JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **202,** 1027–1039 (1996)
ARITCLE NO. 0358

Dynamics of a Periodic Differential Equation with a Singular Nonlinearity of Attractive Type

Pedro Martínez-Amores* and Pedro J. Torres*

Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain

Communicated by Jack K. Hale

Received January 24, 1995; revised November 29, 1995

INTRODUCTION

In this paper we shall consider a general class of equations with a restoring force of attractive type that includes

$$x'' + cx' + \frac{1}{x^{\alpha}} = p(t), \tag{1}$$

where $c \geq 0$, $\alpha > 0$, and p is a continuous T-periodic function for some $T > 0$. We are interested in the existence and stability of positive T-periodic solutions of (1).

The existence of T-periodic solutions of this class of equations where the restoring force is a singular nonlinearity that becomes infinite in zero has been proved by Lazer and Solimini [2] for the case without friction ($c = 0$) and by Habets and Sánchez [1] for the damped case. However, the stability properties of these solutions have been less studied. It is well known that for the autonomous case ($p(t) \equiv p_0 > 0$), this equation has a unique saddle point. In this paper we prove that the dynamics of the periodic equation (1) is similar to the autonomous case. For our study is fundamental a Massera's convergence theorem in $\mathbb{R}^n$ given by Smith in [6].

The paper is divided in three sections. In Section 1, the main results are stated. It is seen that (1) has a unique unstable periodic solution φ.

* Supported by D. G. C. Y. T. PB92-0953, M.E.C., Spain, and E.E.C. project ERBCHRX-CT94-0555.

0022-247X/96 $18.00

Moreover, for the damped case it is proved that there exists a global stable manifold of this periodic solution which is determined by a strictly decreasing map $h: \mathbb{R} \to \mathbb{R}_0^+$ (being $\mathbb{R}_0^+ = [0, +\infty)$) such that this graph divides the plane of initial conditions in two open sets, one of them (set A in Fig. 1) corresponds to the solutions that tend to $+\infty$ as $t \to +\infty$, and the other (set B) to the solutions that goes to zero in finite time, that is, the ones that are "absorbed" by the singularity. When the potential $V(x)$ is infinite as $x \to 0^+$ (case $\alpha \geq 1$), we have that $h: \mathbb{R} \to \mathbb{R}^+$ (being $\mathbb{R}^+ = (0, +\infty)$) is a homeomorphism, whereas if $V(x)$ is finite as $x \to 0^+$ (case $0 < \alpha < 1$) there exists $v_s \in \mathbb{R}$ such that $h(v_s) = 0$.

Similar results are showed for the unstable manifold. As a consequence, there are no homoclinic points.

We prove these results in Section 2. For this, the study of the behavior of the solutions and its derivatives according to its maximal interval of existence (w^-, w^+), $-\infty \leq w^- < w^+ \leq +\infty$, is fundamental. Some properties of comparison of solutions are given which are needed in the proofs.

Finally, in Section 3, we remark that the previous results are true for the undamped case.

1. MAIN RESULTS

Let us consider the forced Liénard equation,

$$x''(t) + f(x(t))x'(t) + g(x(t)) = p(t), \tag{2}$$

where $f, g: \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and $p: \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. Moreover, we assume that f and g have continuous derivative.

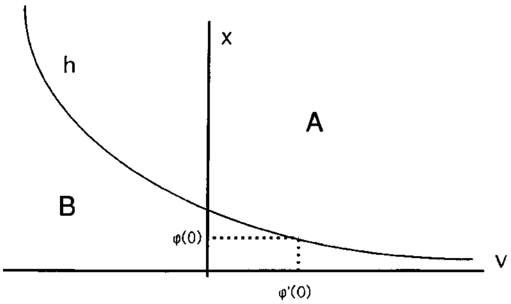

FIG. 1. The semiplane of initial conditions.

In [2] for $f = 0$ and in [1] for arbitrary f and under the condition

$$\lim_{x \to 0^+} g(x) = +\infty, \qquad \lim_{x \to +\infty} g(x) = 0 \tag{3}$$

the method of upper and lower solutions has been used to prove that a necessary and sufficient condition for the existence of positive T-periodic solutions of (2) is that the mean value $\bar{p} = (1/T)\int_0^T p(t)\,dt > 0$.

The study of the stability can be done by using the topological degree as in [5]. However, we shall use the following lemma of linearization of (2) at a periodic solution.

