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Twist Solutions of a Hill's Equation with Singular Term *

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Abstract

The paper deals with the existence of twist solutions of a Hill's equation with singular term (often called Brillouin equation in the related literature) for a given region of parameters involved on the equation. Solutions of twist type are in particular Lyapunov stable and present interesting dynamical features around them. The techniques of proof include upper and lower solutions, topological degree and classical tools on normal forms of area preserving maps.

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1 Introduction and main result

Let us consider the equation

$$x''(t) + w(t)x(t) = \frac{1}{x(t)} \tag{1}$$

where $w(t)$ is a continuous 2π-periodic function such that

$$\overline{w} = \frac{1}{2\pi}\int_0^{2\pi} w(t)dt > 0.$$

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Then, we can write $w(t) = \gamma(1 + \delta p(t))$, where $\gamma > 0$, $\delta \geq 0$, $\overline{p} = 0$ and $p_M = \max\{p(t) : t \in [0, 2\pi]\} = 1$. Our purpose is to find regions of parameters (γ, δ) for which there exists a twist periodic solution.

Since the fifties, equation (1) is known to be relevant in the study of a focusing system for an electron beam immersed in a periodic magnetic field [2]. The first mathematical paper devoted to this equation, often called Brillouin equation, is [4]. Starting with this founding paper, many efforts have been made in order to improve the results about existence of 2π-periodic solutions [14, 15, 16].

On the other hand, the stability of such periodic solutions and in general the dynamics of the equation in its neighborhood seem to be an open problem. The importance of this question is clear if we take into account the type of physical application of the equation. In this direction, a first step was accomplished in [13], where under some conditions the ellipticity (that is, linear stability) and uniqueness in some region of the 2π-periodic solution for a particular case of equation (1) were proved. However, it is known that the stability problem is essentially nonlinear and the first approximation method is not useful in order to prove Lyapunov stability. In fact, by using the theory of Birkhoff normal forms together with the Twist Theorem of Moser, it can be proved [9] that Lyapunov stability is closely related with the third approximation, an idea yet presented in the works of Moser [12]. In this paper, which can be seen as the natural continuation of [13], we use this fact in order to show that in some region of parameters $\gamma - \delta$ the periodic solution is of twist type. A periodic solution is of twist type if the associated twist coefficient of the Poincaré map (which correspond to the first nonlinear term of the Birkhoff normal form) is not zero. This fact implies Lyapunov stability as a consequence of the Twist Theorem of Moser, as well as existence of quasiperiodic solutions and infinitely many subharmonic solutions with minimal period going to infinity in its neighborhood.

Our main result is the following.

Theorem 1 *For each p and any $\gamma \in (0, \frac{1}{8})$, $\gamma \neq \frac{1}{32}, \frac{1}{18}$, there exists a finite number of values $F = \{\delta_1, \ldots, \delta_n\}$ (depending on γ, p) such that there exists a twist periodic solution of equation* (1) *for all $\delta \in [0, \dfrac{1}{8\gamma} - 1]/F$.*

It is important to remark that the result is not of the type of small parameters. The strategy of proof is as follows. First, the existence of an elliptic periodic solution in an adequate region of parameters is shown in Section 2. After that, Section 3 is devoted to the study of the twist coefficient as a function of the parameter δ. This idea appears before in [7]. Another related reference is [6], where it is proved that the twist coefficient $\beta(\delta)$ is analytic except on a finite number of singularities (which correspond to strong resonances up to order four). In our case, we will use a formula deduced in [9] in order to prove that the singularities are poles and in consequence there exists a polynomial $p(\delta)$ such that $p(\delta)\beta(\delta)$ is an analytic function of δ. Finally, in Section 4 a computation of the twist coefficient in $\delta = 0$ (by using the same formula deduced in [9]) finishes the proof.

To conclude this Introduction, it is interesting to notice that the existence of twist solutions for equation (1) is a remarkable fact that depends closely on the nonlinearity.

