PERIODIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS WITH NONLINEAR DAMPING

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Introduction

The scientific legacy of Isaac Newton is remembered mainly for the discovery of infinitesimal calculus, at the same time of Leibniz, and gravitational theory, which explains completely the motion of the planets. After the revolution made by Newton's laws in the knowledge of the universe, the interest of the scientific community was centered mainly on conservative systems, due to the influence of celestial mechanics. However, Newton studied also the motion in resisting media, in which the energy is not constant, and the second book of his famous "Philosophiae Naturalis Principia Mathematica," written in 1682, is devoted to this class of problems. After that, nonconservative systems appeared in a wide variety of applications. Some particular cases of these systems are the ordinary second-order differential equations with a nonlinearity depending on the derivative of the solution; in the present century, many authors have focused their interest on this type of equation; see for instance the classical papers [7], [13] and [25].

In the recent paper [2], Cañada and Drábek have considered a class of equations where the nonlinearity depends only on the derivative,

$$x''(t) + f(x'(t)) = p(t), \qquad t \in [0, T],$$

with some boundary conditions (Dirichlet, Neumann or periodic). In the proofs, shooting and alternative methods are employed. Afterwards, Mawhin [18] improved and extended these results in many directions, by using fixed-point theory.

A nonlinearity depending on the derivative of the solution can be regarded as a nonlinear damping term, and it arises in a wide variety of applications,

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such as physics, engineering or electric circuits (see for instance [20] and [21]). In this sense, one of the most interesting topics is that of dry friction, also called Coulomb friction, in honor of Charles Coulomb (1736–1806), who described it for the first time. Its physical model is the relative motion of two surfaces sliding without lubrication. From the mathematical point of view, the damping term is a multivalued map, so we deal with a differential inclusion, as is shown in [5].

The aim of this paper is to study the existence and uniqueness of periodic solutions of three classes of scalar ordinary differential equations in which the nonlinearity depends on both the solution and its derivative. The first of them extends in some sense the results obtained in [18] for the scalar periodic equation. The second one has an attractive singularity and includes gravitational forces. Finally, the case of two attractive singularities is studied, for reasons of completeness. The main tool is the upper-and-lower-solutions method. Also, a result of uniqueness is given.

Besides, since our main motivation in the elaboration of this paper was dry friction, a specific section is devoted to its study, in which an approximation method by single-valued functions is used, together with uniform bounds of the solutions.

With respect to the notation, we are going to denote by $\overline{p} = \frac{1}{T}\int_0^T p(t)\,dt$ the mean value of p and $\tilde{p}(t) = p(t) - \overline{p}$. Further, I is the closed interval $[0,T]$ and $\widetilde{L^n}(I)$, $n \geq 1$ the subspace of the functions of $L^n(I)$ with mean value zero.

1. The nonsingular equation

Let consider the following equation:

$$x''(t) + F(x(t), x'(t)) = \overline{p} + \tilde{p}(t), \quad t \in [0,T] \tag{1}$$

where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function in the first variable and Lipschitz continuous in the second one. In this context we have the following:

Theorem 1. *Suppose that there exists a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for some $-\infty \leq \alpha < \beta \leq \gamma < \delta \leq +\infty$,*

$$F(x,y) \geq f(y), \qquad \forall x \in (\alpha, \beta] \tag{2}$$

and

$$F(x,y) \leq f(y), \qquad \forall x \in [\gamma, \delta) \tag{3}$$

for each $y \in \mathbb{R}$. Fix a positive number $K \leq \min\{\beta - \alpha, \delta - \gamma\}$. Then, for all $\tilde{p} \in \widetilde{L^2}(I)$ with $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}}{T\sqrt{T}}K$, there exists some $\overline{p}_0 \in \mathbb{R}$ such that equation

(1) *has at least a* T*-periodic solution for any* $\overline{p}$ *such that*

$$\inf_{s\in[\gamma,\delta-K)}\Big\{\max_{\substack{x\in[s,s+K]\\ |y|\le\frac{\sqrt{15}K}{T}}}\{F(x,y)-f(y)\}\Big\}<\overline{p}-\overline{p}_0$$

$$<\sup_{s\in(\alpha,\beta-K]}\Big\{\min_{\substack{x\in[s,s+K]\\ |y|\le\frac{\sqrt{15}K}{T}}}\{F(x,y)-f(y)\}\Big\}.$$

Moreover, if this inf (*respectively sup*) *is a minimum* (*respectively maximum*), *then the respective inequality is not strict.*

Proof. If the following equation,

$$x''(t)+f(x'(t))=\overline{p}_0+\tilde{p}(t), \tag{4}$$

is considered, by using the results in [18], there exists some $\overline{p}_0\in\mathbb{R}$ such that equation (4) has a family of T-periodic solutions $u+C$, with $C\in\mathbb{R}$. Now, multiplying (4) by u'' and integrating over a period, we have that $\|u''\|_2\le\|\tilde{p}\|_2$.

