TWIST PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH A COMBINED ATTRACTIVE-REPULSIVE SINGULARITY

Jifeng Chu

Department of Mathematics, College of Science, Hohai University
Nanjing 210098, China

Pedro J. Torres

Departamento de Matemática Aplicada, Universidad de Granada
18071 Granada, Spain

Feng Wang

Department of Mathematics, College of Science, Hohai University
Nanjing 210098, China

Abstract. We study the existence of twist periodic solutions of second order differential equations with an attractive-repulsive singularity. Such twist periodic solutions are stable in the sense of Lyapunov. The proof is based on the third order approximation method in combination with some location information obtained by the averaging method and the method of upper and lower solutions on the reversed order.

1. Introduction

Historically, differential equations with singularities arose naturally in the study of the motion of particles under the influence of gravitational or electrostatic forces. Nowadays, there is a wide range of nonlinear models involving singular terms, that is, functions which become infinite at certain values of the state variable. If we restrict our attention to periodic boundary conditions, the recent monograph [21] presents a collection of singular models arising in different areas of the applied sciences, as well as a suitable list of references.

Although there are some earlier references, a major landmark on the mathematical treatment of the periodic problem with singularities was the paper by Lazer and Solimini [11], in which it was studied the existence of positive T-periodic solutions for the problem

$$\ddot{x} = \frac{l}{x^{\alpha}} - h(t), \tag{1}$$

where $\alpha > 0$, $l \in \mathbb{R} \setminus \{0\}$ and h is a continuous and T-periodic function. The case of an attractive singularity $l > 0$ is easily handled by means of upper and lower solutions, whereas the repulsive case $l < 0$ is more delicate and the so-called strong force assumption $\alpha \geq 1$ is necessary in some way. After this seminal paper, the

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question of the existence of periodic solutions of singular differential equations has been studied by many researchers. Usually, the proof is based on the method of upper and lower solutions, degree theory and fixed point theorems. See [15, 19, 25] and the references therein.

In this paper, we analyze the singular equation with a small parameter

$$\ddot{x} = \frac{g(t)}{x^\alpha} - \frac{\varepsilon h(t)}{x^\beta}, \tag{2}$$

and the equation without parameter

$$\ddot{x} = \frac{g(t)}{x^\alpha} - \frac{h(t)}{x^\beta}, \tag{3}$$

where $\alpha, \beta > 0$ and g, h are continuous and T-periodic functions.

There are several reasons that make equations (2)-(3) valuable to study. They can be regarded in some sense as an extension of the original Lazer-Solimini equation (1), that arises from (3) when g is constant and $\beta = 0$. Besides, equation (3) is influenced by the action of two different singular terms. A full understanding of the effects of each term in the dynamics of the equation is a highly non-trivial problem. In Section 3, the weight h may in principle change sign. This fact is regarded in the literature as a problem with indefinite weight, which is in general more difficult to handle if compared with the case of coefficients with definite signs. This situation appears for instance in the Gylden-Meshcherskii equation, which can be viewed as a Kepler problem with variable mass and describe a variety of phenomena in Celestial Mechanics and Astrophysics, including the evolution of binary stars, dynamics of particles around pulsating stars, photogravitational effects, solar sails and many others (see [1, 4, 8, 16]). The evolution of the radial component of the motion obeys an equation like (3), where the first term of the right-hand side is the centrifugal force (where $g(t) \equiv L^2$ is the constant angular momentum) and the second term is the gravitational force, that can be modulated by a variation of the luminosity or a change of the cross section of the solar sail. A second potential range of applicability is the pulse propagation in nonlinear optical fibers, as described in [21, Section 5.4] (see in particular equation (5.28) therein).

In the literature, it is said that equations (2)-(3) present an attractive-repulsive singularity. Up to now, there are few results about the existence of periodic solutions of this kind of equations [2, 10, 13, 23].

Compared with the existence of periodic solutions, the study of the Lyapunov stability of periodic solutions for singular differential equations is more recent and there are still few works in the literature up to now. Here we would like to give a short brief. As far as we know, along this line, the first result was proved in [18], in which the second author proves that for each p with $\bar{p} = 0$ and $\gamma \in (0, 1/8) \backslash \{1/32, 1/18\}$, there exists a finite number of values $F = \{\delta_1, \cdots, \delta_n\}$ such that equation

$$\ddot{x} + \gamma(1 + \delta p(t))x = \frac{1}{x}$$

has a twist periodic solution for all $\delta \in [0, \frac{1}{8\gamma} - 1]/F$. Later, such a result was improved in [22]. In [20], it is proved that for any p with $\bar{p}$ large enough, equation

$$\ddot{x} + a(t)x = \frac{b(t)}{x^\alpha} + p(t) \tag{4}$$

has a twist periodic solution if the linear equation $\ddot{x} + a(t) = 0$ is 4-elementary and $b(t) > 0$ for every t. Finally, for the case $p(t) \equiv 0$, in [3, 5], it was proved that (4) has a twist periodic solution if $a, b > 0$ and the L^1-norm of a is small enough.

The stability results contained in the previous cited papers are devoted to differential equations with a repulsive singularity. However, to our knowledge there are no results on the Lyapunov stability of periodic solutions of differential equations which admits an attractive-repulsive singularity. The purpose of this paper is to fill, at least partially, this gap.

After some preliminaries in Section 2, in Section 3 we apply the averaging method to obtain a stability result (Theorem 3.2) for the equation (2) with a small parameter. The result is valid for weight g with definite sign and h with indefinite sign and uses heavily the asymptotic information provided by the averaging method. Thus, it is a kind of perturbative result. To obtain a global stability result for equation (3), Section 4 exploits the location information provided by the theory of upper and lower solutions, at the cost of assuming a definite sign on h.

