# On the interplay between Lorentzian Causality and Finsler metrics of Randers type

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WORKSHOP ON FINSLER GEOMETRY AND ITS APPLICATIONS
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| Global hyperbolicity is equivalent to the following condition $(\mathrm{A}) : \bar{B}^+(p, r) \cap \bar{B}^-(p, r)$ compact $\forall p \in S$ and $\forall r > 0$ for the Randers metric $R$ | $\Rightarrow$ | Condition $(\mathrm{A})$ implies: $(a)$ convexity of $R$; $(b)$ the existence of $f : S \to \mathbb{R}$ such that $R_f = R + \mathrm{d}f$ is forward and backward complete |

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- Randers metrics in a manifold $M$ is a function $R : TM \to \mathbb{R}$ defined as:

$$R(x, v) = \sqrt{h(v, v)} + \omega_x[v]$$

where $h$ is Riemannian and $\omega$ a 1-form with $\|\omega_x\|_h < 1 \ \forall x \in M$,
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- Named after the norwegian physicist Gunnar Randers (1914-1992):

  Randers, G.: On an asymmetrical metric in the fourspace of General Relativity. Phys. Rev. (2) **59**, 195–199 (1941)


Gunnar Randers with Albert Einstein

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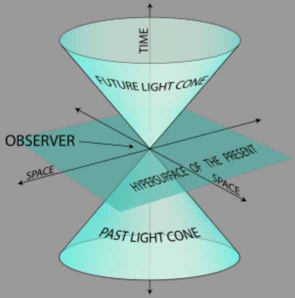

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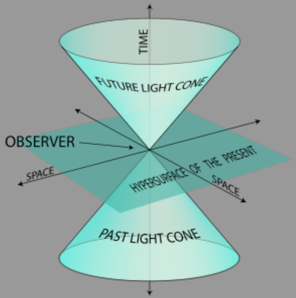

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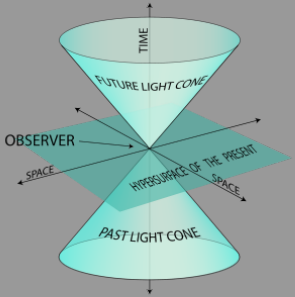

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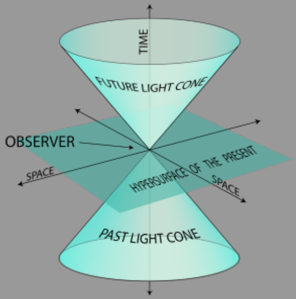

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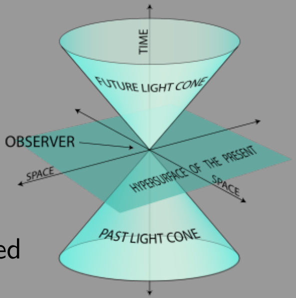

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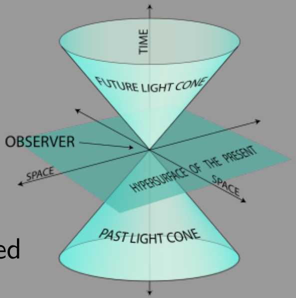

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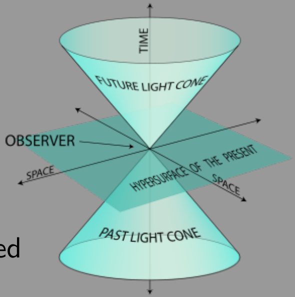

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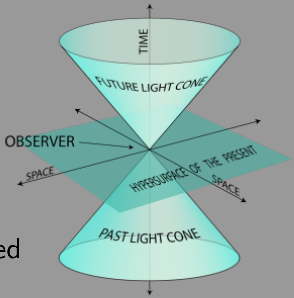
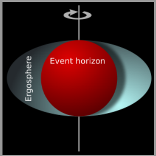
- A stationary spacetime $(M, g)$ is a Lorentzian manifold endowed with a timelike Killing vector field




KERR SPACETIME

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- Analogously we define the chronological past $I^-(p)$ and the causal past $J^-(p)$.

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Causal properties classify spacetimes depending on the behaviour of causal cones. A spacetime is:

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$\Downarrow$

*Causally simple*

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- Globally hyperbolic if it admits a Cauchy hypersurface (a subset $S$ that meets exactly once every inextendible timelike curve)

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M. A. J. and M. Sánchez, *A note on the existence of standard splittings for conformally stationary spacetimes*, Classical Quantum Gravity, 25 (2008), pp. 168001, 7.

