Math. Nachr. **251**, 101 – 107 (2003) / **DOI** 10.1002/mana.200310033

A monotone iterative scheme for a nonlinear second order equation based on a generalized anti–maximum principle

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Received 20 August 2001, accepted 4 January 2002
Published online 24 February 2003

Key words Anti–maximum principle, lower and upper solutions, monotone iterative scheme
MSC (2000) 34C25

In this paper, a generalized anti–maximum principle for the second order differential operator with potentials is proved. As an application, we will give a monotone iterative scheme for periodic solutions of nonlinear second order equations. Such a scheme involves the L^p norms of the growth, $1 \le p \le \infty$, while the usual one is just the case $p = \infty$.

1 Introduction

This paper deals with the existence of solutions of the 2π–periodic boundary value problem

$$\begin{aligned} x'' + g(t,x) \ &= \ 0\,, \\ x(0) \ = \ x(2\pi)\,, \quad x'(0) \ &= \ x'(2\pi)\,, \end{aligned} \tag{1.1}$$

where $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is an L^1–Caratheodory function, that is, a function such that $g(\cdot\,, x)$ is measurable for every x, $g(t, \cdot\,)$ is continuous a.e. $t \in [0, 2\pi]$ and for any $M > 0$, there exists a function $h_M \in L^1(0, 2\pi)$ such that $|g(t, x)| \le h_M(t)$ for all $|x| \le M$ and a.e. t.

The method of lower and upper solutions for this problem is a classical topic with a wide literature (see the monograph [3] and the references therein). The most popular result is the following: If α and β are a couple of lower and upper solutions (see definition in Section 3) with $\alpha < \beta$ then problem (1.1) has a solution. In this sense we speak about "usual" ordering. On the contrary, if this order is reversed ($\beta < \alpha$), it is necessary to include additional conditions in order to get solvability of problem (1.1), as it is shown by the simple example $x'' + x = \sin t$, where constant upper and lower solutions in the reversed order appear and however there is resonance. Several authors have succeed in the search of such a condition by using several strategies, which includes variational techniques [1], topological degree [11] and monotone methods [3, 13, 2]. In this paper we are interested in this last option. In few words, the strategy is to exploit an anti–maximum principle for the linear equation in order to construct a monotone approximation scheme converging to the solution.

Our aim is to prove a new anti–maximum principle based on an L^p–norm criterion (Section 2) which generalizes the known L^∞–anti–maximum principle (see [3, Lemma 4.11]). A less known anti–maximum principle appears in [12], by using an L^1–norm criterion. Our result can be seen as a link between these two results. Such a principle enables us to obtain a new existence result (Section 3) which generalizes in some sense the previous ones. Finally, in Section 4 some applications will be developed illustrating the advantages of our result, specially when the nonlinearity does not admit the decomposition $g(t, x) = g(x) + p(t)$.

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2 An anti–maximum principle

Let us recall some results concerning (anti–) periodic eigenvalues. Consider the eigenvalue problem:

$$u'' + (\lambda + \phi(t))u \;=\; 0 \tag{2.1}$$

subject to the periodic boundary condition:

$$u(0) \;=\; u(2\pi)\,, \quad u'(0) \;=\; u'(2\pi)\,, \tag{2.2}$$

or to the anti–periodic boundary condition:

$$u(0) \;=\; -u(2\pi)\,, \quad u'(0) \;=\; -u'(2\pi)\,, \tag{2.3}$$

where $\phi : [0, 2\pi] \to \mathbb{R}$ is integrable and $\phi(t) \geq 0$ a.e. t, $\phi(t) > 0$ on a subset of positive measure. (We write this as $\phi \succ 0$ for simplicity.) It is well–known that there exist two sequences $\{\underline{\lambda}_k(\phi) : k \in \mathbb{N}\}$ and $\{\bar{\lambda}_k(\phi) : k \in \mathbb{Z}^+\}$ such that:

(a) They have the following order:

$$-\infty \;<\; \bar{\lambda}_0(\phi) \;<\; \underline{\lambda}_1(\phi) \;\leq\; \bar{\lambda}_1(\phi) \;<\; \ldots \;<\; \underline{\lambda}_k(\phi) \;\leq\; \bar{\lambda}_k(\phi) \;<\; \ldots$$

and $\underline{\lambda}_k(\phi) \to \infty$, $\bar{\lambda}_k(\phi) \to \infty$ as $k \to \infty$.