Now, we use the following sharp version of the Sobolev inequality,

$$\|u'\|_\infty\le\frac{\sqrt{T}}{2\sqrt{3}}\|u''\|_2,$$

which is proved for instance on page 207 of the book [22]. Hence, we have that $\|u'\|_\infty\le\frac{\sqrt{T}}{2\sqrt{3}}\|\tilde{p}\|_2\le\frac{\sqrt{15}K}{T}$. Moreover, by mimicking the proof of this inequality in Proposition 7.6 of the cited book (with m=0) but applied to the case where $\mathcal{H}$ is the operator of double primitivation of mean value zero, and using that $\sum_{s=1}s^{-4}=\frac{\pi^4}{90}$, one easily arrives at the Sobolev-type inequality

$$\|\tilde{u}\|_\infty\le\frac{T\sqrt{T}}{12\sqrt{5}}\|u''\|_2.$$

In consequence, we have that

$$|u(t_1)-u(t_0)|\le 2\|\tilde{u}\|_\infty\le\frac{T\sqrt{T}}{6\sqrt{5}}\|\tilde{p}\|_2\le K$$

for all $t_0,t_1\in[0,T]$. Therefore, by the condition imposed over $\overline{p}$, it is possible to take $C_1\le C_2$ such that $x_1(t)=u(t)+C_1\in[\alpha,\beta]$ (or $(\alpha,\beta]$ if $\alpha=-\infty$) and $x_2(t)=u(t)+C_2\in[\gamma,\delta]$ (or $[\gamma,\delta)$ if $\delta=+\infty$) is a couple of ordered lower and upper solutions of (1), so using a known result (see for instance Proposition 1 in [10]), there exists a T-periodic solution of (1) between them.

Remark 1. If $\alpha = -\infty$ and $\delta = +\infty$, there is no condition on the norm of $\tilde{p}$. Some examples of F and f satisfying conditions (2) and (3) are the following:

i) $F(x, y) = f(y) + g(x)$, where $xg(x) \leq 0$ for each $|x| > R$.

ii) $F(x, y) = h(x)f(y)$, where f is positive (respectively negative) and h is bounded above (respectively below) for $x > R$ and bounded below (respectively above) for $x < -R$.

iii) $F(x, y) = h(x)f(y) + g(x)$, with h, f and g as before.

Remark 2. From the proof, it is seen that the set of $\overline{p}$ for which equation (4) has a T-periodic solution is included in the set of $\overline{p}$ for which equation (1) has a T-periodic solution.

Remark 3. If p is continuous, it is possible to consider a Nagumo condition on F instead of Lipschitz continuity (see [17]). For example, $F(x, y) = f(y) + g(x)$ satisfies this condition if f and g are only continuous (see [15]). In this way, our study covers some interesting physical examples like damping produced by the motion of an immersed body in a fluid at high Reynolds numbers: the damping force is very nearly proportional to the square of the velocity of the body; that is, $c\,|x'(t)|\,x'(t)$, where c is a positive constant (see [21]); this is also the effect of air-drag on the descent of bodies, a phenomenon exposed by Newton in the already-cited second book of the Principia (see for instance [4]). Likewise, quadratic friction appears in the study of the problem of the vibrations of a suspended wire (see [23] and the references yherein). Nevertheless, as the main object of this paper will be the study of dry friction, hereafter we will assume Lipschitz continuity, which allows us a greater range of "periodic forces" p.

Remark 4. It is not hard to extend Theorem 1 to nonlinearities F depending also on t.

This theorem leads to an existence result for the pendulum equation with nonlinear friction.

Corollary 1. *Let us consider the equation*

$$x''(t) + f(x'(t)) + a\sin(x(t)) = \overline{p} + \tilde{p}(t), \tag{5}$$

where f is a Lipschitz-continuous function. Fix $0 < K \leq \pi$. Then, for all $\tilde{p}(t) \in \widetilde{L^2}(I)$ with $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}}{T\sqrt{T}}K$, there exists some $\overline{p}_0 \in \mathbb{R}$ not depending on a, such that equation (5) *has at least a T-periodic solution for any $\overline{p} \in [\overline{p}_0 - a\sin(\frac{\pi-K}{2}), \overline{p}_0 + a\sin(\frac{\pi-K}{2})]$.*

In particular, if linear friction is considered, then $\overline{p}_0 = 0$, so in consequence, we have the following:

Corollary 2. *Let us consider the equation*

$$x''(t) + cx'(t) + a\sin(x(t)) = \overline{p} + \tilde{p}(t). \tag{6}$$

Fix $0 < K \leq \pi$. *Then, for all* $\tilde{p}(t) \in \widetilde{L^2}(I)$ *with* $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}}{T\sqrt{T}}K$, *equation* (6) *has at least a* T*-periodic solution for all* $\overline{p}$ *such that* $|\overline{p}| \leq a\sin(\frac{\pi-K}{2})$.

It is interesting to observe that similar results are obtained in references [6], [16], [19] and [24] for the damped pendulum case, and also in [11] for the undamped equation ($c = 0$).

2. The equation with attractive singularities

2.1. **Case of one singularity.**

Let us consider the following equation,

$$x''(t) + F(x(t), x'(t)) = \overline{p} + \tilde{p}(t), \quad t \in [0, T] \tag{7}$$

where $F : (0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the first variable and Lipschitz continuous in the second one, such that,

$$\lim_{x \to 0^+} F(x, 0) = +\infty.$$

Gravitational and electrostatic forces are included on this type of nonlinearity. Equation (7) is said to have an attractive singularity in the origin, and we look for positive T-periodic solutions.

Theorem 2. *Under the previous conditions, suppose that there exists a continuous function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that for some* $0 < \alpha < \beta \leq +\infty$,

$$F(x, y) \leq f(y), \qquad \forall x \in [\alpha, \beta) \tag{8}$$

for each $y \in \mathbb{R}$. *Then, for all* $\tilde{p} \in \widetilde{L^2}(I)$ *bounded above with* $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}(\beta-\alpha)}{T\sqrt{T}}$, *there exists some* $\overline{p}_0$ *such that equation* (7) *has a* T*-periodic solution for all* $\overline{p} \geq \overline{p}_0$.