LEMMA 1. *Let $r \in C^1(\mathbb{R})$, $q \in C(\mathbb{R})$ T-periodic functions such that*

$$r(t) \geq 0, q(t) < 0 \qquad \forall t \in \mathbb{R}.$$

If μ_1, μ_2 are the characteristic multipliers of the equation

$$y'' + (r(t)y)' + q(t)y = 0$$

then $0 < \mu_2 < 1 < \mu_1$.

Proof. Multipliers verify

$$\mu^2 - \Delta\mu + W = 0,$$

where Δ is the discriminant and $W = e^{-\int_0^T r(t)\,dt} \in (0, 1]$. We must prove that $\Delta > 1 + W$.

Let $\bar{r}, \bar{q}$ the mean values of r and q, and

$$r_\lambda(t) = \lambda\bar{r} + (1 - \lambda)r(t), \qquad q_\lambda(t) = \lambda\bar{q} + (1 - \lambda)q(t) \qquad \forall\lambda \in [0, 1].$$

Consider the equation

$$y'' + (r_\lambda(t)y)' + q_\lambda(t)y = 0.$$

$\Delta = \Delta(\lambda)$ is continuous by continuous dependence theorem, and clearly $\Delta(1) > 1 + W$, so we will prove that $\Delta(\lambda) \neq 1 + W$, $\forall\lambda \in [0, 1]$. If it is false, then there exists a nontrivial T-periodic solution ψ. If $\psi > 0$, integrating Eq. (1) obtains a contradiction, so ψ must change sign. In consequence, we can fix $t_0 < t_1$ such that $\psi(t_0) = \psi(t_1) = 0$, $\psi(t) > 0$, $\forall t \in (t_0, t_1)$. Integrating in (t_0, t_1) we have

$$0 > \psi'(t_1) - \psi'(t_0) + \int_{t_0}^{t_1} q_\lambda(s)\psi(s)\,ds = 0,$$

another contradiction.

This lemma shows that if (2) has a unique T-periodic solution then this solution must be unstable. A result of this type was given in [3] for equations of the form $x'' = g(t, x)$ under some conditions on g. In this context we have the following result.

THEOREM 1. *Assume that* $f \geq 0$ *and* (3) *hold. If* $\bar{p} > 0$ *and* g *is strictly decreasing then* (2) *has a unique positive* T*-periodic solution which is unstable.*

Proof. By Lemma 1, it is enough to prove that (2) has a unique T-periodic solution. Suppose, by contradiction, that x_1 and x_2 are two different T-periodic solutions of (2). If $z(t) = x_1(t) - x_2(t)$ does not change of sign in $[0, T]$, we have a contradiction only subtracting the respective equations and integrating in $(0, T)$. If $z(t)$ vanishes at some $t_0 \in [0, T]$ then there must be at least other zero of $z(t)$ on $[0, T]$. Indeed, if $z(t_0) = 0$ and $z(t) > 0$ for $t \neq t_0$, z has a minimum at t_0 and $z'(t_0) = 0$. Hence $x_1(t) = x_2(t)$, $t \in [0, T]$, by uniqueness.

So, let t_0 and t_1 be two successive zeros of z on $[0, T]$ and assume that $z(t) < 0$ over (t_0, t_1). Using the monotonicity of g we obtain

$$z''(t) + f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t) < 0, \qquad t \in (t_0, t_1).$$

Since $z'(t_0) < 0$ and $z'(t_1) > 0$, an integration over (t_0, t_1) gives a contradiction.

The same conclusion is obtained if $z(t) < 0$, $t \in [t_0, t_1]$.

From now we assume that all the conditions in Theorem 1 hold and that $\varphi(t)$ is the T-periodic solution given in this theorem. The following theorem describes the asymptotic behavior of trajectories of (2).

THEOREM 2. *Assume that*

$$0 < m = \inf f \leq \sup f = M < +\infty \tag{4}$$

and

$$M \leq \frac{2 + \sqrt{2}}{2 - \sqrt{2}} m. \tag{5}$$

If $x(t)$ *is a solution of* (2) *defined on* $[t_0, +\infty)$ *then*

$$\lim_{t \to +\infty} [x(t) - \varphi(t)] = 0, \qquad \lim_{t \to +\infty} [x'(t) - \varphi'(t)] = 0,$$

or

$$\lim_{t \to +\infty} x(t) = +\infty.$$

Proof. We show first that the assumptions of Theorem 2 in [6] are satisfied. Making the change of variable $y = x' + F(x)$, where $F'(x) = f(x)$, and taking

$$P = \frac{1}{2}\begin{pmatrix} c & -1 \\ -1 & 0 \end{pmatrix}, \qquad c > 0,$$

one can see that condition (H3) in [6] (see [4]) is equivalent to the matrix

$$A = \begin{pmatrix} g'(x) - cf(x) + \lambda c & \dfrac{f(x) + c}{2} - \lambda \\ \dfrac{f(x) + c}{2} - \lambda & -1 \end{pmatrix}, \qquad \lambda > 0,$$

be negative definite for some λ and c, that is,

$$g'(x) - cf(x) + \lambda c < 0$$

and

$$\det(A) = -g'(x) + cf(x) - \lambda c - \left(\frac{f(x) + c}{2} - \lambda\right)^2 > 0.$$