In this sense, it is known that for the equation with a cubic singularity

$$x'' + \gamma x = \frac{1}{x^3},$$

called Ermakov-Pinney equation on the related literature, a non-linear superposition principle holds (see for instance [11, 1]) which implies its integrability. It turns out that all the solutions are periodic of the same period and in consequence the constant solution $x \equiv \gamma^{-\frac{1}{4}}$ is not of twist type (see also Example 3.4 in [9]). It is also easy to construct similar examples with a nonconstant $w(t)$.

2 Existence of elliptic solutions

We begin by recalling the classical definition of upper and lower solution. Let us consider the periodic problem

$$\begin{aligned} x'' + f(t, x(t)) = 0 \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi), \end{aligned} \qquad (2)$$

with f continuous.

Definition. A function $\alpha \in C^2([0, 2\pi])$ is said to be a *lower solution* of problem (2) if and only if

(i) for all $t \in [0, 2\pi]$, $\alpha''(t) + f(t, \alpha(t)) \geq 0$
(ii) $\alpha(0) = \alpha(2\pi)$, $\alpha'(0) \geq \alpha'(2\pi)$.

Analogously, an upper solution $\beta(t)$ is defined by reversing the respective inequalities in the previous definition. A lower solution (resp. upper solution) is called *strict* if the inequality in (i) is strict for all t.

Let F be the unique 2π-periodic function such that $F'' = p$ and $\overline{F} = 0$, and let define $F_M = \max\{F(t) : t \in [0, 2\pi]\}$. The following result provides lower and upper solutions for equation (1).

Lemma 1 *On the previous conditions, if $\delta > 0$ then*

1. *A constant $\beta < \frac{1}{\sqrt{\gamma(1+\delta)}}$ is a strict upper solution of equation* (1).
2. *For any $K > 1$, $\alpha(t) = \frac{K}{\sqrt{\gamma}} e^{\gamma\delta(F_M - F(t))}$ is a strict lower solution of equation* (1).
3. *If $K > 1$, $\beta < \alpha(t)$ for all t.*

The proof of this lemma follows from elementary computations.

As it is known, a periodic solution is said to be *elliptic* if the linearized equation is elliptic, that is, if the Floquet multipliers have modulus 1 and are different from ± 1. This implies that the linearized equation is stable, although Lyapunov stability of the solution is not assured since it may depend on the nonlinear terms.

Lemma 2 *Let $w(t)$ be a non-constant, continuous and 2π-periodic function such that $\overline{w} > 0$ and $w(t) \leq \frac{1}{4}$ for all t. Then, Hill's equation $x'' + w(t)x = 0$ is elliptic.*

then equation (1) has an elliptic 2π-periodic solution for any $(\gamma, \delta) \in \Sigma$. By a translation of this solution, we can assume that $\varphi \equiv 0$ is an elliptic solution of

$$x'' = f(t, x),$$

where $f(t, 0) = 0$ for all t. In order to study the Lyapunov stability of such solution, the first approximation method does not provide any information, so a classical device [12] is to continue to the third approximation of the Taylor expansion. Hence, we can write the previous equation as

$$x'' + a(t)x + b(t)x^2 + c(t)x^3 + r(t, x) = 0, \tag{3}$$

where $a, b, c \in C(\mathbb{R}/2\pi\mathbb{Z})$, $r \in C^{0,3}(\mathbb{R}/2\pi\mathbb{Z} \times (-\epsilon, \epsilon))$ and $\partial^n r(t, 0) = 0$ for all $t \in \mathbb{R}, n = 0, 1, 2, 3$. In our case, it is easy to compute that $c(t) > 0$ for all t and hence the results of [9] can not be applied.

From the theory of normal forms and the Moser twist theorem it is known that in general (concretely, if the Floquet multiplier of the linearized equation is not a unit root of order less or equal than 4) a periodic solution has a twist coefficient β with the property that if β is not zero, the solution is of twist type, which implies in particular Lyapunov stability. Taking γ fixed, we consider the twist coefficient β as a function of the parameter δ. Then, the main purpose of this section is to prove that there exists a polynomial in δ such that $\beta(\delta)$ multiplied by this polynomial is an analytic function. In the reference [6] the analyticity of $\beta(\delta)$ is proved but out of strong resonances. In our case, we will show that strong resonances appear only for a finite number of values of δ, and an analytic conjugation of the Poincaré map together with a formula in [9] prove that singularities in the resonances are poles and lead to the conclusion.