Throughout this paper, we always assume that $\alpha > \beta > 0$. We remark that such condition is natural in some sense if we want to find Lyapunov stable solutions. In fact, if $\beta > \alpha > 0$ the upper and lower solutions constructed in Section 4 are in the right order, leading to an unstable solution (see [6]). This case has been also considered in [13] by the averaging method, obtaining unstable solutions. In the limiting case $\alpha = \beta$, equations (2)-(3) are singular equations with indefinite weight, with rather different properties. For instance, if g, h are positive periodic functions, no periodic solutions of (2) exist for small values of ε.

2. Preliminaries

2.1. **Notations.** Throughout this paper, for a given T-periodic function e, we use the notations

$$e_* = \inf_{t\in[0,T]} e(t), \quad e^* = \sup_{t\in[0,T]} e(t), \quad \tilde{e}_* = \inf_{t\in[0,T]} |e(t)|,$$

and

$$\bar{e} = \frac{1}{T}\int_0^T e(t)dt.$$

Finally, let

$$\gamma = \frac{1}{\alpha - \beta}.$$

Since $\alpha > \beta > 0$, we have $\gamma \in (0, \infty)$.

2.2. **The twist coefficient.** We summarize some basic facts about the method of the third approximation and the twist coefficient. Consider the scalar equation

$$\ddot{u} + f(t, u) = 0, \tag{5}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is T-periodic in t and has continuous derivatives in u up to order 4. Let $\psi(t)$ be a T-periodic solution of (5). By translating the periodic solution $\psi(t)$ of (5) to the origin, we obtain the third order approximation

$$\ddot{u} + a(t)u + b(t)u^2 + c(t)u^3 + o(u^3) = 0, \tag{6}$$

where

$$a(t) = f_u(t, \psi(t)), \quad b(t) = \frac{1}{2} f_{uu}(t, \psi(t)), \quad c(t) = \frac{1}{6} f_{uuu}(t, \psi(t)).$$

The linearized equation of (6) is the Hill equation

$$\ddot{u} + a(t)u = 0. \tag{7}$$

We say (7) is elliptic if its multipliers λ_1, λ_2 satisfy $\lambda_1 = \overline{\lambda}_2$, $|\lambda_1| = 1$, $\lambda_1 \neq \pm 1$. The T-periodic solution ψ of (5) is called 4-elementary if the multipliers λ of (7) satisfy $\lambda^q \neq 1$ for $1 \leq q \leq 4$. The rotation number ρ is defined by the relation $\lambda = \exp(\pm \mathrm{i}\rho \mathrm{T})$, and for convenience we write $\theta = T\rho$. The T-periodic solution $\psi(t)$ is said to be of twist type if the first twist coefficient

$$\mu = \iint_{[0,T]^2} b(t)b(s)r^3(t)r^3(s)\chi_\theta(|\varphi(t) - \varphi(s)|)dtds - \frac{3}{8}\int_0^T c(t)r^4(t)dt \tag{8}$$

is non-zero, where $\Psi(t) = r(t)(\exp)(\mathrm{i}\varphi(\mathrm{t}))$ is the complex solution of (7) with initial conditions $\Psi(0) = 1, \Psi'(0) = i$ and the kernel χ is given by

$$\chi_\theta(\iota) = \frac{3\cos(\iota - \theta/2)}{16\sin(\theta/2)} + \frac{\cos 3(\iota - \theta/2)}{16\sin(3\theta/2)}, \qquad \iota \in [0, \theta].$$

It is worth to observe that an alternative way to compute the rotation number ρ is given by the formula

$$\rho = \frac{1}{T}\int_0^T \frac{ds}{r^2(s)}.$$

The presented formulation for the twist coefficient is a compact form, obtained in [24] (see also [12]), of the original Ortega's formula [14]. As a consequence of Moser's invariant curve theorem [17], a solution of twist type is Lyapunov stable as a consequence of the presence of quasiperiodic solutions (invariant curves of the Poincaré map) in an arbitrarily small neighbor of the solution. Moreover, near a twist periodic solution the typical KAM scenario arises generically, including stability islands, chaotic regions and the existence of infinitely many subharmonics with minimal periods tending to infinity.

In order to determine the sign of the first twist coefficient, it is important to obtain sharp upper and lower bounds for $r(t)$ and the rotation number $\rho \equiv \rho(a)$, which in general is a difficult task. In [5], when a is nonnegative and has a positive mean $\bar{a}$, the first author and Zhang obtained the following asymptotic behavior.

Lemma 2.1. *Assume that a in (7) is nonnegative and has a positive mean $\bar{a}$. Then $r(t) =: r(t, a)$ and $\theta =: \theta(a)$ in the formula (8) satisfies the asymptotic behavior*

$$r(t) = \bar{a}^{-1/4}(1 + O(\bar{a})), \quad \theta(a) = \bar{a}^{1/2}(1 + O(\bar{a})), \quad \textit{when} \quad \bar{a} \to 0^+.$$

2.3. **Averaging method.** The averaging method is perhaps the most popular method for the determination of periodic solutions on differential equations depending on a small parameter. Here, we will just state the classical result for the sake of completeness. For a complete proof and more information about the history of this topic, we refer for instance to [9, Section V.3].

