3.5. *Let us assume that there exist* $a \in \Lambda^-$ *and* $0<r<R$ *such that*

$$f(t,x) + a(t)x \geqslant 0 \quad \forall x \in \left[-\frac{m}{M} R, -\frac{M}{m} r\right], \quad \text{a.e. } t. \tag{12}$$

Then, if one of the following conditions holds

(i)

$$f(t,x) + a(t)x \geqslant \frac{m}{TM^2} x \quad \forall x \in \left[-r, -\frac{M}{m} r\right], \quad \text{a.e. } t,$$

$$f(t,x) + a(t)x \leqslant \frac{1}{Tm} x \quad \forall x \in \left[-\frac{m}{M} R, -R\right], \quad \text{a.e. } t.$$

(ii)

$$f(t,x) + a(t)x \leqslant \frac{1}{Tm} x \quad \forall x \in \left[-r, -\frac{M}{m} r\right], \quad \text{a.e. } t,$$

$$f(t,x) + a(t)x \geqslant \frac{m}{TM^2} x \quad \forall x \in \left[-\frac{m}{M} R, -R\right], \quad \text{a.e. } t,$$

problem (7) *has a negative solution.*

4. Applications

Below, we are going to perform some illustrating applications of the results previously obtained. This section is divided into two main subsections.

4.1. Nonlinearities defined on the whole real line

Along this subsection, it is supposed that $f \in Car([0,T] \times \mathbb{R}, \mathbb{R})$. The first useful observation is that it is not necessary to know explicitly the value of m, M if it is assumed as in [22] an adequate asymptotic behavior of the nonlinearity.

Corollary 4.1. *Let us assume that* $a \in \Lambda^+$ *and* $f(t,x) \geqslant 0$ *for all* $x \in \mathbb{R}^+$ *and for a.e.* $t \in [0,T]$. *Then, if one of the following conditions holds*:

(i)
$$\lim_{x \to 0^+} \frac{f(t,x)}{x} = +\infty, \quad \lim_{x \to +\infty} \frac{f(t,x)}{x} = 0.$$

(ii)
$$\lim_{x \to 0^+} \frac{f(t,x)}{x} = 0, \quad \lim_{x \to +\infty} \frac{f(t,x)}{x} = +\infty$$

uniformly for a.e. t, *problem* (7) *has a positive solution.*

Proof. It is a direct consequence of Theorem 3.2 taking r and R small and big enough, respectively. □

Analogous corollaries can be derived from Theorem 3.4 and Corollaries 3.3 and 3.5 by imposing suitable asymptotic conditions, leading to the existence of positive or negative solutions. These results can be combined to get multiplicity of solutions, like in the following example.

Example 1. In the paper [6], existence of periodic solutions for the Mathieu–Duffing type equation

$$x'' + (a + b\cos t)x + cx^3 = 0 \tag{13}$$

is studied. The study made in the cited paper is based on Schauder's fixed point theorem, but it does not exclude the trivial solution, which always exists. Now, we can assert the following.

Corollary 4.2. *Let us assume that one of the following hypotheses is satisfied*

(i) $a < b \leqslant 0 < c$.
(ii) $c < 0, \ a + b\cos t \in \Lambda^+$.

Then, Eq. (13) *has at least two nontrivial* 2π*-periodic solutions.*

Proof. The nonlinearity $f(t,x) = -cx^3$ is sublinear in 0 and superlinear in ∞, so Theorem 3.2 and Corollary 3.3 (or alternatively Theorem 3.4 and Corollary 3.5) apply directly for small enough r and big enough R giving a couple of one-signed periodic solutions. □

Remark 3. As pointed out in [32], the parameter region (a,b) such that $a + b\cos t \in \Lambda^+$ is a very close approximation to the first stability region of the linear Mathieu equation.

Example 2. If $a \in \Lambda^+$, the equation

$$x'' + a(t)x = sgn(x)|x|^{\nu}$$

has at least two nontrivial T-periodic solutions for any $\nu \in \mathbb{R}^+ \setminus \{1\}$.