Since $g'(x) < 0$, these inequalities hold if and only if

$$c + 2\lambda + 2\sqrt{\lambda c} \geq M,$$
$$c + 2\lambda - 2\sqrt{\lambda c} \leq m,$$

and

$$\lambda \leq m.$$

Now, taking $c = K\lambda$ and maximizing K, these inequalities hold if (4) and (5) are satisfied.

From [6] it follows that $\lim_{t \to +\infty}[x(t) - \varphi(t)] = 0$, $\lim_{t \to +\infty}[x'(t) - \varphi'(t)] = 0$ or that there exists a sequence $\{t_n\} \to +\infty$ such that some of the following cases hold:

$$\text{(a)} \quad x(t_n) \to 0; \qquad \text{(b)} \quad |x'(t_n)| \to +\infty, \qquad \text{(c)} \quad x(t_n) \to +\infty.$$

If (a) holds, from the equation we obtain that $x''(t_n) + f(x(t_n))x'(t_n) \to -\infty$ as $t_n \to +\infty$. Suppose that there exists a sequence $\tau_n \to +\infty$ such that $x(\tau_n) \to K > 0$. One can interlace $\{t_n\}$ and $\{\tau_n\}$ and take t_n such that $x(t)$ has a local minimum at t_n. Then $x'(t_n) = 0$ and $x''(t_n) \geq 0$, so $x''(t_n)$ cannot tend to $-\infty$ as $t_n \to +\infty$. Hence $x(t) \to 0$ as $t \to +\infty$. Now, by the mean value theorem, there exists other sequence that we call again $\{t_n\}$ such that $x'(t_n) \to 0$. As above, we get $\lim_{t\to+\infty} x'(t) = 0$. A new application of the mean value theorem gives a sequence, called $\{t_n\}$, such that $x''(t_n) \to 0$. Evaluating the equation in t_n and taking limits we have a contradiction.

If (a) does not hold and $x'(t_n) \to +\infty$ (resp. $-\infty$) as $t_n \to +\infty$ then $x''(t_n) + g(x(t_n)) \to -\infty$ (resp. $+\infty$) as $t_n \to +\infty$. Supposing that there exists a sequence $\tau_n \to +\infty$ such that $x'(\tau_n) \to K < +\infty$ (resp. $K > -\infty$) we can take $x''(t_n) = 0$ and then $g(x(t_n)) \to +\infty$ (resp. $-\infty$), which is not possible. Hence $\lim_{t\to+\infty}|x'(t)| = +\infty$. Since $\lim_{t\to+\infty} x(t) \neq 0$, we have that $g(x(t))$ is bounded for $t \geq t_0$ and if $\lim_{t\to+\infty} x'(t) = \pm\infty$ then $\lim_{t\to+\infty} x''(t) = \mp\infty$, a contradiction.

If (c) holds, we are going to prove that $\lim_{t\to+\infty} x(t) = +\infty$. There exists some t_0 such that $z(t) = x(t) - \varphi(t) > 0$ for all $t > t_0$ (see Proposition 1 of Section 2). As $\varphi(t)$ is bounded, we have that $\lim_{t\to+\infty} x(t) = +\infty$ is equivalent to $\lim_{t\to+\infty} z(t) = +\infty$. If this is not true, then there exist sequences $\{z(t_n)\}$ of maxima of z and $\{z(\tau_n)\}$ of minima of z such that

$$\lim_{t_n\to+\infty} z(t_n) = +\infty, \qquad \lim_{\tau_n\to+\infty} z(\tau_n) = K < +\infty,$$

with $t_n < \tau_n < t_{n+1}$ for all n. Since g is strictly decreasing, we have that

$$z''(t) + f(x(t))x'(t) - f(\varphi(t))\varphi'(t) = g(\varphi(t)) - g(x(t)) > 0 \qquad \forall t > t_0.$$

Hence, integrating over $]t_n, \tau_n[$,

$$\int_{x(t_n)}^{x(\tau_n)} f(s)\,ds - \int_{\varphi(t_n)}^{\varphi(\tau_n)} f(s)\,ds = \int_{\varphi(\tau_n)}^{x(\tau_n)} f(s)\,ds - \int_{\varphi(t_n)}^{x(t_n)} f(s)\,ds > 0,$$

so $Mz(\tau_n) > mz(t_n)$, and we have a contradiction when $n \to +\infty$.

