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Necessary and sufficient conditions for existence of periodic motions of forced systems of particles

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Abstract. It is studied the existence of periodic oscillations of a one-dimensional chain of particles periodically perturbed and with nearest-neighbor interaction between particles. It turns out that the behavior of the system is different if the number of particles is finite or infinite. In both cases, necessary and sufficient conditions for existence of periodic solutions are obtained over the mean values of the external forces by using a priori bounds, topological degree and limiting arguments.

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1. Introduction

The object of this note is to continue the study initiated in [3] about existence of T-periodic solutions of some systems of differential equations modeling one-dimensional systems of classical particles with nearest neighbor coupling and external T-periodical forces on each particle. This type of systems are relevant because of its wide variety of physical applications. Our purpose is to obtain necessary and sufficient conditions for existence of T-periodic solutions over the mean values of the external forces. It turns out that this condition is different (more than this, in some sense it is opposite) if the number of particles in the system is finite or infinite, and we will stress this difference trough some illustrative examples at the end of the paper.

As in [3], we consider two main classes of interaction between particles: an interaction of Toda type and a singular repulsive interaction (like electrostatic forces between particles with charge of the same sign). In both cases the interaction is repulsive and the results obtained are the same, with some additional information about the order between particles for singular interactions. The techniques employed here are on the same line of [3]: first, the finite system is studied through a change of variables that enables us a reduction to a simpler sub-system that allows the obtaining of a priori bounds of some homotopic equation. This fact together with classical tools on topological degree enable us to get results on finite systems. Second, we are able to use these same a priori bounds obtained before to pass to the limit and get T-periodic solutions for the lattice. In consequence, we refer to this previous paper for technical details and a more extensive bibliography.

2. Finite systems

Let us consider the finite system of n equations

$$\begin{cases} x_1'' + cx_1' = -g_1(x_2 - x_1) + h_1(t) \\ x_i'' + cx_i' = g_{i-1}(x_i - x_{i-1}) - g_i(x_{i+1} - x_i) + h_i(t), \quad {\scriptstyle i=2,\dots,n-1} \\ x_n'' + cx_n' = g_{n-1}(x_n - x_{n-1}) + h_n(t) \end{cases} \tag{1}$$

where $c \geq 0$ and h_i are continuous T-periodic functions. We are going to look for T-periodic solutions of (1) on the configuration space

$$\mathcal{H}_n = \{x = (x_1, \dots, x_n) \in \mathrm{C}^2(\mathrm{R})^n : \int_0^{\mathrm{T}} x_1(t)dt = 0\}$$

or

$$\mathcal{H}_n^+ = \{x \in \mathcal{H}_n : x_i(t) < x_{i+1}(t), \ \ \forall t \in [0, \mathrm{T}], \ i = 1 \dots, n\}.$$

This last space is called the space of *ordered solutions*. By a T-periodic solution we understand a solution $x \in \mathcal{H}_n$ (or $\mathcal{H}_n^+$) such that $x_i(0) = x_i(\mathrm{T}), x_i'(0) = x_i'(\mathrm{T})$ for each i.

We consider two types of interactions between particles:

- *Toda-type interaction:* $g_i : \mathrm{R} \to \mathrm{R}^+$ is a continuous function such that

$$\lim_{x \to -\infty} g_i(x) = +\infty, \qquad \lim_{x \to +\infty} g_i(x) = 0. \tag{2}$$

- Singular interaction: $g_i : \mathrm{R}^+ \to \mathrm{R}^+$ is a continuous function such that

$$\lim_{x \to 0^+} g_i(x) = +\infty, \qquad \lim_{x \to +\infty} g_i(x) = 0 \tag{3}$$

and

$$\int_0^1 g_i(s)ds = +\infty. \tag{4}$$

Note that if $g_i(x) = e^{-x}$ and $h_i \equiv 0$ for all i, system (1) is the classical unforced Toda chain with free boundaries, which can be integrated. With respect to the singular nonlinearity, the model case is $x^{-\alpha}$ with $\alpha \geq 1$. On the context of singular equations, condition (4) is known as a *strong force condition*, and it is a common device in order to avoid collisions between particles.