(b) λ is an eigenvalue of (2.1) and (2.2) if and only if $\lambda = \underline{\lambda}_k(\phi)$ or $\bar{\lambda}_k(\phi)$ for some even k.

(c) λ is an eigenvalue of (2.1) and (2.3) if and only if $\lambda = \underline{\lambda}_k(\phi)$ or $\bar{\lambda}_k(\phi)$ for some odd k.

These eigenvalues can be characterized using the rotation numbers, cf. [5]. Extend the domain of $\phi(t)$ to the whole $\mathbb{R}$ by 2π–periodicity. Let $u = r\cos\theta$ and $u' = -r\sin\theta$ in (2.1). Then

$$\begin{aligned} r' \;&=\; (\lambda + \phi(t) - 1)r\cos\theta\sin\theta\,, \\ \theta' \;&=\; (\lambda + \phi(t))\cos^2\theta + \sin^2\theta\,. \end{aligned} \tag{2.4}$$

For $t_0,\ \theta_0 \in \mathbb{R}$, let $\theta = \Theta(t; t_0, \theta_0, \lambda)$ be the solution of (2.4) with the initial value: $\theta(t_0) = \theta_0$. Note that $\underline{\lambda}_k(\phi_{t_0}) = \underline{\lambda}_k(\phi)$ and $\bar{\lambda}_k(\phi_{t_0}) = \bar{\lambda}_k(\phi)$ for all t_0, where $\phi_{t_0}(t) \equiv \phi(t + t_0)$. By [5, Proposition 2.2], the first anti–periodic eigenvalue $\underline{\lambda}_1(\phi)$ has the following characterization: $\lambda = \underline{\lambda}_1(\phi)$ if and only if

$$\max_{\theta_0 \in \mathbb{R}}(\Theta(2\pi + t_0; t_0, \theta_0, \lambda) - \theta_0) \;=\; \pi\,, \tag{2.5}$$

where $t_0 \in \mathbb{R}$ is arbitrary.

Lemma 2.1 *Assume that* $\underline{\lambda}_1(\phi) \geq 0$. *Let* $t_0 \in \mathbb{R}$ *and* $u = u(t; t_0)$ *be the solution of*

$$\begin{aligned} u'' + \phi(t)u \;&=\; 0\,, \\ u(t_0) \;=\; 0\,, \quad u'(t_0) \;&=\; 1\,. \end{aligned} \tag{2.6}$$

Then $u(t) > 0$ *for all* $t \in\,]t_0, t_0 + 2\pi[$.

Proof. The equation in (2.6) corresponds to (2.1) with $\lambda = 0$. In the polar coordinates $u = r\cos\theta$, $u' = -r\sin\theta$, $\theta = \theta(t)$ satisfies

$$\begin{aligned} \theta' \;&=\; \phi(t)\cos^2\theta + \sin^2\theta\,, \\ \theta(t_0) \;&=\; -\pi/2\,. \end{aligned} \tag{2.7}$$

Using the notation above, $\theta(t) = \Theta(t; t_0, -\pi/2, 0)$. Since the right–hand side of (2.4) is nondecreasing when λ increases, it follows from the assumption $\underline{\lambda}_1(\phi) \geq 0$ and from the characterization (2.5) that

$$\Theta(2\pi + t_0; t_0, -\pi/2, 0) - (-\pi/2) \;\leq\; \pi\,.$$

Since $\phi \succ 0$, the right–hand side of (2.7) is nonnegative and $\theta(t)$ is nondecreasing. In fact, if $t \in \,]t_0, t_0+2\pi[$, one has

$$\begin{aligned} 0 \;&=\; \theta(t_0) - (-\pi/2) \;<\; \theta(t) - (-\pi/2) \;=\; \Theta(t; t_0, -\pi/2, 0) - (-\pi/2) \\ &<\; \Theta(t_0 + 2\pi; t_0, -\pi/2, 0) - (-\pi/2) \quad \le\; \pi\,, \end{aligned}$$

i.e.,

$$-\pi/2 \;<\; \theta(t) \;<\; \pi/2$$

for $t \in \,]t_0, t_0 + 2\pi[$. Therefore, $u(t) = r(t)\cos\theta(t) > 0$ for $t \in \,]t_0, t_0 + 2\pi[$. □

Remark 2.2 If $u(t_0 + 2\pi; t_0) = 0$ for some t_0, then $u(t_0 + 2\pi; t_0) = 0$ for all t_0. In this case, $\underline{\lambda}_1(\phi) = 0$.