Proof. From [18], it is known that for a given $\tilde{p}$ there exists some $\overline{p}_0$ such that the equation

$$x''(t) + f(x'(t)) = \overline{p}_0 + \tilde{p}(t)$$

has a family of T-periodic solutions $u + C$, for all $C \in \mathbb{R}$. Moreover, as in the proof of Theorem 1, we have that

$$|u(t_1) - u(t_0)| \leq \frac{T\sqrt{T}}{6\sqrt{5}}\|\tilde{p}\|_2 \leq \beta - \alpha$$

for all $t_0, t_1 \in [0,T]$. Therefore, if C is taken such that $u(t)+C \in [\alpha,\beta)$ for all $t \in [0,T]$, we have that

$$u''(t) + F(u(t)+C, u'(t)) \leq u''(t) + f(u'(t)) = \overline{p}_0 + \tilde{p} \leq \overline{p} + \tilde{p}(t),$$

so $u+C$ is an upper solution, and an ordered lower solution can be obtained taking $\epsilon > 0$ such that $F(\epsilon, 0) > \max\{\overline{p} + \tilde{p}(t)\}$ and $\epsilon < \alpha$, so the proof is finished.

Remark 1. Of course, if $\beta = +\infty$, there is no restriction over $\|\tilde{p}\|_2$. Some particular cases are the following:

i) $F(x,y) = f(y) + g(x)$, where

$$\lim_{x\to 0^+} g(x) = +\infty, \qquad \lim_{x\to +\infty} g(x) = 0$$

(for example, the "model case" $g(x) = x^{-a}$, with $a > 0$).

ii) $F(x,y) = \varphi(x)y + g(x)$, where g is as before and there exist $m, M \in \mathbb{R}$ such that $m \leq \varphi(x) \leq M$ for all $x > R$. Then,

$$\varphi(x)y \leq f(y) = \begin{cases} My & \text{if y} > 0 \\ my & \text{if y} \leq 0 \end{cases} \qquad \text{for all } x > R,$$

and clearly f is a continuous function, so (8) holds and therefore the linear case studied in [10],[12] and [14] is extended.

Remark 2. As said in the remarks of the previous subsection, if $\tilde{p}(t)$ is continuous and a Nagumo condition over F is considered instead of Lipschitz continuity, Theorem 2 is still true.

2.2. **The equation with two attractive singularities.** Let us consider the following equation,

$$x''(t) + F(x(t), x'(t)) = \overline{p} + \tilde{p}(t), \quad t \in [0,T] \tag{9}$$

where $F : (0,1) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the first variable and Lipschitz continuous in the second one, such that

$$\lim_{x\to 0^+} F(x,0) = +\infty \quad \text{and} \quad \lim_{x\to 1^-} F(x,0) = -\infty.$$

Then equation (9) is said to have two attractive singularities in 0 and 1, and we look for T-periodic solutions between 0 and 1. From the physical point of view, the model is a couple of charged particles of the same sign fixed in 0 and 1, and a third particle of the opposite sign moving between them. Note that the location of the singularities in 0 and 1 is not restrictive at all.

Theorem 3. *In the previous conditions, equation* (9) *has a T-periodic solution for all $p \in L^\infty(I)$.*

Proof. We take $\epsilon_1 < \epsilon_2$ such that

$$F(\epsilon_1, 0) \geq \|p\|_\infty, \qquad F(\epsilon_2, 0) \leq \|p\|_\infty,$$

so ϵ_1 and ϵ_2 is a couple of ordered upper and lower solutions.

It is interesting to observe that the condition of boundedness over p can be removed if a broader concept of solution, presented in [10], is considered. The main feature of these "generalized" solutions is that they may collide with the singularity in a set of zero measure. We do not insist on this generalization since it can be achieved by using the same arguments developed in [10].

2.3. **Uniqueness.**

Proposition 1. *If F is strictly decreasing in the first variable, equations* (1), (7) *and* (9) *have at most one T-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two different T-periodic solutions with $x_1(t_0) > x_2(t_0)$ for some $t_0 \in [0, T]$. Then, $z(t) = x_1(t) - x_2(t)$ is T-periodic, so $z(t)$ has a positive maximum, say $z(t_1)$; hence, $x_1(t_1) > x_2(t_1)$ and $x_1'(t_1) = x_2'(t_1)$, so in consequence,

$$0 \geq z''(t_1) = F(x_2(t_1), x_1'(t_1)) - F(x_1(t_1), x_1'(t_1)) > 0,$$

a contradiction.

3. Dry friction

Let us consider the following equation,

$$x''(t) + \mu \,\mathrm{sgn}(x'(t)) + g(x(t)) = p(t), \qquad t \in [0, T], \tag{10}$$

where $\mu > 0$ is the so-called kinetic coefficient of friction. We look for T-periodic solutions, in some sense we will make precise in the following. This equation must be understood as the differential inclusion $-x''(t) - g(x(t)) + p(t) \in \mu \,\mathrm{Sgn}(x'(t))$ for almost every $t \in [0, T]$, with

$$\mathrm{Sgn}(x) = \begin{cases} \mathrm{sgn}(x) & \text{if } x \neq 0 \\ \text{[-1,1]} & \text{if } x = 0 \end{cases}.$$

Thus, a T-periodic function $x(t)$ will be a solution of (10) if we can take a measurable selection $s(t) \in \mathrm{Sgn}(x'(t))$ such that equation $x''(t) + \mu s(t) + g(x(t)) = p(t)$ holds for almost every $t \in [0, T]$.