Let us define a continuous function $f : \mathbb{R} \times \Omega \times [0, +\infty) \to \mathbb{R}^n$, where Ω is a domain contained in $\mathbb{R}^n$, such that $f(t, x, \epsilon)$ is T-periodic in t and of class C^1 in x, ϵ. The objective is to find T-periodic solutions of the system

$$\dot{x} = \epsilon f(t, x, \epsilon) \tag{9}$$

for small values of the parameter ϵ. To this aim, let us define the "averaged" function

$$f_0(x) = \frac{1}{T}\int_0^T f(t, x, 0)dt.$$

A zero x_0 of f_0, $f_0(x_0) = 0$, is said to be *non-degenerate* if the determinant of the Jacobian matrix of f_0 evaluated in x_0 is different from zero.

Proposition 1. *Assume that f_0 has a non-degenerate zero $x_0 \in \Omega$. Then there exists $\epsilon_0 > 0$ and a function $x(t, \epsilon)$, continuous for $(t, \epsilon) \in \mathbb{R} \times [0, \epsilon_0]$, such that $x(t, \epsilon)$ is a T-periodic solution of (9) for any $\epsilon \in [0, \epsilon_0]$ and $x(t, 0) = x_0$.*

2.4. **Upper and lower solutions.** The basic theory of upper and lower solutions is exposed in full detail in [7]. For a given second-order scalar equation

$$\ddot{x} + f(t, x) = 0 \tag{10}$$

with T-periodic dependence of t, a T-periodic function $\alpha(t)$ is said to be a *lower solution* if

$$\ddot{\alpha} + f(t, \alpha) \geq 0$$

for all t, whereas a T-periodic function $\beta(t)$ is said to be a *upper solution* if

$$\ddot{\beta} + f(t, \beta) \leq 0.$$

A couple of upper and lower solutions such that $\alpha(t) \leq \beta(t)$ for all t provides a T-periodic solution in between without further assumptions. On the reversed order, however, it is required a suitable non-resonance condition.

Proposition 2. *Assume that there exists upper and lower solutions of (10) such that $\beta(t) \leq \alpha(t)$ for all t. Under the assumption*

$$f_x(t, x) \leq \frac{\pi^2}{T^2}, \qquad \textit{for any } x \in [\beta(t), \alpha(t)],$$

equation (10) has a T-periodic solution x such that $\beta(t) \leq x(t) \leq \alpha(t)$ for every t.

3. Twist periodic solutions of equation (2)

In this section, we will prove that (2) has a twist periodic solution if ε is small enough, g is positive and $\bar{h}$ is positive.

Lemma 3.1. *Assume that $\bar{g} \cdot \bar{h} > 0$. Then equation (2) has a T-periodic solution $x(t, \varepsilon)$ if ε is small enough. Moreover, the following asymptotic behavior holds*

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma} x(t, \varepsilon) = \sigma^{\gamma}, \qquad \textit{uniformly in } t, \tag{11}$$

where

$$\sigma = \frac{\bar{g}}{\bar{h}}.$$

Proof. To this aim, we rewrite the equation as the planar system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{g(t)}{x^{\alpha}} - \frac{\varepsilon h(t)}{x^{\beta}}. \end{aligned} \tag{12}$$

Using the following variables with a small parameter

$$\begin{aligned} x &= u\varepsilon^{-\gamma}, \\ y &= v\varepsilon^{\frac{\gamma(\alpha-1)}{2}}, \\ \epsilon &= \varepsilon^{\frac{\gamma(\alpha+1)}{2}}, \end{aligned}$$

system (12) is equivalent to the system

$$\begin{aligned} \dot{u} &= \epsilon v, \\ \dot{v} &= \epsilon\left(\frac{g(t)}{u^{\alpha}} - \frac{h(t)}{u^{\beta}}\right). \end{aligned} \tag{13}$$

The averaged system corresponding to system (13) is just

$$\begin{aligned} \dot{\xi} &= \epsilon\nu, \\ \dot{\nu} &= \epsilon\left(\frac{\overline{g}}{\xi^{\alpha}} - \frac{\overline{h}}{\xi^{\beta}}\right). \end{aligned} \tag{14}$$

At this point is where we use the assumption $\bar{g}\cdot\bar{h} > 0$. It is a matter of simple computations to verify that the averaged system (14) has a unique constant solution

$$(\xi_0, \nu_0) = (\sigma^{\gamma}, 0)\,,$$

which is non-degenerate, that is, the determinant of the Jacobian matrix evaluated on (ξ_0, ν_0) is different from zero. Then by Proposition 1, the equilibrium (ξ_0, ν_0) is continuable for small ϵ, that is, there exists ϵ_0 such that system (13) has a T-periodic solution $(u(t,\epsilon), v(t,\epsilon))$ for $0 < \epsilon < \epsilon_0$, tending uniformly to (ξ_0, ν_0) as $\epsilon \to 0^+$. Going back to the original variables, equation (2) has a T-periodic solution $x(t,\varepsilon)$ for ε small enough and we have the asymptotic behavior (2). $\square$