Theorem 2.3 *Let $\phi \in L^1(0, 2\pi)$ be such that $\phi \succ 0$. Assume that ϕ satisfies $\underline{\lambda}_1(\phi) \ge 0$. Then the anti–maximum principle holds for the following problem*

$$\begin{aligned} &u'' + \phi(t)u \;=\; h(t)\,, \\ &u(0) \;=\; u(2\pi)\,, \quad u'(0) \;\ge\; u'(2\pi)\,, \end{aligned} \tag{2.8}$$

which means that if $h \succ 0$, then any solution of (2.8) *is positive on* $[0, 2\pi]$.

P r o o f. Let $u : [0, 2\pi] \to \mathbb{R}$ be a solution of (2.8) in the Caratheodory sense. As before, extend ϕ, h and u to $\mathbb{R}$ by 2π–periodicity. Note that $u(t)$ is, in general, not C^1 in $\mathbb{R}$ because

$$u'(2\pi-) \;=\; u'(2\pi)\,, \quad u'(2\pi+) \;=\; u'(0)\,.$$

We assert that $u(t)$ is one–signed. Once this is true, we simply integrate (2.8) over $[0, 2\pi]$ and obtain

$$\int_0^{2\pi} \phi(t)u(t)\,dt \;=\; \int_0^{2\pi} h(t)\,dt + (u'(0) - u'(2\pi)) \;>\; 0\,.$$

It is thus necessary that $u(t) > 0$ for all $t \in [0, 2\pi]$, completing the proof of the theorem.

Now we prove the assertion. Suppose that $u(t)$ has a zero $t_0 \in [0, 2\pi[$. Let $v(t)$ be the solution of

$$\begin{aligned} &v'' + \phi(t)v \;=\; 0\,, \\ &v(t_0) \;=\; 0\,, \quad v'(t_0) \;=\; 1\,. \end{aligned} \tag{2.9}$$

Multiplying (2.8) and (2.9) by v and u respectively, we get the difference

$$vu'' - uv'' \;=\; h(t)v\,.$$

In the case $t_0 \in \,]0, 2\pi[$, integrating the above equality over $[t_0, t_0 + 2\pi]$ we have

$$\begin{aligned} \int_{t_0}^{t_0+2\pi} h(t)v(t)\,dt \;&=\; (vu' - uv')|_{t_0}^{2\pi-} + (vu' - uv')|_{2\pi+}^{t_0+2\pi} \\ &=\; u'(t_0)v(t_0 + 2\pi) - (u'(0) - u'(2\pi))v(2\pi)\,. \end{aligned}$$

By Lemma 2.1, $v(t) > 0$ on $]t_0, t_0 + 2\pi[$. Using the boundary condition $u'(0) - u'(2\pi) \ge 0$ in (2.9), we arrive at

$$0 \;<\; \int_{t_0}^{t_0+2\pi} h(t)v(t)\,dt \;\le\; u'(t_0)v(t_0 + 2\pi)\,.$$

Since $v(t_0 + 2\pi) \ge 0$, we have $u'(t_0) > 0$.

In the case $t_0 = 0$, a similar argument shows that $u'(2\pi) > 0$. Thus $u'(0) \ge u'(2\pi) > 0$ by the boundary condition in (2.8).

We have proved that

$$u(t_0) \;=\; 0 \;\Longrightarrow\; u'(t_0) \;>\; 0\,,$$

which is impossible because $u(t)$ is periodic. The assertion is thus proved. □

Remark 2.4 (i) It is not difficult to prove that if $h(t)$ in (2.8) satisfies only $h(t) \geq 0$ then the solution $u(t)$ of (2.8) satisfies $u(t) \geq 0$ for all t.

(ii) As in [3, Lemma 4.11], the condition $\underline{\lambda}_1(\phi) \geq 0$ is best possible to guarantee the anti–maximum principle holds for problem (2.8).