When the solution $x(t)$ is defined on $[t_0, w^+), w^+ < +\infty$, there exists a sequence $t_n \to w^+$, $t_n \in [t_0, w^+)$ such that some of the following cases hold:

$$(\mathrm{a}') \quad x(t_n) \to +\infty, \qquad (\mathrm{b}') \quad |x'(t_n)| \to +\infty, \qquad (\mathrm{c}') \quad x(t_n) \to 0.$$

As above, one can show that only (c′) holds and that then $\lim_{t\to w^+} x(t) = 0$.

From Corollary 2.1 in [6] we can use the same reasonings for the solutions defined on $(-\infty, t_0]$ or on $(w^-, t_0], w^- > -\infty$.

So, we have the following description of the behavior of the solutions of (2) according to its maximal interval of existence:

(i) On $(-\infty, +\infty)$, $\lim_{t\to+\infty}[x(t) - \varphi(t)] = 0$ or $\lim_{t\to+\infty} x(t) = +\infty$ and, on the other hand, $\lim_{t\to-\infty}[x(t) - \varphi(t)] = 0$ or $\lim_{t\to-\infty} x(t) = +\infty$.

(ii) On $(w^-, +\infty)$, $\lim_{t\to w^-} x(t) = 0$ and either $\lim_{t\to+\infty}[x(t) - \varphi(t)] = 0$ or $\lim_{t\to+\infty} x(t) = +\infty$.

(iii) On $(-\infty, w^+)$, $\lim_{t\to w^+} x(t) = 0$ and either $\lim_{t\to-\infty}[x(t) - \varphi(t)] = 0$ or $\lim_{t\to-\infty} x(t) = +\infty$.

(iv) On (w^-, w^+), $\lim_{t\to w^-} x(t) = 0 = \lim_{t\to w^+} x(t)$.

The behavior of $x'(t)$ is given by the following lemma.

LEMMA 2. *According to the maximal interval of existence of the solution* $x(t)$, *we have that* $\lim_{t\to w^+} x'(t) < 0$, $\lim_{t\to w^-} x'(t) > 0$, *and these limits are finite if and only if the potential* $V(x) = \int_x^1 g(s)\,ds$ *is finite as* $x \to 0^+$. *Moreover*, $\limsup_{t\to\pm\infty} x'(t)$ *and* $\liminf_{t\to\pm\infty} x'(t)$ *are finite for any potential.*

Proof. We only show the first assertion. Integrating the equation over $[t_0, w^+ - \epsilon]$ with $\epsilon > 0$, we obtain

$$x'(w^+ - \epsilon) = x'(t_0) + \int_{t_0}^{w^+ - \epsilon} p(s)\,ds - \int_{x(t_0)}^{x(w^+ - \epsilon)} f(s)\,ds$$

$$- \int_{t_0}^{w^+ - \epsilon} g(x(s))\,ds.$$

It is then clear that $\lim_{\epsilon\to 0} x'(w^+ - \epsilon)$ exists and must be negative. Further, this limit is finite if and only if $V(0^+)$ is finite.

Denote by $x(t, x_0, v)$ the solution of (2) such that $x(0, x_0, v) = x_0$, $x'(0, x_0, v) = v$. The main results of this paper describe the global dynamics of Eq. (2) and its proofs are given in the next section.

THEOREM 3. *Assume that the conditions of Theorem* 1 *and* (4), (5) *are satisfied and that* $V(0^+) = +\infty$. *Then there exists a strictly decreasing continuous function* $h\colon \mathbb{R} \to \mathbb{R}^+$ *such that* $x(t, h(v), v) \to \varphi(t)$ *as* $t \to +\infty$. *If* $x_0 < h(v)$ *then* $\lim_{t\to w^+} x(t) = 0$ *and if* $x_0 > h(v)$ *then* $\lim_{t\to+\infty} x(t) = +\infty$. *Moreover*, $\lim_{v\to-\infty} h(v) = +\infty$ *and* $\lim_{v\to+\infty} h(v) = 0$.