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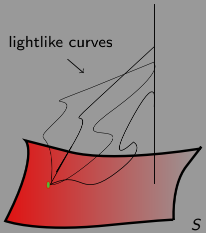


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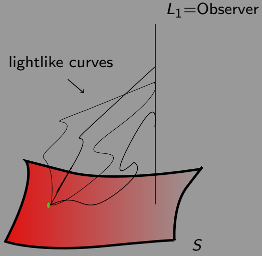
$$t(b)=t(a)+\int_a^b\left(\frac{1}{\beta}g_0(\dot{x},\delta)+\sqrt{\frac{1}{\beta}g_0(\dot{x},\dot{x})+\frac{1}{\beta^2}g_0(\dot{x},\delta)^2}\right)\mathrm{d}s.$$

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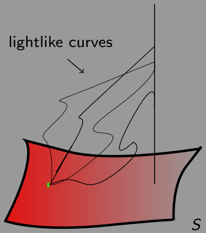
- Thi is just because $g(\dot{\gamma}, \dot{\gamma}) = 0$, that is

$$g_0(\dot{x}, \dot{x}) + 2g_0(\delta(x), \dot{x})\dot{t} - \beta(x)\dot{t}^2 = 0$$

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*A curve $s \to \gamma(s) = (s, x(s))$ is a lightlike pregeodesic of $(\mathbb{R} \times S, g)$ iff $s \to x(s)$ is a Fermat geodesic with unit speed.*

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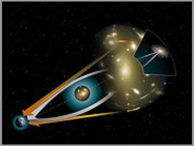

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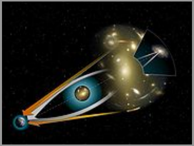
  - Existence of $t$-periodic lightlike geodesics is equivalent to existence of Fermat closed geodesics


Einstein Ring



Gravitational lensing

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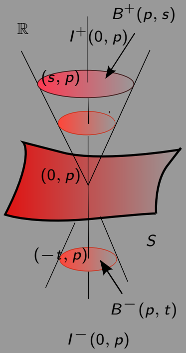

# Causality through the Fermat metric

Theorem

*Globally hyperbolic*

$\Downarrow$

*Causally simple*

$\Downarrow$

*Causally continuous*

$\Downarrow$

*Stably causal*

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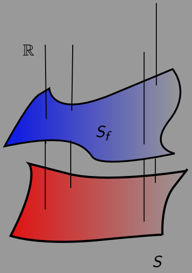


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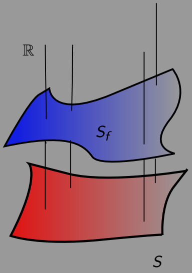


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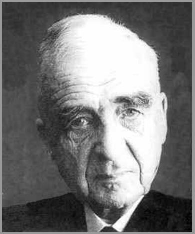

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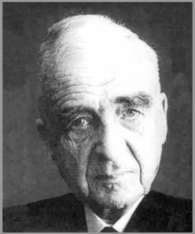

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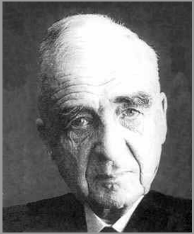

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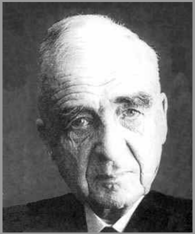

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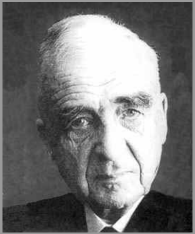

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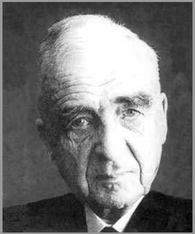

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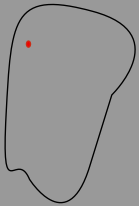

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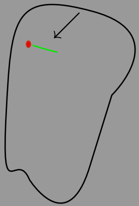


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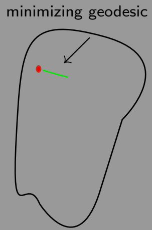


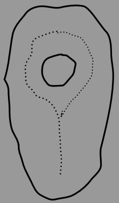

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Y. Li and L. Nirenberg, *The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations*, Comm. Pure Appl. Math.,(2005).

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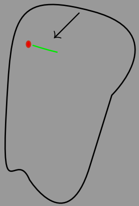

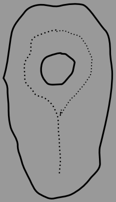

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More information in:


E. Caponio, M. A. Javaloyes and M. Sánchez, *The interplay between Lorentzian causality and Finsler metrics of Randers type.*, arxiv: 0903.3501, preprint 2009.

E. Caponio, M. A. Javaloyes and A. Masiello, *On the energy functional on Finsler manifolds and applications to stationary spacetimes*, arxiv: 0702323, preprint 2007.

# Open problems

(1) Does it hold Generalized Hopf-Rinow theorem for any Finsler metric?
(2) and the results for the distance $\rho_C$ from a closed subset?

More information in:

E. Caponio, M. A. Javaloyes and M. Sánchez, *The interplay between Lorentzian causality and Finsler metrics of Randers type.*, arxiv: 0903.3501, preprint 2009.

E. Caponio, M. A. Javaloyes and A. Masiello, *On the energy functional on Finsler manifolds and applications to stationary spacetimes*, arxiv: 0702323, preprint 2007.

Thank you for your attention!!!!!!