Proof. The nonlinearity $f(t,x) = sgn(x)|x|^{\nu}$ is sublinear in ∞ and superlinear in 0 if $0 < \nu < 1$ and vice versa if $\nu > 1$. In any case, the conclusion is the same as in the previous example. □

Example 3. In the recent decades, jumping nonlinearities has become a very popular subject, specially in connection with the modelling of oscillations of suspension bridges (see for instance [18] and their references). In our context, the following result can be proved.

Corollary 4.3. *Let us consider the equation*

$$x'' + a_1(t)x^+ - a_2(t)x^- = f(t,x) \tag{14}$$

with $f \in Car([0,T] \times \mathbb{R}, \mathbb{R})$ *and* $a_1, a_2 \in \Lambda^+$. *Let us suppose that*

$$sgn(x)f(t,x) \geqslant 0$$

for all $x \in \mathbb{R}$ *and a.e.* $t \in [0,T]$ *and one of the following assumptions holds*:

(i)
$$\lim_{x\to 0} \frac{f(t,x)}{x} = 0, \qquad \lim_{|x|\to+\infty} \frac{f(t,x)}{x} = +\infty.$$

(ii)
$$\lim_{x\to 0} \frac{f(t,x)}{x} = +\infty, \qquad \lim_{|x|\to+\infty} \frac{f(t,x)}{x} = 0$$

uniformly in t. *Then, Eq.* (14) *has at least two nontrivial* T*-periodic solutions.*

Note that Example 1 is a particular case when $a_1 \equiv a_2$. In both cases, it is interesting to remark that a_1, a_2 can be unbounded functions, so they may cross the eigenvalues of the linear periodic problem $x''(t) + \lambda x(t) = 0$. This result is to be compared with other results about jumping nonlinearities (see, for instance, [27, Theorem 4.2] as well as [5,10,16,29] and their references).

4.2. Singular nonlinearities

In order to get applications of the results of Section 3, an important observation is that the required hypotheses must be verified only over a compact interval (for instance $[\frac{m}{M}r, \frac{M}{m}R]$ in Theorem 3.2). This fact makes it possible to truncate the equation and cover a wide variety of new equations, not necessarily defined on the whole real line. In this sense, we will pay attention to the so-called *singular equations*. Generally speaking, an equation is called *singular* if the nonlinearity is defined in $\mathbb{R}^+$ for the dependent variable x and tends to infinity when x tends to zero. The opening work of this research line was done by Lazer and Solimini [19], in which the model equations

$$x'' \pm \frac{1}{x^\alpha} = p(t)$$

were studied. Since then, many researchers have studied the existence of periodic solutions for this type of problems and a long list of references is available (a complete bibliography is out of the purpose of this work, see [4,14,9,20,21,23–25,30,31] and their bibliographies only to mention some of them).

Let us consider the periodic problem

$$\begin{aligned} &x'' + a(t)x = f(t,x), \\ &x(0) = x(T), \quad x'(0) = x'(T), \end{aligned} \tag{15}$$

where $a \in L^1(0,T)$ and $f \in Car([0,T] \times \mathbb{R}^+, \mathbb{R})$. It is said that problem (15) has an *attractive singularity* if

$$\lim_{x \to 0^+} f(t,x) = -\infty, \quad \text{unif. a.e. } t \in [0,T] \tag{16}$$

and has a *repulsive singularity* if

$$\lim_{x \to 0^+} f(t,x) = +\infty, \quad \text{unif. a.e. } t \in [0,T]. \tag{17}$$

For attractive singularities, the classical technique for proving existence of periodic solution is the lower and upper solution method. Now, the following result can be proved as a direct corollary of Theorem 3.4. As in the previous sections, m and M are the minimum and the maximum values of the Green's function of $x'' + a(t)x = 0$ with periodic boundary conditions.

Theorem 4.4. *Let us assume* (16) *and that* $a \in \Lambda^-$. *Then, if there exists* $R > 0$ *such that*

$$f(t,x) \leqslant 0, \quad \forall x \in \left(0, \frac{m}{M} R\right]$$

and

$$f(t,x) \geqslant \frac{1}{TM} x, \quad \forall x \in \left[R, \frac{m}{M} R\right],$$

a.e. $t \in [0,T]$, *problem* (15) *has a positive solution.*

Example 4. If $a \in \Lambda^-$, the equation

$$x'' + a(t)x + b(t)x^{\nu} + \frac{1}{x^{\lambda}} = 0$$

with $\lambda > 0$, $0 < \nu < 1$ and $b(t) \succ 0$ has a T-periodic solution.