THEOREM 4. *Assume that the conditions of Theorem* 1 *and* (4), (5) *are satisfied and that* $V(0^+) < +\infty$. *Then there exists a function* $h\colon (-\infty, v_s] \to \mathbb{R}_0^+$ *in the same conditions of the previous theorem, but now there exists a "critical velocity"* v_s *such that* $h(v_s) = 0$.

Remark 1. Theorems 3 and 4 give a function such that its graph has the property that the solutions of (2) with initial conditions in the graph approach the periodic solution $\varphi(t)$ as $t \to +\infty$. This graph is the section on $t = 0$ of the global stable manifold of $\varphi(t)$, that can be defined as

$$W^s(\varphi) = \bigcup_{t \in \mathbb{R}} \Phi_t(\text{graph}(h)),$$

where Φ_t denotes the flow generated by (2) and

$$\text{graph}(h) = \{(x_0, v) \in \mathbb{R}_0^+ \times \mathbb{R}: h(v) = x_0\}.$$

The existence of the stable manifold can be obtained from general results of hyperbolic manifolds via the Poincaré map since $\varphi(t)$ is a hyperbolic solution (see [5]) in the sense that the Floquet multipliers have modulus different from 1.

However, our proofs are done working directly on the Eq. (2) and, moreover, we describe the geometry of these hyperbolic manifolds.

Remark 2. Note that the choice of the initial time (0 in our case) is not essential in the proofs. So, for any other initial time, that is, any other section of the global stable manifold $W^s(\varphi(t))$, the plane of initial conditions has the same structure.

Analogous results can be proved looking towards the past. In this case, we have the following theorem.

THEOREM 5. *Assume that the conditions of Theorem* 1 *and* (4), (5) *are satisfied and that* $V(0^+) = +\infty$. *Then the set*

$$\{(x_0, v) \in \mathbb{R}^+ \times \mathbb{R}: x(t, x_0, v) - \varphi(t) \to 0; \text{ as } t \to -\infty\}$$

is the set of initial conditions (*the section on* $t = 0$) *of the global unstable manifold of* $\varphi(t)$ *and is the graph of a strictly increasing function* $\xi: \mathbb{R} \to \mathbb{R}^+$ *such that* $\lim_{v \to -\infty} \xi(v) = 0$, $\lim_{v \to +\infty} \xi(v) = +\infty$. *Moreover, if* $x_0 < \xi(v)$ *then* $x(t, x_0, v) \to 0$ *as* $t \to w^-$ *and if* $x_0 > \xi(v)$ *then* $x(t, x_0, v) \to +\infty$ *as* $t \to -\infty$.

When $V(0^+)$ *is finite, the function* $\xi(v)$ *has the property that there exists* $v_u \in \mathbb{R}$ *such that* $\xi(v_u) = 0$.

Note that the intersection of h and ξ determine the initial conditions of the periodic solution.

2. PROOFS

To prove the theorems of Section 1 we need to point out some facts. The first is a comparison result of two solutions of (2). In this section we shall assume that all the conditions on f, g in Theorems 1 and 2 hold.

PROPOSITION 1. *Any couple of different solutions of* (2) *has at most one point in common.*

Proof. Suppose $x_1(t), x_2(t)$ are two different solutions of (2) such that

$$x_1(t_0) = x_2(t_0), \qquad x_1(t_1) = x_2(t_1)$$

and $x_1(t) > x_2(t)$ for all $t \in]t_0, t_1[$. Then, $z(t) = x_1(t) - x_2(t)$ satisfies

$$z''(t) + f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t) = g(x_2(t)) - g(x_1(t)) > 0 \qquad \forall t \in]t_0, t_1[,$$

so integrating over $]t_0, t_1[$ we have that $z'(t_1) > z'(t_0)$, a contradiction because $z(t) > 0$ for any $t \in]t_0, t_1[$.

PROPOSITION 2. *Let* $x(t) = x(t, x_0, v_0)$, $x_1(t) = x(t, x_1, v_1)$ *be solutions of* (2) *such that* $x_0 < x_1$. *If* $x_1 - x_0 \geq (v_0 - v_1)/m$ *then*

$$x(t) < x_1(t) \qquad \forall t \geq 0.$$

Proof. If the conclusion fails to hold, then there must be some $t_1 > 0$ such that $x(t_1) = x_1(t_1)$. Set $z(t) = x(t) - x_1(t)$. Then $z(t) < 0$, $t \in [0, t_1)$ and $z(t_1) = 0$. As in the proof of Theorem 1 we have

$$z''(t) + f(x(t))x'(t) - f(x_1(t))x_1'(t) < 0 \qquad \forall t \in [0, t_1)$$

and an integration over $[0, t_1)$ gives

$$z'(t_1) - v_0 + v_1 + \int_{x_0}^{x(t_1)} f(s)\, ds - \int_{x_1}^{x_1(t_1)} f(s)\, ds < 0.$$

So $z'(t_1) < v_0 - v_1 - \int_{x_0}^{x_1} f(s)\, ds \leq v_0 - v_1 - m(x_1 - x_0) \leq 0$, but it is not possible.