Theorem 3.2. *Assume that g and $\bar{h}$ are positive. Then, the T-periodic solution $x(t,\varepsilon)$ obtained in Lemma 3.1 is of twist type if ε is small enough and the following inequalities are satisfied*

$$2\alpha^2 + 2\beta^2 + 7\alpha\beta + \alpha + \beta - 1 \neq 0, \tag{15}$$

$$\frac{\alpha g_*}{\beta h^*} \geq \sigma. \tag{16}$$

Proof. For simplicity, we use $x(t)$ to denote the periodic solution $x(t,\varepsilon)$. Let us fix

$$f(t,x) = \frac{\varepsilon h(t)}{x^{\beta}} - \frac{g(t)}{x^{\alpha}}.$$

Then the coefficients of the third-order approximation are

$$a(t) = a(t,\varepsilon) = \frac{\alpha g(t)}{x^{\alpha+1}} - \frac{\varepsilon\beta h(t)}{x^{\beta+1}}, \tag{17}$$

$$b(t) = b(t,\varepsilon) = \frac{1}{2}\left[\frac{\varepsilon\beta(\beta+1)h(t)}{x^{\beta+2}} - \frac{\alpha(\alpha+1)g(t)}{x^{\alpha+2}}\right], \tag{18}$$

and

$$c(t) = c(t,\varepsilon) = \frac{1}{6}\left[\frac{\alpha(\alpha+1)(\alpha+2)g(t)}{x^{\alpha+3}} - \frac{\varepsilon\beta(\beta+1)(\beta+2)h(t)}{x^{\beta+3}}\right]. \tag{19}$$

By inserting the limit (11) into (17)-(19), we have

$$\lim_{\varepsilon\to 0}\frac{a(t)}{\varepsilon^{\gamma(\alpha+1)}} = \frac{\alpha g(t)}{\sigma^{\gamma(\alpha+1)}} - \frac{\beta h(t)}{\sigma^{\gamma(\beta+1)}}, \tag{20}$$

$$\lim_{\varepsilon\to 0}\frac{b(t)}{\varepsilon^{\gamma(\alpha+2)}} = \frac{1}{2}\left[\frac{\beta(\beta+1)h(t)}{\sigma^{\gamma(\beta+2)}} - \frac{\alpha(\alpha+1)g(t)}{\sigma^{\gamma(\alpha+2)}}\right], \tag{21}$$

and

$$\lim_{\varepsilon\to 0}\frac{c(t)}{\varepsilon^{\gamma(\alpha+3)}} = \frac{1}{6}\left[\frac{\alpha(\alpha+1)(\alpha+2)g(t)}{\sigma^{\gamma(\alpha+3)}} - \frac{\beta(\beta+1)(\beta+2)h(t)}{\sigma^{\gamma(\beta+3)}}\right]. \tag{22}$$

Using condition (16), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{a(t)}{\varepsilon^{\gamma(\alpha+1)}} &= \frac{\alpha g(t)}{\sigma^{\gamma(\alpha+1)}}-\frac{\beta h(t)}{\sigma^{\gamma(\beta+1)}}\\ &\geq \frac{\alpha g_*}{\sigma^{\gamma(\alpha+1)}}-\frac{\beta h^*}{\sigma^{\gamma(\beta+1)}}\\ &\geq \frac{\sigma\beta h^*}{\sigma^{\gamma(\alpha+1)}}-\frac{\beta h^*}{\sigma^{\gamma(\beta+1)}}=0,\end{aligned}$$

which implies that $a(t)$ is nonnegative if ε is small enough. This step requires the hypothesis that g is positive (and not only its mean value).

Moreover,

$$\lim_{\varepsilon\to 0}\frac{\bar a}{\varepsilon^{\gamma(\alpha+1)}}=(\alpha-\beta)\frac{\bar h^{\gamma(\alpha+1)}}{\bar g^{\gamma(\beta+1)}}.$$

By Lemma 2.1 and straighforward computations, we obtain

$$\lim_{\varepsilon\to 0}\frac{\theta}{\sqrt{\varepsilon^{\gamma(\alpha+1)}}}=T\sqrt{\alpha-\beta}\sqrt{\frac{\bar h^{\gamma(\alpha+1)}}{\bar g^{\gamma(\beta+1)}}},\tag{23}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{\gamma(\alpha+1)}{4}}r(t)=\frac{1}{\sqrt[4]{\alpha-\beta}}\sqrt[4]{\frac{\bar g^{\gamma(\beta+1)}}{\bar h^{\gamma(\alpha+1)}}}.\tag{24}$$

From [12, Lemma 3.6], it follows that (7) is elliptic and 4-elementary if ε is small enough. Moreover, it is proved in [24] that $\chi_\theta(\iota)$ is symmetric with respect to the line $\iota=\theta/2$, $\chi_\theta(x)$ is strictly increasing on $[0,\theta/2]$ and strictly decreasing on $[\theta/2,\theta]$. Therefore,

$$\min_{\iota\in[0,\theta]}\chi_\theta(\iota)=\chi_\theta(0)=\frac{3\cos(\theta/2)}{16\sin(\theta/2)}+\frac{\cos(3\theta/2)}{16\sin(3\theta/2)},\tag{25}$$

and

$$\max_{\iota\in[0,\theta]}\chi_\theta(\iota)=\chi_\theta(\theta/2)=\frac{3}{16\sin(\theta/2)}+\frac{1}{16\sin(3\theta/2)}.\tag{26}$$

It follows from (25)-(26) that when ε is small enough, we obtain

$$\chi_\theta(\iota)=\frac{5}{12\theta}\left(1+O(\theta^2)\right)=\frac{5}{12}(T\sqrt{\bar a})^{-1}+O(\bar a),$$

in which we have used the fact $\theta=T\rho$ and the rotation number $\rho=\sqrt{\bar a}+O(\bar a)$ when $\bar a\to 0^+$. Therefore,

$$\lim_{\varepsilon\to 0}[\varepsilon^{\frac{1}{2}\gamma(\alpha+1)}\chi_\theta(|\varphi(t)-\varphi(s)|)]=\frac{5}{12T}\frac{1}{\sqrt{(\alpha-\beta)}}\sqrt{\frac{\bar g^{\gamma(\beta+1)}}{\bar h^{\gamma(\alpha+1)}}}.$$