Let us recall a lower bound for $\underline{\lambda}_1(\phi)$ from [16]. Suppose that $\phi \succ 0$ and $\phi \in L^p(0, 2\pi)$ for some $1 \leq p \leq \infty$. If the L^p norm $\|\phi\|_p$ satisfies

$$\|\phi\|_p \leq K(2p^*), \quad (p^* = p/(p-1))$$

then

$$\underline{\lambda}_1(\phi) \geq \frac{1}{4}\left(1 - \frac{\|\phi\|_p}{K(2p^*)}\right), \tag{2.10}$$

see (13) in [16]. Here $K(q)$ is the best Sobolev constant in the following inequality:

$$C\,\|u\|_q^2 \leq \|u'\|_2^2 \quad \text{for all} \quad u \in H_0^1(0, 2\pi).$$

Explicitly,

$$K(q) = \begin{cases} \dfrac{1}{q(2\pi)^{2/q}}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)}\right)^2, & \text{if} \quad 1 \leq q < \infty, \\ \dfrac{2}{\pi}, & \text{if} \quad q = \infty. \end{cases} \tag{2.11}$$

See Talenti [14]. Thus we have the following

Corollary 2.5 *Assume that $\phi \succ 0$ and $\phi \in L^p(0, T)$ for some $1 \leq p \leq \infty$. If*

$$\|\phi\|_p \leq K(2p^*), \tag{2.12}$$

then the anti–maximum principle holds for (2.8). *In particular, when $p = \infty$, we arrive at the usual criterion*

$$\|\phi\|_\infty \leq K(2) = 1/4,$$

yet used in [3, 13].

3 Lower and upper solutions in the reversed order

We begin this section with the classical definition of upper and lower solution.

Definition 3.1 A function $\alpha \in C^2(]0, 2\pi[) \cap C^1([0, 2\pi])$ is a *lower solution* of the periodic problem (1.1) if

$$\begin{aligned} &\alpha''(t) + g(t, \alpha(t)) \geq 0 \quad \text{a.e.} \quad t \in]0, 2\pi[, \\ &\alpha(0) = \alpha(2\pi), \ \alpha'(0) \geq \alpha'(2\pi). \end{aligned} \tag{3.1}$$

A function $\beta \in C^2(]0, 2\pi[) \cap C^1([0, 2\pi))$ is an *upper solution* of the periodic problem (1.1) if

$$\begin{aligned} &\beta''(t) + g(t, \beta(t)) \leq 0 \quad \text{a.e.} \quad t \in]0, 2\pi[, \\ &\beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi). \end{aligned} \tag{3.2}$$

Our main result is the following.

Theorem 3.2 *Let us assume the existence of a couple of lower and upper solutions α and β such that $\beta \leq \alpha$. Suppose that there exists a function $\phi \in L^1(0, 2\pi)$ such that $\phi \succ 0$ and*

$$g(t, v) - g(t, u) \leq \phi(t)(v - u) \tag{3.3}$$

for a.e. $t \in [0, 2\pi]$ and all $\beta(t) \leq u \leq v \leq \alpha(t)$. Then, if

$$\|\phi\|_p \leq K(2p^*) \tag{3.4}$$

for some $p \in [1, +\infty]$, problem (1.1) *has a solution $x \in [\beta, \alpha]$.*

Remark 3.3 This theorem generalizes Theorem 4.12 of [3], which is obtained for $p = +\infty$.

P r o o f. The proof consists in the construction of two monotone sequences of upper and lower solutions by a repetitive application of the anti–maximum principle. Although this technique is quite standard, it will be sketched for convenience of the reader.

Step 1. Construction of the monotone sequences.

Let us call $\alpha_0 = \alpha$. Define α_1 as the unique 2π–periodic solution of the linear equation

$$\alpha_1'' + \phi(t)\alpha_1 \;=\; -g(t,\alpha_0) + \phi(t)\alpha_0 \,.$$

As α_0 is a lower solution, we have

$$\alpha_1'' + \phi(t)\alpha_1 \;\leq\; \alpha_0'' + \phi(t)\alpha_0 \,,$$

so if $u = \alpha_0 - \alpha_1$, then $u'' + \phi(t)u \geq 0$. Moreover, $u(0) = u(2\pi)$ and

$$u'(0) \;=\; \alpha_0'(0) - \alpha_1'(0) \;\geq\; \alpha_0'(2\pi) - \alpha_1'(2\pi) \;=\; u'(2\pi)\,.$$