Remark 1. By Proposition 1, Proposition 2 holds true if $x_0 = x_1$ and $v_0 < v_1$.

Remark 2. The same conclusion is obtained for $t \leq 0$ if $x_1 - x_0 \leq (v_0 - v_1)/M$.

The following lemma asserts that two solutions of (2) with the same initial velocity cannot be very close.

LEMMA 3. *Let* $x(t) = x(t, x_0, v)$, $x_1 = x(t, x_1, v)$ *be solutions of* (2) *defined on* $[0, +\infty)$. *Given* $\delta > 0$, *there exists* $\epsilon > 0$ *such that if* $x_1 - x_0 > \delta$, *then there exists a sequence* $\tau_n \to +\infty$ *such that* $x_1(\tau_n) - x(\tau_n) > \epsilon$ *for large* n.

The same conclusion is true for $x(t) = x(t, x_0, v_0)$ *and* $x_1(t) = x(t, x_0, v_1)$ *with* $v_1 - v_0 > \delta$.

Proof. Set $z(t) = x_1(t) - x(t)$. Then $z(t) > 0 \; \forall t \in [0, +\infty)$, by Proposition 2. If the conclusion is not true, $z(t) \to 0$ when $t \to +\infty$. By the mean value theorem there must be a sequence $\tau_n \to +\infty$ such that $z'(\tau_n) \to 0$ and $z'(\tau_n) < 0$. As in the previous proof, an integration on $(0, \tau_n)$ gives

$$z'(\tau_n) - z'(0) + \int_{x_1}^{x_1(\tau_n)} f(s)\, ds - \int_{x_0}^{x(\tau_n)} f(s)\, ds > 0.$$

Since $z'(0) = 0$, we get for large n

$$\int_{x(\tau_n)}^{x_1(\tau_n)} f(s)\, ds \geq \int_{x_0}^{x_1} f(s)\, ds$$

and, hence, $M(x_1(\tau_n) - x(\tau_n)) \geq m(x_1 - x_0)$. Taking $\epsilon < \delta m/M$ we are done.

If $x(t) = \varphi(t)$ and $x(t) - \varphi(t) \to 0$ as $t \to +\infty$, we get $0 \geq m(x_1 - x_0)$, but this is not possible.

When $x_0 = x_1$ and $v_1 - v_0 > \delta$ we obtain $M(x_1(\tau_n) - x(\tau_n)) \geq v_1 - v_0$ and it is enough to take $\epsilon < \delta/M$.

LEMMA 4. *Fixed* $v \in \mathbb{R}$, *the set*

$$A = \left\{ x_0 \in \mathbb{R}^+ : \lim_{t \to +\infty} x(t, x_0, v) = +\infty \right\}$$

is a nonempty open interval.

Proof. If A is empty then, we can choose $x_1 \in \mathbb{R}^+$ such that $x_1 - \varphi(0) \geq (\varphi'(0) - v)/m$, and applying Proposition 2 and Theorem 2 we obtain a contradiction.

If $x_0 \in A$ and $x_1 > x_0$ then $x(t, x_1, v) \to +\infty$ as $t \to +\infty$, by Proposition 2. Hence A is an interval.

To prove that A is open, let $x_0 \in A$ and

$$C = \limsup_{t \to +\infty} x'(t, x_0, v), \qquad c = \liminf_{t \to +\infty} x'(t, x_0, v).$$

Given $R > (v - c + 1)/m + x_0$, we have $x(nT, x_0, v) > R$ and $c - 1 < x'(nT, x_0, v) < C + 1$ for some positive integer n. By continuous dependence there exists $\delta > 0$ such that if $|x_0 - x_1| < \delta$ then $x(nT, x_1, v) > R$ and $c - 1 < x'(nT, x_1, v) < C + 1$. If we define $x_1(t) = x(t + nT, x_1, v)$ then

$$x_1(0) - x_0 > R - x_0 > \frac{v - c + 1}{m} > \frac{v - x_1'(0)}{m}.$$

Hence, by Proposition 2, $x_1(t) \to +\infty$ as $t \to +\infty$ and $x_1 \in A$.

