On the semi-Riemannian bumpy theorem

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New developments in Lorentzian Geometry

My collaborators

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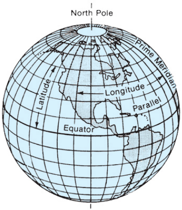

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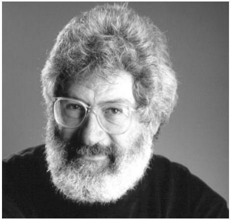

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- The dynamical approach does not apply in the semi-Riemmanian version (for example the unit tangent bundle is not meaningful)

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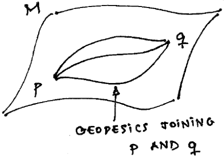

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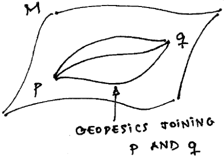

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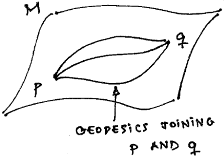

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2) a certain transversality condition is not satisfied in the iterates of a geodesic

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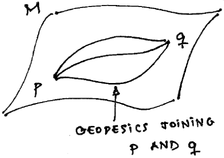

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1) The first problem will be overcome obtaining a $\mathbb{S}^1$-equivariant genericity theorem
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3) Seps 1) and 2) will give a weak bumpy theorem
4) To conclude the "authentic" semi-Riemannian bumpy theorem we will use Anosov's ideas

Our main tool: the Genericity theorem

X separable Banach manifold and Y a separable Hilbert manifold and $\Pi : X \times Y \to X$ the projection.

Theorem

Let $f : A \subset X \times Y \to \mathbb{R}$ be C^2. Assume that for every $(x_0, y_0) \in A$ with $\frac{\partial f}{\partial y}(x_0, y_0) = 0$ it holds:

- $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ *is (self-adjoint)-Fredholm in* $T_{y_0} Y$
- *for all* $v \in \ker \left[\frac{\partial^2 f}{\partial y^2}(x_0, y_0) \setminus \{0\}\right]$, $\exists\ w \in T_{x_0} X$ *such that*

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)(v, w) \neq 0 \quad \textit{(Transversality condition)}$$

For $x \in \Pi(A)$ set $A_x = \{y \in Y : (x, y) \in A\}$. Then, the set of $x \in X$ such that $A_x \ni y \to f(x, y) \in \mathbb{R}$ is a Morse function is generic in $\Pi(A)$.

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- E_g is invariant by this action!!

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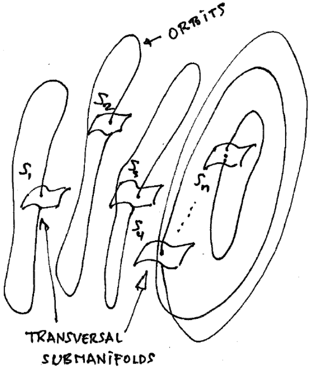

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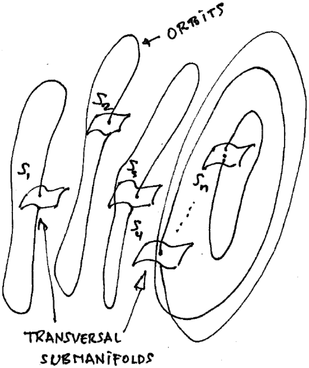

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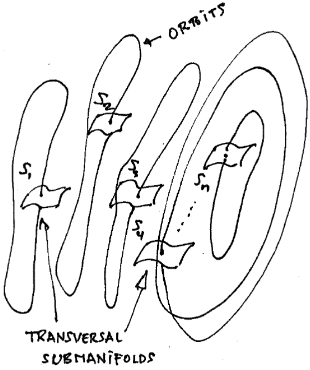

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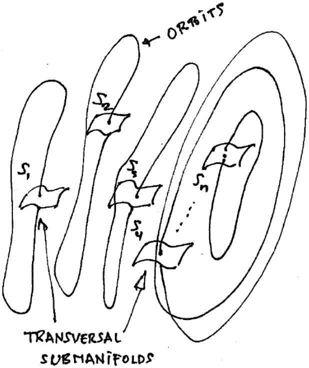

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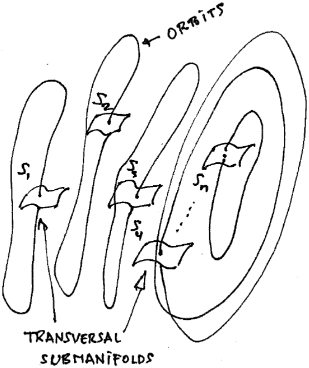

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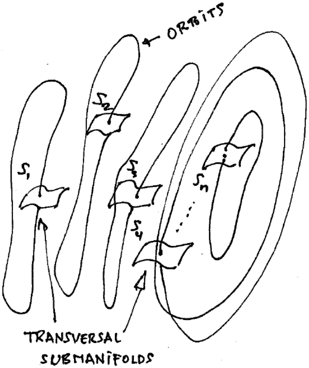
- the countable intersection of generic subsets is generic



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- But Transversality condition is satisfied just when consider prime closed geodesics
- In this way we obtain the genericity of metrics with all the prime closed geodesics non-degenerate, that is, the weak bumpy theorem

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Let us introduce the notation:

$$\mathcal{M}(a,b) = \Big\{ \mathbf{g} \in \mathrm{Met}(M,i;k) : \text{every closed } \mathbf{g}\text{-geodesic } \gamma \text{ with } \mathrm{E}_{\min}(\gamma) \le a \text{ and } \mathrm{E}(\gamma) \le b \text{ is nondegenerate} \Big\}.$$

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- to conclude the semi-Riemannian bumpy theorem it is enough to show that every $\mathcal{M}(n,n)$ is generic in $\mathrm{Met}(M,i;k)$

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Proof: take $\{g_n\} \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$ and show that $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$.