In some cases, this type of equation admits constant solutions, also called stationary solutions. If $x(t) \equiv c$ is a stationary solution, the measurable selection is $s(t) = \frac{p(t) - g(c)}{\mu} \in [-1, 1]$. From this fact, it is easy to describe the set of stationary solutions.

Proposition 2. *A necessary condition for existence of stationary solutions of* (10) *is that* $p \in L^\infty(I)$. *In such a case, the set of stationary solutions is*

$$\Upsilon = \{c \in \mathbb{R} : \max_{t\in[0,T]} p(t) - \mu \leq g(c) \leq \min_{t\in[0,T]} p(t) + \mu\}.$$

Proof. We know that the measurable selection is $-1 \leq \frac{p(t)-g(c)}{\mu} \leq 1$, so we have that $-\mu \leq p(t) - g(c) \leq \mu$, and some easy computations prove the proposition.

From this result, a necessary condition for existence of stationary solutions is that

$$\max_{t\in[0,T]} p(t) - \min_{t\in[0,T]} p(t) \leq 2\mu,$$

that is, a friction coefficient large enough. However, it is interesting to study the existence of T-periodic solutions with arbitrary μ and p, and this is the aim of the following subsections.

3.1. **Dry friction on the nonsingular equation.** In order to simplify, let us consider equation (10) with $g : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function such that for some $-\infty \leq \alpha < \beta \leq \gamma < \delta \leq +\infty$,

$$g(x) \geq 0, \text{ for all } x \in (\alpha, \beta] \quad \text{and} \quad g(x) \leq 0, \text{ for all } x \in [\gamma, \delta).$$

Theorem 4. *For all* $\tilde{p} \in \widetilde{L^2}(I)$ *with* $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}}{T\sqrt{T}} \min\{\beta - \alpha, \delta - \gamma\}$, *there exists some* $\overline{p} \in [-\mu, \mu]$ *such that equation* (10) *has at least a* T*-periodic solution.*

Proof. We are going to approach the multivalued equation (9) by a sequence of single-valued equations. We define the following sequence of Lipschitz-continuous functions, often known as the Yosida approximation:

$$f_n(x) = \begin{cases} 1 & \text{if x} > \frac{1}{\text{n}} \\ nx & \text{if } |\text{x}| \leq \frac{1}{\text{n}} \\ -1 & \text{if x} < -\frac{1}{\text{n}} \end{cases}, \qquad \text{for each } n \in \mathbb{N}.$$

If we consider the sequence of single-valued equations

$$x''(t) + \mu f_n(x'(t)) + g(x(t)) = \overline{p} + \tilde{p}(t), \qquad t \in [0, T], \tag{11}$$

it is known from Section 1 that for each n there exists some $\overline{p}_n$ such that (11) has a T-periodic solution $x_n(t)$. Now, in order to get the convergence of this sequence to a T-periodic solution of (10), we find uniform bounds for x_n and its derivative.

By using the results in [18], for each n there exists some $\overline{p}_n$ such that the equation

$$x''(t) + \mu f_n(x'(t)) = \overline{p}_n + \tilde{p}(t), \qquad t \in [0, T] \tag{12}$$

has a family of T-periodic solutions. For convenience, suppose that u_n is the element of this family such that $\min_{t\in[0,T]} u_n(t) = 0$. Integrating (12) over a period, we have that $|\overline{p}_n| < \mu$. Now by the same reasoning as in Theorem 1,

$$\|u_n\|_\infty \leq \frac{T\sqrt{T}}{6\sqrt{5}} \|\tilde{p}\|_2.$$

Hence, taking $C_1 = \beta - \frac{T\sqrt{T}}{6\sqrt{5}} \|\tilde{p}\|_2$ and $C_2 = \gamma$, we have that $u_n + C_1$ and $u_n + C_2$ are two ordered lower and upper solutions, so there exists a T-periodic solution of (11), and moreover

$$C_1 \leq u_n(t) + C_1 \leq x_n(t) \leq u_n(t) + C_2 \leq \frac{T\sqrt{T}}{6\sqrt{5}} \|\tilde{p}\|_2 + C_2 = C_3, \quad t \in [0, T],$$

so there exist bounds of the sequence $\{x_n\}$ not depending on n. Now, uniform bounds for the derivatives are obtained. Let $K = \max_{x\in[C_1,C_3]} |g(x)|$. Multiplying equation (11) by $x_n''(t)$ and integrating over a period, we have that

$$\|x_n''\|_2^2 = \int_0^T \tilde{p}(t) x_n''(t)\, dt - \int_0^T g(x_n(t)) x_n''(t)\, dt \leq (\|\tilde{p}\|_2 + K\sqrt{T}) \|x''\|_2,$$

so $\|x_n''\|_2 \leq (\|\tilde{p}\|_2 + K\sqrt{T})$, and in consequence

$$\|x_n'\|_\infty \leq \sqrt{T} \|x_n''\|_2 \leq \sqrt{T} (\|\tilde{p}\|_2 + K\sqrt{T}).$$

From these bounds and by using Ascoli's theorem, there exists a subsequence of x_n, which we again index by n, such that x_n converges to some $x_0(t)$ uniformly in C^1, and moreover, taking a subsequence if necessary, there exists some $\overline{p} \in [-\mu, \mu]$ such that $\overline{p}_n \to \overline{p}$. If we take limits in (11), then

$$-x_0''(t) - g(x_0(t)) + \overline{p} + \tilde{p}(t) = \mu \lim_{n\to+\infty} f_n(x_n'(t)) \in \mu\, \mathrm{Sgn}(x_0'(t)),$$

and therefore $x_0(t)$ is a T-periodic solution of equation (10).