LEMMA 5. *Fixed $v \in \mathbb{R}$, the set*

$$B = \left\{ x_0 \in \mathbb{R}^+ : \lim_{t \to w^+} x(t, x_0, v) = 0 \right\}$$

is an open interval. Moreover, if $V(0^+)$ is infinite then B is nonempty.

Proof. Let $p_M = \max_{t \in [0,T]} p(t)$ and $x_M > 0$ such that $g(x_M) = p_M$. We claim that a local minimum of any solution $x(t)$ of (2) is larger than x_M. If not, $x(t)$ would have a minimum at $t = t_0$ with $x(t_0) < x_M$, then $x'(t_0) = 0$, $x''(t_0) \geq 0$ and $g(x(t_0)) > p_M$. But this is impossible, because

$$x''(t_0) + f(x(t_0))x'(t_0) + g(x(t_0)) = p(t_0) \leq p_M.$$

Now, if $x_0 \in B$, for $R > 0$ there exists t_0 such that $x(t_0, x_0, v) < R$ and $x'(t_0, x_0, v) < 0$. By continuous dependence, there exists $\delta > 0$ such that if $|x_0 - x_1| < \delta$ then $x(t_0, x_1, v) < R$ and $x'(t_0, x_1, v) < 0$. Taking $R = x_M$ we obtain that $x(t, x_1, v)$ has not a minimum for $t > t_0$ and, hence, $x(t, x_1, v) \to 0$ as $t \to w^+$. So B is open.

To prove that B is nonempty we see that if we have $\int_{x_0}^{x_M} g(s)\,ds > p_M x_M + v^2/2$ then $x(t) \to 0$ as $t \to w^+$. In fact, if this is not the case, let $t_0 > 0$ be such that $x(t) < x_M$, $x'(t) > 0$ $\forall t \in (0, t_0)$. Since

$$x'(t) = \frac{p(t) - g(x(t)) - x''(t)}{f(x(t))} > 0 \qquad \forall t \in (0, t_0),$$

we have $x''(t) + g(x(t)) < p_M$, $t \in (0, t_0)$. An integration over $[0, t_0]$ after multiplying by $x'(t)$ gives

$$\int_{x_0}^{x_M} g(s)\,ds < p_M(x_M - x_0) - \frac{x'(t_0)^2}{2} + \frac{v^2}{2} < p_M x_M + \frac{v^2}{2}$$

and we get a contradiction.

In a similar way, one can obtain that for a fixed $x_0 \in \mathbb{R}^+$, the set

$$A_1 = \left\{ v \in \mathbb{R} : \lim_{t \to +\infty} x(t, x_0, v) = +\infty \right\}$$

is a nonempty open interval.

COROLLARY 1. *If the potential is infinite, for any fixed $v \in \mathbb{R}$ (resp. $x_0 \in \mathbb{R}^+$) there exists a unique $x_0 \in \mathbb{R}^+$ (resp. $v \in \mathbb{R}$) such that $x(t, x_0, v) - \varphi(t) \to 0$ as $t \to +\infty$.*

Proof. By Lemmas 4 and 5, A and B are nonempty open intervals, and by Proposition 2 we have that $\sup B \leq \inf A$. If we take $x_0 \in [\sup B, \inf A]$, then $x(t, x_0, v)$ is bounded and defined until $+\infty$, so using Theorem 2 we conclude that $x(t, x_0, v) - \varphi(t) \to 0$ as $t \to +\infty$. Moreover, by Lemma 3, $\sup B = \inf A$, so x_0 exists and it is unique.

Proof of Theorem 3. Using the previous lemmas, the function $h: \mathbb{R} \to \mathbb{R}^+$ defined by

$$h(v) = \sup B = \inf A$$

is well defined and $\lim_{t \to +\infty}[x(t, h(v), v) - \varphi(t)] = 0$.

Let see that h is strictly decreasing. If it is not true, there exists $v_0 < v_2$ such that $h(v_0) \leq h(v_2)$. Let $x_0 = h(v_0)$. Then $\lim_{t \to +\infty}[x(t, x_0, v_0) - \varphi(t)] = 0$ and since $x_0 < h(v_2)$ (the equality is not possible because of Corollary 1), $\lim_{t \to w^+} x(t, x_0, v_2) = 0$, a contradiction with Proposition 1.