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1) $\mathcal{M}(a,b)$ is open in $\mathrm{Met}(M,i;k)$ for every $a \leq b$

Proof: take $\{g_n\} \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$ and show that $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$.

2) $\mathcal{M}(\frac{3}{2}a, \frac{3}{2}a) \cap \mathcal{M}(a,2a)$ is dense in $\mathcal{M}(a,2a)$

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3) $\mathcal{M}(a,2a)$ is dense in $\mathcal{M}(a,a)$ Proof: a perturbation argument

Steps of the proof

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1) $\mathcal{M}(a,b)$ is open in $\mathrm{Met}(M,i;k)$ for every $a \leq b$

Proof: take $\{g_n\} \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$ and show that $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$.

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5) $\mathcal{M}(b,b)$ is dense in $\mathcal{M}(a,a)$

Proof: Apply step 4 to obtain: $\mathcal{M}(\left(\frac{3}{2}\right)^n a, \left(\frac{3}{2}\right)^n a)$ is dense in $\mathcal{M}(a,a)$

Steps of the proof

1) $\mathcal{M}(a,b)$ is open in $\mathrm{Met}(M,i;k)$ for every $a \leq b$

Proof: take $\{g_n\} \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$ and show that $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$.

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Steps of the proof

1) $\mathcal{M}(a,b)$ is open in $\mathrm{Met}(M,i;k)$ for every $a \leq b$

Proof: take $\{g_n\} \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$ and show that $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M,i;k) \setminus \mathcal{M}(a,b)$.

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6) $\mathcal{M}(b,b)$ is dense in $\mathrm{Met}(M,i;k)$ for every $b \in \mathbb{R}$

Proof: Step 5 and the fact that for a fix metric g all the closed geodesics have g_R-energy greater than $\bar{a} > 0$

Bumpy theorem in the C^∞-topology

Introduce the notations:

$$\mathrm{Met}^*_N(M, i; k) = \big\{\mathbf{g} \in \mathrm{Met}(M, i; k) : \text{all closed } \mathbf{g}\text{-geodesics } \gamma \text{ with } E(\gamma) \le N \text{ are nondegenerate}\big\}$$

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- $\mathrm{Met}^*(M, i; \infty) = \cap_{N=1}^{\infty} \mathrm{Met}^*_N(M, i; \infty)$
- it is enough to prove that every $\mathrm{Met}^*_N(M, i; \infty)$ is open and dense in $\mathrm{Met}(M, i; \infty)$.

Bumpy theorem in the C^∞-topology

Bumpy theorem in the C^∞-topology

- $\mathrm{Met}^*_N(M, i; k)$ is open in $\mathrm{Met}(M, i; k)$ for $k = 2, \ldots, \infty$

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Proof: Again consider $g_n \in \mathrm{Met}(M, i; k) \setminus \mathrm{Met}^*_N(M, i; k)$, then $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M, i; k) \setminus \mathrm{Met}^*_N(M, i; k)$

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Proof:

1) $\mathrm{Met}^*(M,i;k)$ is dense in $\mathrm{Met}(M,i;k)$ (Bumpy theorem)

Bumpy theorem in the C^∞-topology

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Proof:

1) $\mathrm{Met}^*(M, i; k)$ is dense in $\mathrm{Met}(M, i; k)$ (Bumpy theorem)
2) $\mathrm{Met}^*(M, i; k) \subset \mathrm{Met}^*_N(M, i; k)$ (trivial)

Bumpy theorem in the C^∞-topology

- $\mathrm{Met}^*_N(M,i;k)$ is open in $\mathrm{Met}(M,i;k)$ for $k=2,\dots,\infty$

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Bumpy theorem in the C^∞-topology

- $\mathrm{Met}^*_N(M, i; k)$ is open in $\mathrm{Met}(M, i; k)$ for $k = 2, \ldots, \infty$

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4) $\mathrm{Met}(M, i; \infty) \cap \mathrm{Met}^*_N(M, i; k) = \mathrm{Met}^*_N(M, i; \infty)$ is dense in $\mathrm{Met}(M, i; k)$ for all $k \geq 2$:

$$\text{dense} \cap (\text{open and dense}) = \text{dense}$$

- $\mathrm{Met}^*_N(M, i; k)$ is open in $\mathrm{Met}(M, i; k)$ for $k = 2, \dots, \infty$

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5) Step 4) implies that $\mathrm{Met}^*_N(M, i; \infty)$ is dense in $\mathrm{Met}(M, i; \infty)$

Bumpy theorem in the C^∞-topology

- $\mathrm{Met}^*_N(M,i;k)$ is open in $\mathrm{Met}(M,i;k)$ for $k=2,\dots,\infty$

Proof: Again consider $g_n \in \mathrm{Met}(M,i;k)\setminus\mathrm{Met}^*_N(M,i;k)$, then $\lim_{n\to\infty} g_n = g_\infty \in \mathrm{Met}(M,i;k)\setminus\mathrm{Met}^*_N(M,i;k)$

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Vielen Dank for Ihre freundliche Aufmerksamkeit

Thank you very much for your kind attention