Define

$$\mu_1=\iint_{[0,T]^2}b(t)b(s)r^3(t)r^3(s)\chi_\theta(|\varphi(t)-\varphi(s)|)dtds,$$

and

$$\mu_2=\int_0^T c(t)r^4(t)dt.$$

Using (20)-(22), (23), (24) and the above facts, we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\mu_2}{\varepsilon^{2\gamma}} &= \lim_{\varepsilon\to 0}\int_0^T \frac{c(t)}{\varepsilon^{\gamma(\alpha+3)}}\cdot \varepsilon^{\gamma(\alpha+1)} r^4(t)dt \\
&= \int_0^T \frac{1}{6}\left[\frac{\alpha(\alpha+1)(\alpha+2)g(t)}{\sigma^{\gamma(\alpha+3)}} - \frac{\beta(\beta+1)(\beta+2)h(t)}{\sigma^{\gamma(\beta+3)}}\right]\cdot \frac{1}{\alpha-\beta}\frac{\bar{g}^{\gamma(\beta+1)}}{\bar{h}^{\gamma(\alpha+1)}}dt \\
&= \frac{T}{6(\alpha-\beta)}\frac{\bar{g}^{\gamma(\beta+1)}}{\bar{h}^{\gamma(\alpha+1)}}\left[\frac{\alpha(\alpha+1)(\alpha+2)\bar{g}}{\sigma^{\gamma(\alpha+3)}} - \frac{\beta(\beta+1)(\beta+2)\bar{h}}{\sigma^{\gamma(\beta+3)}}\right] \\
&= \frac{T}{6(\alpha-\beta)}\left[\frac{\alpha(\alpha+1)(\alpha+2)}{\sigma^{2\gamma}} - \frac{\beta(\beta+1)(\beta+2)}{\sigma^{2\gamma}}\right] \\
&= \frac{T}{6\sigma^{2\gamma}}(\alpha^2+\alpha\beta+\beta^2+3\alpha+3\beta+2).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\mu_1}{\varepsilon^{2\gamma}} = \lim_{\varepsilon\to 0}&\iint_{[0,T]^2}\left[\frac{b(t)}{\varepsilon^{\gamma(\alpha+2)}}\right]\left[\frac{b(s)}{\varepsilon^{\gamma(\alpha+2)}}\right]\left[\varepsilon^{\frac{3}{4}\gamma(\alpha+1)}r^3(t)\right] \\
\times &\left[\varepsilon^{\frac{3}{4}\gamma(\alpha+1)}r^3(s)\right]\left[\varepsilon^{\frac{1}{2}\gamma(\alpha+1)}\chi_\theta(|\varphi(t)-\varphi(s)|)\right]dtds.
\end{aligned}
$$

By the same procedure, we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\mu_1}{\varepsilon^{2\gamma}} &= \iint_{[0,T]^2}\frac{1}{4}\left[\frac{\beta(\beta+1)h(t)}{\sigma^{\gamma(\beta+2)}} - \frac{\alpha(\alpha+1)g(t)}{\sigma^{\gamma(\alpha+2)}}\right]\left[\frac{\beta(\beta+1)h(s)}{\sigma^{\gamma(\beta+2)}} - \frac{\alpha(\alpha+1)g(s)}{\sigma^{\gamma(\alpha+2)}}\right] \\
&\times \frac{1}{(\alpha-\beta)^{3/2}}\left[\frac{\bar{g}^{\gamma(\beta+1)}}{\bar{h}^{\gamma(\alpha+1)}}\right]^{3/2}\frac{5}{12T}\frac{1}{\sqrt{(\alpha-\beta)}}\sqrt{\frac{\bar{g}^{\gamma(\beta+1)}}{\bar{h}^{\gamma(\alpha+1)}}}dtds \\
&= \frac{5}{48T}\frac{1}{(\alpha-\beta)^2}\frac{\bar{g}^{2\gamma(\beta+1)}}{\bar{h}^{2\gamma(\alpha+1)}}\iint_{[0,T]^2}\left[\frac{\beta(\beta+1)h(t)}{\sigma^{\gamma(\beta+2)}} - \frac{\alpha(\alpha+1)g(t)}{\sigma^{\gamma(\alpha+2)}}\right] \\
&\times \left[\frac{\beta(\beta+1)h(s)}{\sigma^{\gamma(\beta+2)}} - \frac{\alpha(\alpha+1)g(s)}{\sigma^{\gamma(\alpha+2)}}\right]dtds \\
&= \frac{5}{48T}\frac{1}{(\alpha-\beta)^2}\frac{\bar{g}^{2\gamma(\beta+1)}}{\bar{h}^{2\gamma(\alpha+1)}}\left[\frac{\beta(\beta+1)T\bar{h}}{\sigma^{\gamma(\beta+2)}} - \frac{\alpha(\alpha+1)T\bar{g}}{\sigma^{\gamma(\alpha+2)}}\right]^2 \\
&= \frac{5T}{48\sigma^{2\gamma}}(\alpha+\beta+1)^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\mu}{\varepsilon^{2\gamma}} &= \lim_{\varepsilon\to 0}\frac{\mu_1-\frac{3}{8}\mu_2}{\varepsilon^{2\gamma}} \\
&= \frac{T}{\sigma^{2\gamma}}\left[\frac{5(\alpha+\beta+1)^2}{48} - \frac{\alpha^2+\alpha\beta+\beta^2+3\alpha+3\beta+2}{16}\right] \\
&= \frac{T}{48\sigma^{2\gamma}}(2\alpha^2+2\beta^2+7\alpha\beta+\alpha+\beta-1).
\end{aligned}
$$