Theorem 2.1. *Let us assume that system* (1) *is of Toda type or singular. Then,* *there exists at least one* T *-periodic solution* $x \in \mathcal{H}_n$ *if and only if*

$$\begin{aligned} &\sum_{i=1}^{n} \overline{h}_i = 0, \\ &\sum_{i=1}^{k} \overline{h}_i > 0, \qquad {\scriptstyle k=1,\dots,n-1.} \end{aligned} \tag{5}$$

Moreover, if the system is singular, then $x \in \mathcal{H}_n^+$.

Proof. In order to see that condition (5) is necessary, we only have to add the first k equations with $k = 1, \dots, n$ and integrate over a period. For the sufficiency, we follow the method of proof in [3], where a particular case of this system is studied. Through the change of variables

$$\left\{ \begin{array}{l} y(t) = x_1(t) \\ d_i(t) = x_{i+1}(t) - x_i(t), \qquad {\scriptstyle i=1,\dots,n-1} \end{array} \right. \tag{6}$$

it is obtained an equivalent system which can be deformed through an homotopy to an autonomous system with Brouwer degree different from 0 . The proof is finished with the obtention of a priori bounds for the T -periodic solutions of the homotopic system, which is done basically as in [3], by using now that

$$\int_0^{\mathrm{T}} g_i(d_i(t))dt = \sum_{j=1}^{i} \overline{h}_j \mathrm{T}, \qquad {\scriptstyle i=1,\dots,n-1.}$$

In the case of a singular system, it is possible also to find a priori bounds from below as in Lemma 3 in [3], proving that $x \in \mathcal{H}_n^+$. □

3. Infinite systems

In this section, we are concerned with the existence of T -periodic solutions of the infinite system of non-autonomous differential equations

$$x_i'' + cx_i' = g_{i-1}(x_i - x_{i-1}) - g_i(x_{i+1} - x_i) + h_i(t), \qquad i \in \mathbf{Z} \tag{7}$$

where $c \geq 0$ and h_i are continuous T -periodic functions. We are going to look for T -periodic solutions of (7) on the configuration space

$$\mathcal{H} = \{x = \{x_i\}_{i\in\mathbf{Z}} \in \mathrm{C}^2(\mathrm{R})^{\mathbf{Z}} : \int_0^{\mathrm{T}} x_0(t)dt = 0\}$$

or

$$\mathcal{H}^+ = \{x = \{x_i\}_{i\in\mathbf{Z}} \in \mathcal{H} : x_i(t) < x_{i+1}(t), \ \ \forall i \in \mathbf{Z}, t \in [0, \mathrm{T}]\}.$$

Theorem 3.1. *Let us assume that system* (7) *is of Toda type or singular. Then, there exists at least one* T *-periodic solution* $x \in \mathcal{H}$ *of system* (7) *if and only if* *there exists a positive constant* K_0 *such that*

$$K_0 > \sup_{n\in\mathbb{N}}\{-\sum_{i=1}^{n}\overline{h}_i, \sum_{i=0}^{-n}\overline{h}_i\}. \tag{8}$$

Moreover, in this case we have that for any $K \geq K_0$ *there exists a* T*-periodic solution* $x \in \mathcal{H}$ *of* (7) *such that*

$$\int_0^T g_0(x_1(t) - x_0(t))dt = KT. \tag{9}$$

Finally, if the system is singular, then $x \in \mathcal{H}^+$.

Proof. In order to prove that condition (8) is necessary for existence of T-periodic solutions, fix $K_0 = \frac{1}{T}\int_0^T g_0(x_1(t) - x_0(t))dt$. Then, we get condition (8) by an addition of the equations from $i = 1$ to n and from $i = -n$ to 0 respectively and a further integration over a period.