We will need two elementary lemmas.

Lemma 3 *Let us define the function*

$$\begin{aligned} \mathcal{F} : \Sigma &\longrightarrow \mathbb{R}^2 \\ (\gamma, \delta) &\mapsto (\varphi(2\pi, \gamma, \delta), \varphi'(2\pi, \gamma, \delta)), \end{aligned}$$

where $\varphi(t, \gamma, \delta)$ *is the corresponding* 2π*-periodic solution of* (1) *such that* $\varphi(t) \geq \sqrt{\dfrac{1}{\gamma(1+\delta)}}$.

Then, $\mathcal{F}$ *is analytic.*

Proof. It is clear that the Poincaré map

$$(x_0, v_0; \gamma, \delta) \mapsto \mathcal{P}(x_0, v_0; \gamma, \delta)$$

is an analytic function. The initial conditions of $\varphi(t; \gamma, \delta)$ coincide with the fixed point of the Poincaré map (which is unique on certain region by Theorem 3)

$$(x_0, v_0) = \mathcal{P}(x_0, v_0; \gamma, \delta).$$

Now, the elliptic character of the solution makes possible an application of the Implicit Function Theorem in its analytic version, which close the proof. ■

Let us denote by $R[\theta]$ the rotation of angle θ in $\mathbb{R}^2$.

Lemma 4 *Let M_δ be a family of elliptic matrices analytic on the parameter δ. Then, there exists a family of symplectic matrices Q_δ analytic on δ and an analytic function θ_δ such that $Q_\delta^{-1}M_\delta Q_\delta = R[\theta_\delta]$.*

Proof. As M_δ is elliptic, it is conjugate to a rotation, and we only have to prove that there is an analytic choice of the step matrix Q_δ. If $\lambda_\delta, \overline{\lambda}_\delta$ are the eigenvalues of M_δ, it is clear that λ_δ is analytic in δ. If $\lambda_\delta = e^{i\theta_\delta}$, then the angle θ_δ can be chosen analytic in δ. Also the corresponding eigenvector v_δ can be chosen analytic in δ, by imposing to the matrix $Q_\delta = (\text{Re } v_\delta | - \text{Im } v_\delta)$ the normalization condition $\det Q_\delta = \pm 1$. If $\det Q_\delta = 1$, the result follows from [8, Lemma 4]. If on the contrary $\det Q_\delta = -1$, we change the role of λ_δ and $\overline{\lambda}_\delta$. ∎

From now on, we say that an elliptic matrix M has no strong resonances if the eigenvalues are $\lambda = e^{\pm i\theta}$ with $\theta \neq \frac{\pi}{2}, \frac{2\pi}{3}$.

Proposition 1 *For a given domain in $U \subset \mathbb{R}^2$, let us consider an analytic function $F : U \times [0, \Delta] \to \mathbb{R}^2$. For each $\delta \in [0, \Delta]$, denote $F_\delta(x) = F(x, \delta)$ and let us assume that*

(i) $F_\delta(0) = 0$,

(ii) $\det D_x F_\delta(x) = 1, \quad \forall x \in U$,

(iii) The matrix $D_x F_\delta(0)$ is elliptic.

In addition, assume that $D_x F_\delta(0)|_{\delta=0}$ has no strong resonances. Then, there exist numbers $\delta_1, \ldots, \delta_n \in (0, \Delta]$, $\mu_1, \ldots \mu_n \in \mathbb{N}$ and an analytic function $\gamma(\delta) : [0, \Delta] \to \mathbb{R}$ such that if $\delta \neq \delta_i$ for $i = 1, \ldots, n$, then $D_x F_\delta(0)$ has no strong resonances and the twist coefficient is given by

$$\beta(\delta) = \frac{1}{\prod_{i=1}^{n} (\delta - \delta_i)^{\mu_i}} \gamma(\delta).$$