Under the condition (15), we know

$$
\lim_{\varepsilon\to 0}\frac{\mu}{\varepsilon^{2\gamma}} \neq 0,
$$

which means that the twist coefficient μ is non-zero when ε is small enough. Now the proof is finished. □

Remark 1. As a consequence of Theorem 3.2, we can obtain that the following equation

$$\ddot{x} = \frac{\lambda g(t)}{x^\alpha} - \frac{h(t)}{x^\beta} \tag{27}$$

has a twist T-periodic solution $x(t,\lambda)$ if λ is large enough and α,β satisfy (15)-(16). In fact, if we introduce the variable

$$x = \lambda^{\frac{1}{\alpha+1}} y,$$

then (27) is changed to the equation

$$\ddot{y} = \frac{g(t)}{y^\alpha} - \frac{\varepsilon h(t)}{x^\beta},$$

where

$$\varepsilon = \lambda^{-\frac{\beta+1}{\alpha+1}}.$$

Note that $\varepsilon \to 0$ if and only if $\lambda \to +\infty$.

4. Twist periodic solutions of equation (3)

Although in the last section we have obtained a stability result for equation (2), we do not know whether there exists a twist periodic solution for (3) in which no small parameters are involved. In this section, we will show that (3) also may admit stable periodic solutions. However, we have to assume that both g and h are positive functions.

The following stability result for (5) was proved in [22]. The original result deals with the equation with the period 2π, however, after checking the details of the proof, we note that the following condition (iii) remains unchanged for any period T.

Lemma 4.1. [22, Theorem 3.1] *Assume that there exists a T-periodic solution ψ of (5) such that* (i) $0 < a_* \leq a^* < (\frac{\pi}{2T})^2$; (ii) $c_* > 0$; (iii) $10\tilde{b}_*^2 a_*^{3/2} > 9c^*(a^*)^{5/2}$. *Then the solution $\psi(t)$ of (5) is of twist type.*

Lemma 4.2. *Assume that g, h are positive, T-periodic functions and the following inequality holds*

$$\frac{(g^*)^{\gamma(\beta+1)}}{h_*^{\gamma(\alpha+1)}} > \left(\frac{T}{\pi}\right)^2 \left(\alpha\Delta^{\gamma(\alpha+1)} - \beta\right), \tag{28}$$

where

$$\Delta_g = \frac{g^*}{g_*}, \quad \Delta_h = \frac{h^*}{h_*}, \quad \Delta = \Delta_g \cdot \Delta_h.$$

Then equation (3) has at least one T-periodic solution such that

$$\left(\frac{g_*}{h^*}\right)^\gamma < x(t) < \left(\frac{g^*}{h_*}\right)^\gamma. \tag{29}$$

Proof. Note that

$$\psi_1(t) = \left(\frac{g_*}{h^*}\right)^\gamma$$

is a constant strict upper function and

$$\psi_2(t) = \left(\frac{g^*}{h_*}\right)^\gamma$$

is a constant strict lower function on the reserved order $\psi_2 > \psi_1$. Let us fix

$$f(t,x) = \frac{h(t)}{x^\beta} - \frac{g(t)}{x^\alpha}. \tag{30}$$

Note that

$$\begin{aligned} f_x(t,x) &= \frac{\alpha g(t)}{x^{\alpha+1}} - \frac{\beta h(t)}{x^{\beta+1}} \\ &\leq \frac{\alpha g(t)}{\psi_1^{\alpha+1}} - \frac{\beta h(t)}{\psi_2^{\beta+1}} \\ &\leq \frac{\alpha g^*}{\left(\frac{g_*}{h^*}\right)^{\gamma(\alpha+1)}} - \frac{\beta h_*}{\left(\frac{g^*}{h_*}\right)^{\gamma(\beta+1)}}. \end{aligned}$$

By Proposition 2, a sufficient condition for the existence of T-periodic solution of (3) is

$$\frac{\alpha g^*}{\left(\frac{g_*}{h^*}\right)^{\gamma(\alpha+1)}} - \frac{\beta h_*}{\left(\frac{g^*}{h_*}\right)^{\gamma(\beta+1)}} < \left(\frac{\pi}{T}\right)^2,$$

which is equivalent to condition (28). □

Theorem 4.3. *Let us assume that*

$$\frac{(g^*)^{\gamma(\beta+1)}}{h_*^{\gamma(\alpha+1)}} > \left(\frac{2T}{\pi}\right)^2 \left(\alpha\Delta^{\gamma(\alpha+1)} - \beta\right), \tag{31}$$

and

$$\Delta_g < \left(\frac{5[\alpha(\alpha+1) - \beta(\beta+1)]^2}{3(\alpha-\beta)[\alpha(\alpha+1)(\alpha+2) - \beta(\beta+1)(\beta+2)]}\right)^{2/7}. \tag{32}$$

Then there exists a constant $\Delta_0 > 1$ such that the T-periodic solution $x(t)$ of (3) obtained in Lemma 4.2 is of twist type if $\Delta < \Delta_0$.