An immediate consequence of this theorem is the following corollary about the pendulum equation with dry friction.

Corollary 3. *Let us consider the equation*

$$x''(t) + \mu\ sgn(x'(t)) + a \sin(x(t)) = p(t). \tag{13}$$

Then, for all $\tilde{p}(t) \in \widetilde{L^2}(I)$ *with* $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}}{T\sqrt{T}}\pi$*, there exists some* $\overline{p} \in [-\mu, \mu]$ *not depending on* a*, such that equation* (13) *has at least a* T*-periodic solution.*

Note. It is possible to get a whole interval of admissible mean value, on the line of Section 1. However, we have chosen the previous formulation for reasons of brevity. Also, it is interesting to note that it is possible to add a "viscous damping" term $cx'(t)$ with c constant in equation (10), and the results of this section are still true without modification of the proofs.

3.2. **Dry friction on the singular equation.** Let us consider equation (10), but now $g : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function such that

$$\lim_{x \to 0^+} g(x) = +\infty,$$

and moreover, there exist $0 < \alpha < \beta \leq +\infty$ and $K \in \mathbb{R}$ such that $g(x) \leq K$ for all $x \in [\alpha, \beta)$.

Theorem 5. *In the previous assumptions, for all* $\tilde{p}(t) \in \widetilde{L^2}(I)$ *bounded above with* $\|\tilde{p}\|_2 \leq \frac{6\sqrt{5}}{T\sqrt{T}}(\beta - \alpha)$*, there exists some* $\overline{p}_0 \in [-\mu + K, \mu + K]$ *such that equation* (10) *has at least a* T*-periodic solution for all* $\overline{p} \geq \overline{p}_0$*.*

Proof. Again, we use the Yosida approximations as in the previous subsection. By using the results of [18], for each n there exists some $\overline{p}_n$ such that the equation

$$x''(t) + \mu f_n(x'(t)) + K = \overline{p}_n + \tilde{p}(t)$$

has a T-periodic solution, and integrating this equation over a period, $\overline{p}_n \in [-\mu + K, \mu + K]$. Similar reasonings to that employed in Section 2 prove that equation (11) has a T-periodic solution for all $\overline{p} \geq \overline{p}_n$. Taking a subsequence if necessary, we have that $\{\overline{p}_n\} \to \overline{p}_1 \in [-\mu + K, \mu + K]$, and there exists a T-periodic solution of (11) for all $\overline{p} > \overline{p}_1$, for n great enough. If $\overline{p}_1 = \mu + K$, then there exists a T-periodic solution of (11) for all $\overline{p} \geq \overline{p}_1$, for n great enough, so we take $\overline{p}_0 = \overline{p}_1$. On the other hand, if $\overline{p}_1 \neq \mu + K$, we only have to take $\overline{p}_0 = \frac{\overline{p}_1 + \mu + K}{2}$. In any case, the question is reduced to finding uniform bounds for the sequence of solutions of (11), which is done by using the same techniques employed in the last subsection.

As in Section 2, if $\beta = +\infty$ there is no restriction about $\|\tilde{p}\|_2$. This is the case of $g(x) = x^{-a}$, with $a > 0$. Besides, from the previous theorem, it is easy to prove the following corollary.

Corollary 4. *Let us consider* $g : \mathbb{R}^+ \to \mathbb{R}$ *a continuous function such that* $\lim_{x \to 0^+} g(x) = +\infty$*. Then, for all* $\tilde{p}(t) \in \widetilde{L^2}(I)$ *bounded above, there exists* *some* $\overline{p}_0 \in \mathbb{R}$ *such that equation* (10) *has at least a* T*-periodic solution for all* $\overline{p} \geq \overline{p}_0$.

3.3. **Uniqueness and stability.** In the presence of dry friction, it is possible to have an interval of stationary solutions, so in general there is no uniqueness even if g is assumed strictly decreasing. However, the following result can be proved.

Theorem 6. *Let us consider equation* (10) *with* g *strictly decreasing. Then, if there exist stationary solutions, they are the unique* T*-periodic solutions. On the other hand, if there don't exist stationary solutions, there is at most one* T*-periodic solution, and it is unstable.*

Proof. Suppose that $x_1(t), x_2(t)$ is a couple of different T-periodic solutions, one of them not stationary. Let $z(t) = x_1(t) - x_2(t)$ with $z(t_1) = \max_{t\in[0,T]} z(t)$. It is not restrictive to assume that $z(t) > 0$. From the equation and the fact that g is strictly decreasing, we get that

$$\operatorname{sgn}(x_1'(t_1)) - \operatorname{sgn}(x_2'(t_1)) > 0,$$

so $x_1'(t_1) \geq 0$ and $x_2'(t_1) \leq 0$, and moreover $z'(t_1) = x_1'(t_1) - x_2'(t_1) = 0$. Hence, we have that $x_1'(t_1) = 0$ and $x_2'(t_1) = 0$.