Proof. Note that if $a \in \Lambda^-$, then $m < M < 0$. The nonlinearity $f(t,x) = -\frac{1}{x^{\alpha}} - b(t)x^{\nu}$ is negative, singular in 0 and sublinear in $+\infty$. So it is easy to verify the conditions of Theorem 4.4. □

Remark 4. Observe that a can be unbounded and zero at some points in the interval $[0,T]$. So it is difficult to find upper and lower solutions by using the standard tricks available in the related literature.

In the following, we consider the repulsive case.

Theorem 4.5. *Let us assume that* (17) *holds and that there exists* $a \in \Lambda^+$ *and* $R > 0$ *such that*

$$f(t,x) \geqslant 0, \quad \forall x \in \left(0, \frac{M}{m} R\right] \tag{18}$$

and

$$f(t,x) \leqslant \frac{1}{TM} x, \quad \forall x \in \left[R, \frac{M}{m} R\right], \tag{19}$$

a.e. $t \in [0,T]$, *problem* (15) *has a positive solution.*

Proof. The proof follows from a direct application of Theorem 3.2 since (17) together with (19) implies that hypothesis (i) holds for r small enough. □

In the context of repulsive singularities, it is usual to assume some kind of 'strong force' condition, which means roughly that the potential in zero is infinity. Typically, this condition is employed to obtain a priori bounds of the solutions. In the founding paper [19], it is proved that the strong singularity condition cannot be dropped without further assumptions, and in fact such a condition has become standard in the related literature. Recently, Rachunková et al. [25] have obtained for the first time existence results in the presence of weak singularities, by using topological degree arguments. In our case, the method of proof enables to deal also with weak singularities, obtaining existence of periodic solutions for a new type of singular equations. Our intention is to check our Theorem 4.5 in two particular cases of singular equations studied in [25,31] and compare the results.

Example 5. The first equation we want to analyze is

$$x'' - \frac{a}{u^\lambda} + k^2 u = e(t) \tag{20}$$

with $a, k, \lambda \in \mathbb{R}^+$ and $e \in L[0,1]$. Let us recall

$$e^* = \sup \text{ess}\, e(t), \quad e_* = \inf \text{ess}\, e(t).$$

Eq. (20) is Example 3.9 in [25], where it is proved that there is a positive periodic solution if $k \in (0, \pi]$ and the following inequality holds:

$$e_* > -\left(\frac{\pi^2 - k^2}{\lambda a}\right)^{\frac{\lambda}{\lambda+1}} (\lambda + 1)a. \tag{21}$$

In particular, there exists a periodic solution if $e_* \geqslant 0$.

Now, a new result can be proved.

Corollary 4.6. *Let us assume that* $e \in L^\infty[0,1]$, $k \in (0, \pi)$ *and let us suppose that* $e_* < 0$ *and*

$$e^* \leqslant \frac{e_*}{\cos^\lambda(\frac{k}{2})} + k \sin(k) \left(\frac{a}{|e_*|}\right)^{\frac{1}{\lambda}}. \tag{22}$$

Then, there is a positive 1*-periodic solution of Eq.* (20).

Proof. If $k \in (0, \pi)$, then $k^2 \in \Lambda^+$ and the minimum and maximum of the corresponding Green's function have been computed in (6), and this results in

$$m = \frac{1}{2k} \cot\left(\frac{k}{2}\right), \quad M = \frac{1}{2k \sin(\frac{k}{2})}.$$

Then, conditions (18)–(19) are reduced to finding $R > 0$ such that

$$\frac{a}{x^\lambda} + e_* \geqslant 0, \quad 0 < x \leqslant \frac{R}{\cos(\frac{k}{2})} \tag{23}$$

and

$$\frac{a}{x^\lambda} + e^* \leqslant 2k \sin\left(\frac{k}{2}\right)x, \quad R \leqslant x \leqslant \frac{R}{\cos(\frac{k}{2})}. \tag{24}$$

By using the monotonocity of the left-hand side, (23) is fulfilled by fixing

$$R := \left(\frac{a}{|e_*|}\right)^{\frac{1}{\lambda}} \cos\left(\frac{k}{2}\right).$$