Proof. We will apply Lemma 4.1. Let us fix $f(t,x)$ as in (30). Then the coefficients of the third-order approximation are

$$a(t) = \frac{\alpha g(t)}{x^{\alpha+1}} - \frac{\beta h(t)}{x^{\beta+1}},$$

$$b(t) = \frac{1}{2}\left[\frac{\beta(\beta+1)h(t)}{x^{\beta+2}} - \frac{\alpha(\alpha+1)g(t)}{x^{\alpha+2}}\right],$$

$$c(t) = \frac{1}{6}\left[\frac{\alpha(\alpha+1)(\alpha+2)g(t)}{x^{\alpha+3}} - \frac{\beta(\beta+1)(\beta+2)h(t)}{x^{\beta+3}}\right].$$

Using the estimates (29), we have

$$a(t) > \frac{\alpha g_*^{\gamma(\alpha+1)} h_*^{\gamma(\alpha+1)} - \beta (g^*)^{\gamma(\alpha+1)}(h^*)^{\gamma(\alpha+1)}}{g_*^{\gamma(\beta+1)}(g^*)^{\gamma(\alpha+1)}},$$

and

$$a(t) < \frac{\alpha (g^*)^{\gamma(\alpha+1)}(h^*)^{\gamma(\alpha+1)} - \beta g_*^{\gamma(\alpha+1)} h_*^{\gamma(\alpha+1)}}{g_*^{\gamma(\alpha+1)}(g^*)^{\gamma(\beta+1)}}. \tag{33}$$

Note that if

$$\Delta < \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\gamma(\alpha+1)}} =: \Delta_1,$$

then

$$\begin{aligned} a_* &> \frac{\alpha g_*^{\gamma(\alpha+1)} h_*^{\gamma(\alpha+1)} - \beta (g^*)^{\gamma(\alpha+1)} (h^*)^{\gamma(\alpha+1)}}{g_*^{\gamma(\beta+1)} (g^*)^{\gamma(\alpha+1)}} \\ &= \frac{\left(\alpha - \beta \Delta^{\gamma(\alpha+1)}\right) g_* h_*^{\gamma(\alpha+1)}}{(g^*)^{\gamma(\alpha+1)}} > 0. \end{aligned}$$

Using this estimate together with (31) in (33), it is easy to verify that (i) of Lemma 4.1 holds.

Since

$$c(t) > \frac{\alpha(\alpha+1)(\alpha+2) g_*^{\gamma(\alpha+3)} h_*^{\gamma(\alpha+3)} - \beta(\beta+1)(\beta+2)(g^*)^{\gamma(\alpha+3)} (h^*)^{\gamma(\alpha+3)}}{6 g_*^{\gamma(\alpha+3)} (g^*)^{\gamma(\beta+3)}},$$

then (ii) of Lemma 4.1 is satisfied if

$$\frac{g_*^{\gamma(\alpha+3)} h_*^{\gamma(\alpha+3)}}{(g^*)^{\gamma(\alpha+3)} (h^*)^{\gamma(\alpha+3)}} > \frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)},$$

which is equivalent to

$$\Delta^{\gamma(\alpha+3)} < \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)},$$

and it holds if

$$\Delta < \left(\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \right)^{1/\gamma(\alpha+3)} =: \Delta_2.$$

Note that

$$\begin{aligned} b(t) &< \frac{\beta(\beta+1)(g^*)^{\gamma(\alpha+2)} (h^*)^{\gamma(\alpha+2)} - \alpha(\alpha+1) g_*^{\gamma(\alpha+2)} h_*^{\gamma(\alpha+2)}}{2 g_*^{\gamma(\beta+2)} (g^*)^{\gamma(\alpha+2)}} \\ &= \frac{[\beta(\beta+1)\Delta^{\gamma(\alpha+2)} - \alpha(\alpha+1)] g_* h_*^{\gamma(\alpha+2)}}{2(g^*)^{\gamma(\alpha+2)}} \\ &< \frac{[\beta(\beta+1)\Delta_3^{\gamma(\alpha+2)} - \alpha(\alpha+1)] g_* h_*^{\gamma(\alpha+2)}}{2(g^*)^{\gamma(\alpha+2)}}. \end{aligned}$$

Then $b(t) < 0$ for all t if

$$\Delta < \left(\frac{\alpha(\alpha+1)}{\beta(\beta+1)} \right)^{1/\gamma(\alpha+2)} =: \Delta_3,$$

and therefore,

$$\tilde{b}_* > \frac{\alpha(\alpha+1) g_*^{\gamma(\alpha+2)} h_*^{\gamma(\alpha+2)} - \beta(\beta+1)(g^*)^{\gamma(\alpha+2)} (h^*)^{\gamma(\alpha+2)}}{2 g_*^{\gamma(\beta+2)} (g^*)^{\gamma(\alpha+2)}} > 0,$$

because

$$b(t) < \frac{\beta(\beta+1)(g^*)^{\gamma(\alpha+2)} (h^*)^{\gamma(\alpha+2)} - \alpha(\alpha+1) g_*^{\gamma(\alpha+2)} h_*^{\gamma(\alpha+2)}}{2 g_*^{\gamma(\beta+2)} (g^*)^{\gamma(\alpha+2)}}.$$

Finally, we have

$$c(t) > \frac{\alpha(\alpha+1)(\alpha+2) g_*^{\gamma(\alpha+3)} h_*^{\gamma(\alpha+3)} - \beta(\beta+1)(\beta+2)(g^*)^{\gamma(\alpha+3)} (h^*)^{\gamma(\alpha+3)}}{6 g_*^{\gamma(\alpha+3)} (g^*)^{\gamma(\beta+3)}}$$

and

$$c(t) < \frac{\alpha(\alpha+1)(\alpha+2)(g^*)^{\gamma(\alpha+3)}(h^*)^{\gamma(\alpha+3)} - \beta(\beta+1)(\beta+2)g_*^{\gamma(\alpha+3)}h_*^{\gamma(\alpha+3)}}{6g_*^{\gamma(\alpha+3)}(g^*)^{\gamma(\beta+3)}}.$$