From this fact, and by using that at most one of the solutions is not stationary, it is deduced that there exists $t_2 > t_1$ such that for all $t \in (t_1, t_2)$, $z(t) > 0$, $z'(t) < 0$ and either $x_1'(t) < 0$ or $x_2'(t) > 0$. In consequence,

$$\int_{t_1}^{t_2} \operatorname{sgn}(x_1'(t))\, dt \leq \int_{t_1}^{t_2} \operatorname{sgn}(x_2'(t))\, dt,$$

but a contradiction is obtained, since the reverse inequality is obtained only subtracting the equations and integrating over $[t_1, t_2]$.

Finally, supposing that $\varphi(t)$ is the unique T-periodic solution, we are going to prove that it is unstable. Let us take $x_1(t)$ with initial conditions $x_1(0) > \varphi(0)$, $x_1'(0) = \varphi'(0) > 0$ (if this last condition does not hold, a translation of time can be done). Note that for all $z(t) = x_1(t) - \varphi(t) > 0$ we obtain from equation (10) that

$$z''(t) + \mu \operatorname{sgn} x_1'(t) - \mu \operatorname{sgn} \varphi'(t) = g(\varphi(t)) - g(x_1(t)) > 0, \qquad (14)$$

so the function $z(t)$ has a minimum in 0.

Now, we claim that $z(t)$ is a strictly increasing function for all $t > 0$. If this is not true, there would be some $z(t_1)$ maximum of z with $t_1 > 0$. By using (14), sign $x_1'(t_1) >$ sign $\varphi'(t_1)$. In consequence, $x_1'(t_1) \geq 0$ and $\varphi'(t_1) \leq 0$, but if one of these inequalities is strict, then we would have $z'(t_1) > 0$, so we get that $x_1'(t_1) = 0 = \varphi'(t_1)$. Now, let us take $t_2 > t_1$ such that $x_1'(t) < 0$ for all $t \in (t_1, t_2)$ or $\varphi'(t) > 0$ for all $t \in (t_1, t_2)$; one of these possibilities holds because $z(t_1)$ is a maximum (maybe there is a "deadzone" in the solution, that is, an interval in which the derivative of $z(t)$ vanishes, but then we take the extreme of this interval as t_1). Then,

$$\int_{t_1}^{t_2} \operatorname{sgn}(x_1'(t))\, dt \le \int_{t_1}^{t_2} \operatorname{sgn}(\varphi(t))\, dt,$$

but if (14) is integrated we get the reverse inequality, so a contradiction is obtained.

Therefore, $z(t)$ has no maxima for $t > 0$. Let (w^-, w^+) be the maximal interval of existence of $x_1(t)$. If $w^+ < +\infty$, then $z(t) \to +\infty$ as $t \to w^+$. On the other hand, if $w^+ = +\infty$, we also have that $z(t) \to +\infty$ as $t \to +\infty$. In fact, if $z(t) \to K > 0$ and $K < +\infty$, then there exists t_3 such that $z''(t) < 0$ for all $t > t_3$. Now, from (14) we get that $\operatorname{sgn}(x_1'(t)) > \operatorname{sgn}(\varphi'(t))$ for all $t > t_3$, so $\varphi'(t) \le 0$ for all $t > t_3$, but this is not possible since φ is T-periodic and not stationary. In conclusion, the instability of φ is proved.

4. Additional results

In this section, techniques used before are employed in order to study some classical situations in the presence of nonlinear damping. From now on, the following equation,

$$x''(t) + f(x'(t)) + g(x(t)) = \overline{p} + \tilde{p}(t), \tag{15}$$

is considered, with g continuous and two possibilities:

(1) f continuous and $\tilde{p}(t)$ continuous and T-periodic of mean value zero, and
(2) f Lipschitz continuous and $\tilde{p}(t) \in \widetilde{L^2}(I)$.

4.1. **A Landesman–Lazer-type result.**

Theorem 7. *Suppose that*

$$g(+\infty) = \limsup_{x\to+\infty} g(x) < \liminf_{x\to-\infty} g(x) = g(-\infty). \tag{16}$$

Then, there exists some $\overline{p}_0 \in \mathbb{R}$ such that equation (15) has at least a T-periodic solution for any $\overline{p}$ such that $g(+\infty) < \overline{p} - \overline{p}_0 < g(-\infty)$.

Proof. From [18], there is some $\overline{p}_0$ such that the equation $x''(t) + f(x'(t)) = \overline{p}_0 + \tilde{p}(t)$ has a family of T-periodic solutions $x_0 + C$. For C_1 great enough, we have that $g(x_0(t) + C_1) < \overline{p} - \overline{p}_0$, and some easy computations show that

$x_0(t) + C_1$ is an upper solution of equation (15). In a similar way, a lower solution $x_0(t) + C_2$ is obtained for a suitable $C_2 < C_1$. Hence, we have a couple of ordered lower and upper solutions, so the proof is finished.

Observe that if the linear case $f(y) = cy$ with c constant is considered, then $\overline{p}_0 = 0$, so we extend a classical Landesman–Lazer result to equations with nonlinear damping. Moreover, g may be unbounded (that is, $g(-\infty) = +\infty$ and (or) $g(+\infty) = -\infty$), so we improve Corollary 2 in [11] in two ways. However, if the inequality (16) is reversed, a similar result does not seem easy, and it would require a different approach.

Nevertheless, if an equality appears in (16), the following result can be stated.