If h is not continuous at v, let x_0 such that $h(v^+) < x_0 < h(v^-)$. By Corollary 1, there exists $v_0 \in \mathbb{R}$ such that $h(v_0) = x_0$. If $v_0 < v$ (resp. $v_0 > v$) then $h(v_0) > h(v^-) > x_0$ (resp. $h(v_0) < h(v^+) < x_0$), a contradiction.

h is decreasing and bounded below, so $\exists \lim_{v \to +\infty} h(v)$, and it is zero because in the other case A_1 would be empty for any x_0 minor that this limit.

In the same way the limit $\lim_{v \to -\infty} h(v)$ exists, and modifying slightly Lemma 3, we see that it is infinite.

Finally, for the proof of Theorem 4 we prove that if the potential is finite in zero then there exists $v_s \in \mathbb{R}$ such that for all $v \geq v_s$ we have that B is empty.

In fact, let $(x_1, v_1) \in \mathbb{R}^+ \times \mathbb{R}$ be the initial conditions of $\varphi(t)$ at $t = 0$. Take $x_0 < x_1$ such that $x_1 - x_0 \leq (v_0 - v_1)/M$, where $v_0 = h^{-1}(x_0)$, $v_1 = h^{-1}(x_1)$. By Proposition 2, $x(t, x_0, v_0) < \varphi(t)$, $t \leq 0$. Hence $x(t, x_0, v_0) \to 0$ as $t \to w^-(x_0, v_0)$.

By continuation of solutions theorem, we have that

$$\limsup_{x \to x_1^-} w^-\big(x, h^{-1}(x)\big) \leq w^-(x_1, v_1) = -\infty.$$

Hence, there exists $\bar{v}$ and $\bar{x}_0 = h(\bar{v})$ such that $w^-(\bar{x}_0, \bar{v}) = -kT$, a multiple of T.

Now, if we take a sequence $v_n \to \bar{v}$ as $n \to +\infty$ with $v_n < \bar{v}$, for the solutions $x(t, h(v_n), v_n)$ we have

$$\limsup_{n \to +\infty} w^-\big(h(v_n), v_n\big) \leq -kT.$$

So, the points $x(-kT, h(v_n), v_n), x'(-kT, h(v_n), v_n)$ are the initial conditions at $t = 0$ of $x(t - kT, h(v_n), v_n)$ and

$$\begin{aligned} &x\big(t, x(-kT, h(v_n), v_n), x'(-kT, h(v_n), v_n)\big) - \varphi(t) \\ &\quad = x\big(t - kT, h(v_n), v_n\big) - \varphi(t) \to 0 \end{aligned}$$

as $t \to +\infty$. Hence, if we take limits in n, by continuous dependence of initial conditions, the point $(x(-kT, \bar{x}_0, \bar{v}), x'(-kT, \bar{x}_0, \bar{v}))$ is on graph(h). But $x(-kT, \bar{x}_0, \bar{v}) = 0$ and $x'(-kT, \bar{x}_0, \bar{v})$ is finite by Lemma 2, so if we let $x'(-kT, \bar{x}_0, \bar{v}) = v_s$, then $h(v_s) = 0$ and, consequently, for all $v \geq v_s$ the set B is empty.

3. CASE WITHOUT FRICTION

All the previous results can be obtained for the case without friction $(f \equiv 0)$ by similar reasonings. For the sake of brevity, we will not repeat all the proofs because they are very similar; for example, in Theorem 1 we only have to take

$$\lambda = 0, \qquad P = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and we have the assumptions of Theorem 1 in [6].

ACKNOWLEDGMENTS

We thank the referee for his suggestions that have improved the presentation of the paper. We also are grateful to R. Ortega for his suggestions and comments on the contents of this paper.

REFERENCES

1. P. Habets and L. Sanchez, Periodic solutions of some Liénard equations with singularities, *Proc. Amer. Math. Soc.* **109** (1990), 1035–1044.
2. A. C. Lazer and S. Solimini, On periodic solutions of nonlinear differential equations with singularities, *Proc. Amer. Math. Soc.* **99** (1987), 109–114.
3. J. Mawhin and C. Muñoz, Application du degré topologique à l'estimation du nombre des solutions périodiques d'équations differentielles, *Ann. Mat. Pura Appl.* **46**, No. 4 (1973), 1–19.
4. R. Ortega, Topological degree and stability of periodic solutions for certain differential equations, *J. London Math. Soc.* **42**, (1990), 505–516.
5. R. Ortega, Some applications of the topological degree to stability theory, *in* "Topological Methods in Differential Equations and Inclusions," NATO ASI Ser. 6, Kluwer Academic, Dordrecht, 1994.
6. R. A. Smith, Massera's convergence theorem for periodic nonlinear differential equations, *J. Math. Anal. Appl.* **120** (1986), 679–708.