Using the above facts and by direct computations, we can get that (iii) of Lemma 4.1 holds if the following inequality holds

$$P_1(\Delta) > P_2(\Delta), \tag{34}$$

where

$$P_1(\Delta) = 5\frac{1}{\Delta_g^{7/2}}[\alpha(\alpha+1) - \beta(\beta+1)\Delta^{\gamma(\alpha+2)}]^2[\alpha - \beta\Delta^{\gamma(\alpha+1)}]^{3/2},$$

and

$$P_2(\Delta) = 3[\alpha(\alpha+1)(\alpha+2)\Delta^{\gamma(\alpha+3)} - \beta(\beta+1)(\beta+2)][\alpha\Delta^{\gamma(\alpha+1)} - \beta]^{5/2}.$$

Under the condition (32), we know that $P_1(1) > P_2(1)$. Therefore by continuity there exists Δ_4 such that (34) holds whenever $\Delta < \Delta_4$.

Define

$$\Delta_0 = \min\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}.$$

Then all conditions of Lemma 4.1 are satisfied if $\Delta < \Delta_0$. Finally Δ_0 depends only on the exponents α, β. □

References

[1] A.A. Bekov, Periodic solutions of the Gylden-Meshcherskii problem, Astron. Rep. 37 (1993), 651-654.

[2] J. L. Bravo, P. J. Torres, Periodic solutions of a singular equation with indefinite weight, Adv. Nonlinear Stud. 10 (2010), 927-938.

[3] J. Chu and M. Li, Twist periodic solutions of second order singular differential equations, J. Math. Anal. Appl. 355 (2009), 830-838.

[4] J. Chu, P. J. Torres, F. Wang, Radial stability of periodic solutions of the Gylden-Meshcherskii-type problem, Discrete Contin. Dyn. Syst. 35 (2015), 1921-1932.

[5] J. Chu and M. Zhang, Rotation number and Lyapunov stability of elliptic periodic solutions, Discrete Contin. Dyn. Syst. 21 (2008), 1071-1094.

[6] E.N. Dancer, R. Ortega, The index of Lyapunov stable fixed points in two dimensions, J. Dynam. Differential Equations 6 (1994), 631-637.

[7] C. De Coster and P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results, in: Nonlinear Analysis and Boundary Value Problems for Ordinary Differential Equations, edited by F. Zanolin, CISM-CICMS 371 (Springer-Verlag, New York, 1996), pp. 1-78.

[8] A. Deprit, The secular accelerations in Gylden's problem, Celestial Mechanics, 31 (1983), 1-22.

[9] J.K. Hale, Ordinary differential equations, Krieger Publishing Company, 1969.

[10] R. Hakl, P. J. Torres, On periodic solutions of second-order differential equations with attractive-repulsive singularities, J. Differential Equations, 248 (2010), 111-126.

[11] A. C. Lazer and S. Solimini, On periodic solutions of nonlinear differential equations with singularities, Proc. Amer. Math. Soc. 99 (1987), 109-114.

[12] J. Lei, X. Li, P. Yan and M. Zhang, Twist character of the least amplitude periodic solution of the forced pendulum, SIAM J. Math. Anal. 35 (2003), 844-867.

[13] Q. Liu and D. Qian, Nonlinear dynamics of differential equations with attractive-repulsive singularities and small time-dependent coefficients, Math. Methods Appl. Sci. 36 (2013), 227-233.

[14] R. Ortega, Periodic solution of a Newtonian equation: stability by the third approximation, J. Differential Equations, 128 (1996), 491-518.

[15] I. Rachunková, M. Tvrdý and I. Vrkoč, Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems, J. Differential Equations, 176 (2001), 445-469.

[16] D. Selaru, C. Cucu-Dumitrescu and V. Mioc, On a two-body problem with periodically changing equivalent gravitational parameter, Astron. Nachr. 313 (1993), 257-263.

[17] C. Siegel and J. Moser, Lectures on Celestial Mechanics, Springer-Verlag, Berlin, 1971.

[18] P. J. Torres, Twist solutions of a Hill's equations with singular term, Adv. Nonlinear Stud. 2 (2002), 279-287.

[19] P. J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, J. Differential Equations, 190 (2003), 643-662.

[20] P. J. Torres, Existence and stability of periodic solutions for second order semilinear differential equations with a singular nonlinearity, Proc. Royal Soc. Edinburgh Sect. A. 137 (2007), 195-201.

[21] P.J. Torres, Mathematical Models with singularities - A Zoo of Singular Creatures, Atlantis Press, 2015.

[22] P. J. Torres and M. Zhang, Twist periodic solutions of repulsive singular equations, Nonlinear Anal. 56 (2004), 591-599.

[23] A.J. Urena, Periodic solutions of singular equations, Topol. Methods Nonlinear Anal., to appear.

[24] M. Zhang, The best bound on the rotations in the stability of periodic solutions of a Newtonian equation, J. London Math. Soc. 67 (2003), 137-148.

[25] M. Zhang, Periodic solutions of equations of Ermakov-Pinney type, Adv. Nonlinear Stud. 6 (2006), 57-67.

E-mail address: jifengchu@126.com
E-mail address: ptorres@ugr.es
E-mail address: fengwang188@163.com