Theorem 8. *Let* $K = \frac{T\sqrt{T}}{6\sqrt{5}}\|\tilde{p}\|_2$. *Suppose that* $g(-\infty) = g(+\infty) = L$. *If*

$$x(g(x) - L) \leq 0, \qquad |x| > R \tag{17}$$

for some $R > 0$, *then there exists some* $\overline{p}_0 \in \mathbb{R}$ *such that equation* (15) *has at least a* T*-periodic solution for any* $\overline{p}$ *such that*

$$\inf_{s\in[R,+\infty)} \left\{ \max_{x\in[s,s+K]} \{g(x)\} \right\} < \overline{p} - \overline{p}_0 < \sup_{s\in(-\infty,-R]} \left\{ \min_{x\in[s,s+K]} \{g(x)\} \right\}.$$

Moreover, if this inf (*respectively supremum*) *is a minimum* (*respectively maximum*), *then the respective inequality is not strict. Also, if the inequality* (17) *is strict, this interval of "admissible" mean values is not degenerate.*

Proof. It is an immediate consequence of Theorem 1. With respect to the last assertion, note that condition (17) implies

$$\inf_{s\in[R,+\infty)} \left\{ \max_{x\in[s,s+K]} \{g(x)\} \right\} \leq L \leq \sup_{s\in(-\infty,-R]} \left\{ \min_{x\in[s,s+K]} \{g(x)\} \right\}$$

and that these inequalities are strict if the inequality (17) is strict.

An interesting example in which the latter theorem applies is $g(x) = \frac{-x}{1+x^2}$, which has been studied for the undamped case in [11].

4.2. **Oscillating-expansive nonlinearities.** In [8] is proposed the following nonlinearity,

$$g_0(x) = \begin{cases} \sin(\sqrt{\ln x}) & x \geq 1 \\ 0 & 0 \leq x < 1 \\ -g_0(-x) & x < 0 \end{cases}$$

for an elliptic problem, as an example of the so-called "expansive functions," which were defined in the mentioned paper. We are going to state a new definition that also includes the previous example and is more appropriate in order to apply our method.

Definition. A nonconstant continuous function g is said to be oscillating-expansive if for each $s \in \mathbb{R}$ such that $\inf_{x\in\mathbb{R}} g(x) < s < \sup_{x\in\mathbb{R}} g(x)$, there exists a sequence $\{a_n\}$ with $\lim_{n\to+\infty}\{a_{n+1} - a_n\} = +\infty$ and some positive integer n_0 such that

$$\min_{x\in[a_{2n},a_{2n+1}]} g(x) \geq s \geq \max_{x\in[a_{2n+1},a_{2n+2}]} g(x)$$

for all $n \geq n_0$.

This definition extends in many ways the concept of expansive function. Thus, an expansive function must be bounded and odd, assumptions that are not required in our definition. Besides, if g is oscillating-expansive, $-g$ also is, a fact that is not true for expansive functions. Another interesting feature is that the assumption $\lim_{n\to+\infty}\{\frac{a_{n+1}}{a_n}\} = +\infty$, supposed in the definition of expansive function, implies that $\lim_{n\to+\infty}\{a_{n+1} - a_n\} = +\infty$, but the reverse is not true, so in this sense we have a weaker condition.

However, $g(x) = x$ is expansive but not oscillating-expansive. It is easy to verify the following:

Proposition 3. *If g and $-g$ are expansive functions, then g is an oscillating-expansive function.*

Now, the following theorem can be stated.

Theorem 9. *Let be g an oscillating-expansive nonlinearity. Then, there exists some $\overline{p}_0 \in \mathbb{R}$ such that equation* (15) *has an infinite number of T-periodic solutions for any $\overline{p}$ such that*

$$\inf_{x\in\mathbb{R}} g(x) < \overline{p} - \overline{p}_0 < \sup_{x\in\mathbb{R}} g(x).$$

Proof. Again, $\overline{p}_0$ is the mean value for which the equation $x''(t)+f(x'(t)) = \overline{p}_0 + \tilde{p}(t)$ has a family of T-periodic solutions $x_0 + C$. If $\{a_n\}$ is the sequence of the definition of oscillating-expansive function corresponding to $s = \overline{p} - \overline{p}_0$, then for every n great enough we can take adequate constants so that $x_0 + C_{1n} \in [a_{2n}, a_{2n+1}]$ and $x_0 + C_{2n} \in [a_{2n+1}, a_{2n+2}]$ are a lower and an upper solution of equation (15). In this way, there exists an entire sequence of solutions $x_n \in [a_{2n}, a_{2n+2}]$ of equation (15) for n great enough.

Corollary 5. *There exists some $\overline{p}_0 \in \mathbb{R}$ such that equation* (15) *with $g \equiv g_0$ has an infinite number of T-periodic solutions for all $\overline{p}$ such that $|\overline{p}-\overline{p}_0| < 1$.*

Another interesting consequence is that for a given oscillating-expansive nonlinearity such that $\inf_{x\in\mathbb{R}} g(x) = -\infty$ and $\sup_{x\in\mathbb{R}} g(x) = +\infty$ (for instance, $g(x) = xg_0(x)$), equation (15) has an infinite number of T-periodic solutions without restrictions on $\overline{p}$.

4.3. **Other examples.** The hypothesis (17) considered in the first subsection is a sign condition very frequent in the related literature. Nevertheless, when (17) does not hold, it is still possible to obtain some information by applying our method, as is shown by the following examples. Proofs are omitted for reasons of brevity. Again, we fix the constant $K = \frac{T\sqrt{T}}{6\sqrt{5}}\|\tilde{p}\|_2$.