Analogously, (23) holds if

$$e^* \leqslant 2k \sin\left(\frac{k}{2}\right)R - \frac{a}{R^\lambda}.$$

Now, it is easy to prove through basic operations that this inequality and (22) are equivalent. □

Remark 5. Assumptions (21) and (22) are independent and it is interesting to compare them. Assumption (21) is a uniform lower bound on e. It does not impose any restriction above, but it does not include some rather natural cases: for instance, if e is constant, it is clear that there exists a periodic (constant) solution for any value of $e \in \mathbb{R}$. This trivial case is not covered by (21). On the contrary, (22) cannot handle unbounded forcing terms, but imposes some kind of restriction over the "oscillation" of e (that is, the difference $e^* - e_*$) in which e_* can be under the bound given by (21). Now, the trivial case where e is constant is covered.

Our method can deal with more complex equations (adding, for instance, time-varying coefficients) that seem not to be covered by the results in [25], as in the following example.

Corollary 4.7. *If $a(t) \in \Lambda^+$, $\lambda > 0$ and $b \in L^\infty(0, T)$ is such that $b_* > 0$, then equation*

$$x'' + a(t)x - \frac{b(t)}{x^\lambda} = e(t)$$

has a T-periodic solution for any $e \in L^\infty(0, T)$ verifying that $e_* \geqslant 0$.

Example 6. As an application of the latter result, let us consider the Brillouin beam-focusing equation

$$x'' + a(1 + \cos t)x = \frac{1}{x^\lambda}. \tag{25}$$

This equation has been widely studied as a model for the motion of a magnetically focused axially symmetric electron beam with Brillouin flow (see [1] for a description of the model). The problem of existence of a positive periodic solution has been considered in several papers [2,26,28,30,31], which have improved successively the estimation of a. In the more recent paper, it is proved that Eq. (25) has a 2π-periodic solution if $0<a<0.16488$ and $\lambda \geqslant 1$ by a careful study of a priori bounds and Leray–Schauder degree. The condition over λ is the 'strong force' condition, which is necessary to get the a priori bounds. Now, this condition can be dropped without further assumptions.

Corollary 4.8. *If* $0<a<0.16488$ *and* $\lambda>0$, *then Eq.* (25) *has a* 2π*-periodic solution.*

Proof. It is direct from the previous corollary if we take into account that $a(t) = a(1+\cos t)x \in \Lambda^+$ if

$$a < \max_{1<p<+\infty} \frac{K(2p^*)}{||1+\cos t||_p} \approx 0.16488$$

(the maximum is attained at $p \approx 2.1941$). □

We finish with a multiplicity result for equations with a repulsive singularity.

Theorem 4.9. *Let us assume* (17), $a(t) \in \Lambda^+$ *and the following hypotheses hold*:

1.

$$f(t,x) \geqslant 0 \quad \forall x>0.$$

2.

$$\lim_{x \to +\infty} \frac{f(t,x)}{x} = +\infty.$$

3. *There exists* $R>0$ *such that*

$$f(t,x) \leqslant \frac{1}{TM} x \quad \forall x \in \left(R, \frac{M}{m} R\right)$$

a.e. $t \in [0,T]$. *Then, problem* (15) *has at least two positive solutions.*

Proof. By using the asymptotic behavior of f, it is possible to apply Theorem 2.1 twice, obtaining two periodic solutions x_1, x_2. Let us see that these solutions are different. From the proof of Theorem 2.1, it is deduced that if $x_1 \equiv x_2$, then $R \leqslant x_1(t) \leqslant \frac{M}{m} R$ for all t, but by using assumption 3, we arrive to a contradiction, since

$$x_1(t) = \int_0^T G(t,s) f(s, x_1(s))\, ds \leqslant \frac{1}{TM} \int_0^T G(t,s) x_1(s)\, ds < \frac{1}{T} \int_0^T x_1(s)\, ds$$

for all t, but integrating over a period, the contradiction is aimed at. Therefore, x_1 and x_2 are different solutions and the proof is concluded. □

Other multiplicity results in the presence of a repulsive singularity have been obtained previously in [21, Theorem 6.3]; [12,24]. In these cases, a strong force assumption was essential for the proof, in contrast with our result.

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