Hence, by the anti–maximum principle $\alpha_0 \geq \alpha_1$. If α_{n+1} is defined recursively as the 2π–periodic solution of

$$\alpha_{n+1}'' + \phi(t)\alpha_{n+1} \;=\; -g(t,\alpha_n) + \phi(t)\alpha_n \,, \tag{3.5}$$

we get a decreasing sequence $\{\alpha_n\}_n$. Analogously, if $\beta_0 = \beta$ and β_{n+1} is defined as the 2π–periodic solution of

$$\beta_{n+1}'' + \phi(t)\beta_{n+1} \;=\; -g(t,\beta_n) + \phi(t)\beta_n \,, \tag{3.6}$$

we have an increasing sequence $\{\beta_n\}_n$.

Step 2. $\beta_n \leq \alpha_n$ for every n.

If we define $u_n = \alpha_n - \beta_n$, we need to prove that $u_n \geq 0$ for all n. By hypothesis, $u_0 \geq 0$. By an argument of induction, let us assume that $u_n \geq 0$. Since $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ are 2π–periodic solutions of (3.5) and (3.6), $u_{n+1}(t) = \alpha_{n+1}(t) - \beta_{n+1}(t)$ is 2π–periodic. Moreover,

$$u_{n+1}'' + \phi(t)u_{n+1} \;=\; -g(t,\alpha_n) + \phi(t)\alpha_n + g(t,\beta_n) - \phi(t)\beta_n \;\geq\; 0$$

because of (3.3). Therefore, $u_{n+1} \geq 0$ by the anti–maximum principle.

Step 3. For every n, β_n is an upper solution and α_n a lower solution.

By definition of α_n and condition (3.3),

$$\alpha_n'' + g(t,\alpha_n) \;=\; g(t,\alpha_n) - g(t,\alpha_{n-1}) - \phi(t)(\alpha_n - \alpha_{n-1}) \;\geq\; 0\,,$$

so α_n is a lower solution. An analogous argument proves that β_n is an upper solution.

Step 4. Conclusion.

We have constructed two sequences of upper solutions $\{\beta_n\}_n$ and lower solutions $\{\alpha_n\}_n$ such that

$$\beta_0 \;\leq\; \beta_1 \;\leq\; \ldots \;\leq\; \beta_n \;\leq\; \ldots \;\leq\; \alpha_n \;\leq\; \ldots \;\leq\; \alpha_1 \;\leq\; \alpha_0\,.$$

Now, a direct application of the Ascoli–Arzela theorem proves that $\alpha_n \to x_M$ and $\beta_n \to x_m$, where x_m, x_M are solutions of problem (1.1) (they could be the same, so no information about multiplicity can be deduced). Moreover these solutions are extremal in the sense that every solution lying between β_0 and α_0 will be also in $[x_m, x_M]$. □

4 Applications

4.1 Class C^1 nonlinearities

Let us consider the problem (1.1) and assume that the derivative of g with respect to the second variable $g_x(t,x)$ is continuous. Several authors have studied conditions over $g_x(t,x)$ which imply the solvability of problem (1.1). A classical result is obtained in [8], which generalizes a previous result by Loud [9]. The Loud–Leach theorem states that if there exists an integer $N \geq 0$ and numbers $a,\ b$ such that

$$N^2 \ < \ a \ \leq \ g_x(t,x) \ \leq \ b \ < \ (N+1)^2$$

for all $t,\ x$, then there exists a unique solution of problem (1.1). Now, we can prove the following results.

Proposition 4.1 *Let us assume the existence of a couple of lower and upper solutions α and β such that $\beta \leq \alpha$. Suppose that there exists a function $\phi \in L^1(0,2\pi)$ such that $\phi \succ 0$ and*

$$g_x(t,x) \ \leq \ \phi(t)$$

for all x and a.e. $t \in [0,2\pi]$. Then, if $\|\phi\|_p \leq K(2p^*)$ for some $p \in [1,+\infty]$, problem (1.1) has a solution $x \in [\beta,\alpha]$.