Example 1. (see [1], example 6.6) Let $g(x) = \frac{1}{1+x^2}$. Then, there exists some $\overline{p}_0 \in \mathbb{R}$ such that equation (15) has at least a T-periodic solution for any $\overline{p}$ such that $\overline{p}_0 < \overline{p} \leq \overline{p}_0 + \frac{4}{4+K^2}$.

Example 2. (vanishing nonlinearity, [11]) Let $g(x) = -e^{-x^2}$. Then, there exists some $\overline{p}_0 \in \mathbb{R}$ such that equation (15) has at least a T-periodic solution for any $\overline{p}$ such that $\overline{p}_0 - e^{-\frac{K^2}{4}} \leq \overline{p} < \overline{p}_0$.

Example 3. ([3], [11]) Let $g(x) = -xe^x$ and suppose $K > 0$. Then, there exists some $\overline{p}_0 \in \mathbb{R}$ such that equation (15) has at least a T-periodic solution for any $\overline{p}$ such that $\overline{p} \leq \overline{p}_0 + g(\frac{Ke^K}{1-e^K})$.

As a final remark, we point out that all the results of this section are true if dry friction is considered in equation (15) instead of f, as can be proved by using the same approximation method of Section 3, and obviously in that case $\overline{p}_0 \in [-\mu, \mu]$.

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References

[1] A. Ambrosetti and G. Mancini, *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance,* J. Differential Equations, 28 (1978), 220–245.

[2] A. Cañada and P. Drábek, *On semilinear problems with nonlinearities depending only on derivatives,* to appear in SIAM J. Math. Analysis.

[3] A. Cañada and P. Martinez-Amores, *Periodic solutions of nonlinear vector ordinary differential equations of higher order at resonance,* Nonlinear Anal., 7 (1983), 747–761.

[4] S. Chandrasekhar, "Newton's Principia for the Common Reader," Clarendon Press, Oxford, 1995.

[5] K. Deimling, *Resonance and Coulomb friction,* Differential and Integral Equations, 7 (1994), 759–765.

[6] G. Fournier and J. Mawhin, *On periodic solutions of forced pendulum-like equations,* J. Differential Equations, 68 (1985), 381–395.

[7] P.O. Frederickson and A.C. Lazer, *Necessary and sufficient damping in a second order oscillator,* Journal of Differential Equations, 5 (1969), 262–270.

[8] S. Fučik and M. Krbec, *Boundary value problems with bounded nonlinearity and general null-space of the linear part,* Math. Z., 155 (1977), 129–138.

[9] C. Gupta, J.J. Nieto, and L. Sánchez, *Periodic solutions of some Liénard and Duffing equations,* J. Math. Anal. Appl., 140 (1989), 67–82.

[10] P. Habets and L. Sánchez, *Periodic solutions of some Liénard equations with singularities,* Proc. Amer. Math. Soc., 109 (1990), 1035–1044.

[11] R. Kannan and R. Ortega, *Periodic solutions of pendulum-type equations,* J. Differential Equations, 59 (1985), 123–144.

[12] A.C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities,* Proc. Amer. Math. Soc., 99 (1987), 109–114.

[13] N. Levinson, *Transformation theory of non-linear differential equations of the second order,* Annals of Mathematics, 45 (1944), 723–737.

[14] P. Martínez-Amores and P.J. Torres, *Dynamics of a periodic differential equation with a singular nonlinearity of attractive type,* J. Math. Anal. Appl., 202 (1996), 1027–1039.

[15] J. Mawhin, *Boundary value problems for a nonlinear second order vector differential equations,* J. Diff. Eq., 16 (1974), 257–269.

[16] J. Mawhin, *Periodic solutions of forced pendulum-like equations,* in "Ordinary and Partial Differential Equations," Dundee, Everitt and Sleeman eds., Lecture Notes in Math., 964, Springer, Berlin, 1982, 458–476.

[17] J. Mawhin, *Points fixes, points critiques et problèmes aux limites,* Semin. Math. Sup., no. 92 (1985), Presses Univ. Montreal.

[18] J. Mawhin, *Some remarks on semilinear problems at resonance where the nonlinearity depends only on the derivatives,* Acta Mathematica et Informatica Universitatis Ostraviensis, 2 (1994), 61–69.

[19] J. Mawhin and M. Willem, *Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations,* J. Differential Equations, 52 (1984), 264–287.

[20] N. Minorski, "Nonlinear Oscillations," Robert E. Krieger Publishing Company, Huntington, New York, 1974.

[21] A.H. Nayfeh and D.T. Mook, "Nonlinear Oscillations," J. Wiley & Sons, New York-Chichester-Brisbane-Toronto, 1979.

[22] N. Rouche and J. Mawhin, "Ordinary differential equations, Stability and Periodic Solutions," Pitman, Boston, 1980.

[23] F. Zanolin, *Periodic solutions for second order differential systems with damping,* Rend. Sem. Mat. Univ. Padova, 65 (1981), 223–234.

[24] F. Zanolin, *Remarks on multiple solutions for nonlinear ordinary differential equations of Liénard type,* Boll. Un. Mat. Ital., 6, 1-B (1982), 683–698.

[25] N.A. Zelezcov, *The method of point transformation and the problem of the forced vibrations of an oscillator with combined friction,* Translation Amer. Math. Soc., Serie 1, vol. 4 (1962), 159–204.