Proof. Let us expand F_δ up to third approximation

$$F_\delta(x) = M_\delta x + F_{2,\delta}(x) + F_{3,\delta}(x) + \cdots,$$

where the dots denote terms of order higher that 3. By applying Lemma 4, there exists a family of symplectic matrices Q_δ analytic on δ and an analytic function θ_δ such that the function $G_\delta = Q_\delta^{-1} F_\delta Q_\delta$ has the expansion

$$G_\delta(x) = R[\theta_\delta] + G_{2,\delta}(x) + G_{3,\delta}(x) + \cdots.$$

Now, Proposition 2.4 of [9] proves that $(1 - \cos\theta_\delta)(1 - \cos 3\theta_\delta)\beta_\delta$ is analytic in δ, since in the explicit formula (2.6) (see also (5) below) the functions A, C, N involved are analytic. As the twist coefficient is a symplectic invariant, the same can be said for the twist coefficient of F_δ. The proof is now complete because the number of zeroes of $1 - \cos\theta_\delta$ and $1 - \cos 3\theta_\delta$ is finite. Here one is applying that $D_x F_\delta(0)|_{\delta=0}$ has no strong resonances. ∎

Corollary 1 *For a given $\gamma \in (0, \frac{1}{8})$, $\gamma \neq \frac{1}{32}, \frac{1}{18}$, there exists a finite number of values $\delta_1, \ldots, \delta_n \in [0, \dfrac{1}{8\gamma} - 1]$ and $\mu_1, \ldots, \mu_n \in I\!N$ an analytic function $\gamma(\delta) : [0, \Delta] \to I\!R$ such that the twist coefficient associated to the Poincaré map is given by*

$$\beta(\delta) = \frac{1}{\prod_{i=1}^{n}(\delta - \delta_i)^{\mu_i}}\gamma(\delta).$$

Proof. Let us consider the variational equation

$$y'' + (\gamma(1 + \delta p(t)) + \frac{1}{\varphi^2(t, \gamma, \delta)})y = 0. \tag{4}$$

If $\Psi(t) = \phi_1(t) + i\phi_2(t)$ is the solution of (4) with initial conditions $\Psi(0) = 1, \Psi'(0) = i$, by Lemma 3 and the analytic dependence with respect to parameters of the linearized equation (4), the discriminant $\Delta(\delta) = \phi_1(2\pi) + \phi_2'(2\pi)$ is an analytic function of δ.

On the other hand, if $\delta = 0$ it is possible to compute explicitly the discriminant. In this case, the variational equation is

$$y'' + 2\gamma y = 0,$$

so

$$\phi_1(t) = \cos(\sqrt{2\gamma}t), \qquad \phi_2(t) = \frac{1}{\sqrt{2\gamma}}\sin(\sqrt{2\gamma}t)$$

and in consequence

$$\Delta(0) = 2\cos(2\pi\sqrt{2\gamma}).$$

The characteristic multipliers of the variational equation (4) are $e^{\pm i\theta}$ with some $0 < \theta < \pi$. Let us study where strong resonances appear. It is known that $\theta = \frac{\pi}{2}$ if and only if $\Delta(\delta) = 0$ and $\theta = \frac{2\pi}{3}$ if and only if $\Delta(\delta) = -1$. Note that in our case, $\Delta(0)$ is explicitly known, and it is different of $0, -1$ if $\gamma \neq \frac{1}{32}, \frac{1}{18}$. As the discriminant is an analytic function of δ, we can conclude that $\Delta(\delta)$ is 0 or -1 in a finite number of values $\delta_1, \ldots, \delta_n$ with multiplicities $\mu_1, \ldots, \mu_n$. On the other hand, as the solution is elliptic the linear part of the Poincaré map is conjugate to a rotation. Hence, Proposition 1 can be applied, leading to the conclusion. ∎