Now, let us prove the sufficiency. This is done passing to the limit from a finite system. If $K \geq K_0$, let us consider the finite system of $2n + 1$ equations

$$\begin{cases} x''_{-n} + cx'_{-n} = -g_{-n}(x_{-n+1} - x_{-n}) + h_{-n}(t) + K - \sum_{i=0}^{-n}\overline{h}_i \\ x''_i + cx'_i = g_{i-1}(x_i - x_{i-1}) - g_i(x_{i+1} - x_i) + h_i(t), \quad {\scriptstyle i=-n+1,\ldots,n-1} \\ x''_n + cx'_n = g_{n-1}(x_n - x_{n-1}) + h_n(t) - K - \sum_{i=1}^{n}\overline{h}_i \end{cases}$$

This system has been studied in Section 1, and it is easy to check that condition (8) assures the existence of a T-periodic solution $\{x_i^{(n)}\}_{i=-n,\ldots,n}$. Also, a trivial rescalling enable us to assume that $\int_0^T x_0(t)dt = 0$. Finally, the bounds deduced in the proof of Theorem 1 can be used as in [3] in order to prove the convergence of this sequence to a solution of the infinite system. In the singular case, the a priori bounds from below provide the order on the lattice. □

4. Further remarks and open problems

In spite of our method, in which solutions of the lattice are found from finite systems through a pass to the limit, it is remarkable the difference of behavior between finite and infinite systems, that is, between conditions (5) and (8). If $(\overline{h}_i)_i$ is the vector (with finite or infinite components) of mean values, we will say that it is *admissible* if the respective system has a T-periodic solution. Then, the vector $(1, \ldots, 1, -1, \ldots, -1) \in \mathbb{R}^{2n}$ is admissible for the finite system, but $(-1, \ldots, -1, 1, \ldots, 1)$ is not. On the contrary, the infinite vector

$$(\ldots, -1, \ldots, -1, 1, \ldots, 1, \ldots)$$

is admissible for the lattice, but $(\ldots,1,\ldots,1,-1,\ldots,-1,\ldots)$ is not. In other words, for a finite system the average of the external forces should be directed inward the system; in contrast, the situation in the lattice is opposite: it is possible to direct the external forces outside. However, a lattice admits also T-periodic solutions if the series $\sum_{i\geq 1}\overline{h}_i$ and $\sum_{i\leq 0}\overline{h}_i$ are convergent, for example, if

$$(\overline{h}_i)_i = \begin{cases} \frac{1}{i^2} & \text{if } i<0 \\ 0 & \text{if } i=0 \\ -\frac{1}{i^2} & \text{if } i>0 \end{cases}$$

then the lattice has a T-periodic solution. Further, although it is necessary for a finite system that the sum of all mean values is zero (see condition (5)), this is not necessary at all for a lattice. For instance, the infinite vector of mean values defined by

$$(\overline{h}_i)_i = \begin{cases} \frac{1}{i^2} & \text{if } i<0 \\ 1 & \text{if } i\geq 0 \end{cases}$$

is admissible for the lattice. I must say that I have not been able to find a reasonable physical explanation for this phenomenon.

We conclude with a brief exposition of some open problems and future research lines:

- The energy of the solutions found with our method is infinite. In contrast, variational methods like in [1] lead to solutions of finite energy. Thus, it is interesting to find additional conditions over the external forces leading to finite energy solutions.
- We have proved the existence of an infinite number of periodic solutions for the lattice. However, the question of multiplicity of solutions of finite systems is open. Also, stability of such solutions seems a difficult problem.
- It would be very interesting to study the existence of other special kind of solutions like travelling waves [2], solitons or breathers which are relevant for their physical significance. Also, the study of systems with a more complicate interaction between particles is open.

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