P r o o f. It is a direct consequence of Theorem 3.2, since condition (3.3) is clearly satisfied by an elementary application of the mean value theorem. □

Theorem 4.2 *If there exists $a > 0$ and $\phi \in L^1(0,2\pi)$ such that*

$$a \ \leq \ g_x(t,x) \ \leq \ \phi(t) \tag{4.1}$$

for all x and a.e. $t \in [0,2\pi]$, and $\|\phi\|_p < K(2p^*)$ for some $p \in [1,+\infty]$, then problem (1.1) has a unique solution which is linearly stable.

P r o o f. By (4.1) it is easy to find constant lower and upper solutions in the reversed order, and then the previous proposition is applied, so there exists a 2π–periodic solution $x(t)$.

As for the uniqueness of the periodic solution, we note that if $x(t),\ y(t)$ are 2π–periodic solutions of (1.1), then $z(t) = x(t) - y(t)$ is a 2π–periodic function satisfying

$$z'' + w(t)z \ = \ 0\,, \tag{4.2}$$

where $w(t) = g_x(t,\xi(t)) \in [a,\ \phi(t)]$ for all t. By the comparison result for eigenvalues, we have $\bar{\lambda}_0(w) \leq -a < 0$ and $\underline{\lambda}_1(w) \geq \underline{\lambda}_1(\phi) > 0$, cf. (2.10). Thus equation (4.2) has only the trivial 2π–periodic solution, which implies that $x(t) = y(t)$.

Finally, the variational equation associated to this periodic solution is

$$z'' + g_x(t,x(t))z \ = \ 0\,,$$

and the stability is a consequence of the results in [16] because $a \leq g_x(t,x(t)) \leq \phi(t)$. □

The idea of this proof comes from [15], where the classical monotone method is used in the study of a singular equation. In the future, it would be interesting to look for conditions leading to Lyapunov stability by using KAM theory and twist maps, in the line of [10].

4.2 Jumping nonlinearities

The equation

$$x'' + \mu(t)x^+ - \nu(t)x^- \ = \ e(t)\,, \tag{4.3}$$

where $x^+ = \max\{x,0\},\ x^- = \max\{-x,0\}$, is very popular since the works of Lazer and McKenna (see [7]) as a simple model for investigating vertical oscillations of long–span suspension bridges. The coefficients $\mu,\ \nu$ are related with the stiffness coefficient of the suspension cables (in our case it would be time–dependent) and then the positiveness is a reasonable assumption, as in the following result.

Theorem 4.3 *Let us assume that* $\mu,\ \nu \in L^1(0, 2\pi)$ *and that there exists* $\epsilon > 0$ *such that*

$$\mu(t)\,,\,\nu(t)\ >\ \epsilon\quad a.e.\quad t\in[0,2\pi]\,. \tag{4.4}$$

Moreover, suppose that $e(t)$ *is continuous. If there exists* $p \in [1, +\infty]$ *such that*

$$\left\|\max_t\{\mu(t),\ \nu(t)\}\right\|_p\ \le\ K(2p^*)\,,$$

then equation (4.3) *has a* 2π*–periodic solution.*

P r o o f. If $\bar e$ is the mean value of e, then we can write $e = \bar e + \tilde e$ with $\int_0^{2\pi} \tilde e(t)\,dt = 0$. Let $E(t)$ be the unique function with zero mean value such that $E'' = \tilde e$. Then, by using condition (4.4) it is easy to prove that $\alpha = M + E(t)$ is a lower solution and $\beta = -M + E(t)$ is an upper solution for M big enough. Note that $\beta < \alpha$. In this case, $g(t, x) = \mu(t)x^+ - \nu(t)x^- - e(t)$, and some easy computations prove that assumption (3.3) holds for $\phi(t) = \max_t\{\mu(t), \nu(t)\}$, and therefore Theorem 3.2 applies. It is interesting to compare this theorem with other results appearing in the literature, as for example [6], [3, Theorem 4.5] or [4, Theorem 3.1]. ☐

Acknowledgements The first author is supported by C.I.C.T. PB98–1294, M.E.C., Spain. The second author is supported by the NNSF, the National 973 Project and the Excellent Personnel Supporting Plan of the Ministry of Education, China. The final version of this manuscript was completed when the second author was visiting Memorial University of Newfoundland, Canada. He would like to thank Professor Xingfu Zou for his kind hospitality during the visit.

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