4 Proof of the main result

In order to prove Theorem 1, we are going to prove that $\beta(\delta) \neq 0$ except possibly in a finite number of $\delta \in (0, \dfrac{1}{8\gamma} - 1]$. As it is proved that $\prod_{i=1}^{n}(\delta - \delta_i)^{\mu_i}\beta(\delta)$ is an analytic function of δ, all we have to prove is that this function is not identically zero, and for this it is sufficient to prove that $\beta(0) \neq 0$. Hence, from now on $\delta = 0$. Then, the periodic solution of equation (1) is $\varphi(t) \equiv \dfrac{1}{\sqrt{\gamma}}$. After the translation $y = x - \varphi$ and a Taylor expansion, the equivalent equation is

$$y'' + 2\gamma y - \gamma\sqrt{\gamma}y^2 + \gamma^2 y^3 + \cdots = 0.$$

By rescaling the time with the change $\tau = \sqrt{2\gamma}t$, the later equation becomes periodic of period $T = 2\pi\sqrt{2\gamma}$ and takes the form

$$y'' + y - \frac{\sqrt{\gamma}}{2}y^2 + \frac{\gamma}{2}y^3 + \cdots = 0.$$

Let consider the equation $y'' + y = 0$. The solution $\Psi = \phi_1 + i\phi_2$ with initial conditions $\Psi(0) = 1, \Psi'(0) = i$ is

$$\Psi(t) = \cos t + i \sin t = e^{it}.$$

Following the notation of [9], for $\delta \neq \delta_i$ $(i = 1, \ldots, n)$ the twist coefficient is

$$\beta(\delta) = \mathcal{I}m(\overline{\lambda}N) + \frac{3\sin\theta}{1-\cos\theta}|A|^2 + \frac{\sin 3\theta}{1-\cos 3\theta}|C|^2, \tag{5}$$

as

$$\begin{aligned}
&\lambda = e^{-i\theta}\\
&A = -\tfrac{i\lambda}{4}\textstyle\int_0^T b(t)\overline{\Psi}(t)^2\Psi(t)dt\\
&C = -\tfrac{i\lambda}{4}\textstyle\int_0^T b(t)\Psi(t)^3 dt\\
&N = -\tfrac{3i\lambda}{8}\textstyle\int_0^T c(t)\left|\Psi(t)\right|^4 dt-\\
&\quad -\tfrac{i\lambda}{4}\textstyle\int\int_{\Delta_T} G(t,s)b(t)b(s)\left[2\left|\Psi(t)\right|^2\left|\Psi(s)\right|^2 + \Psi(t)^2\overline{\Psi}(s)^2\right]dsdt,
\end{aligned} \tag{6}$$

where $G(t,s) = \phi_1(t)\phi_2(s) - \phi_2(t)\phi_1(s)$, $\Delta_T = \{(t,s) : 0 < s < t, 0 < t < T\}$ and a, b, c come from the expansion (3).

In our particular case, $\lambda = e^{-iT}$. Some computations in the formulas (6) lead to

$$\begin{aligned}
&A = \tfrac{i\lambda\sqrt{\gamma}}{8}\textstyle\int_0^T e^{-it}dt = \tfrac{\sqrt{\gamma}}{8}\left(e^{-iT} - e^{-2iT}\right),\\
&C = \tfrac{i\lambda\sqrt{\gamma}}{8}\textstyle\int_0^T e^{3it}dt = \tfrac{\sqrt{\gamma}}{24}\left(e^{2iT} - e^{-iT}\right),\\
&N = -\tfrac{3i\lambda\gamma T}{16} - \tfrac{i\lambda\gamma}{16}\textstyle\int\int_{\Delta_T}\sin(s-t)\left[2 + e^{2i(t-s)}\right]dsdt,
\end{aligned}$$

and in consequence

$$\begin{aligned}
|A|^2 &= \frac{\gamma}{32}(1-\cos T),\\
|C|^2 &= \frac{\gamma}{288}(1-\cos 3T),\\
\mathcal{I}m(\overline{\lambda}N) &= -\frac{3\gamma T}{16} + \frac{\gamma}{16}\int\int_{\Delta_T}\sin(t-s)\left[2 + \cos 2(t-s)\right]dsdt\\
&= -\frac{\gamma T}{12} - \frac{3\gamma}{32}\sin T - \frac{\gamma}{288}\sin 3T.
\end{aligned}$$

Taking into account that $\theta = T$, after some simplifications the twist coefficient β is

$$\beta(0) = -\frac{\gamma T}{12},$$

and this coefficient is not zero for all $\gamma \in (0, \frac{1}{8})$. ∎

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