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On a new set of polynomials representing the wave functions of the quantum relativistic harmonic oscillator

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Dedicated to Professor Luigi Gatteschi on the occasion of his seventieth birthday

The wave functions of the quantum relativistic harmonic oscillator in configuration space have recently been shown by Aldaya et al. to be expressed by means of a one-parameter family of polynomials $\{H_n^{(N)}(\xi)\}_{n=0}^{\infty}$. These polynomials are to be called Relativistic Hermite Polynomials (briefly RHP) because they reduce to the well-known classical Hermite polynomials in the non-relativistic limit ($N \to \infty$). Here, several algebraic and spectral properties of these polynomials are investigated. As to the former ones, a Rodrigues-type formula, a generating function, various recurrence relations and sum rules are found. On the other hand, the density of zeros $\Omega_n(\xi)$ (i.e. the number of zeros per interval) of the nth-degree RHP is studied by means of the so-called Newton sum rules. The exact values of these quantities, which are closely related to the moments around the origin of $\Omega_n(\xi)$, are explicitly given in terms of the parameter N.

Keywords: Differential equations, zeros, special functions, hypergeometric-type polynomials.

1. Introduction

Aldaya et al. [2] have recently obtained a quantum symmetry algebra of a relativistic harmonic oscillator in $1+1$ dimensions generated by the energy, position and momentum operators. This algebra, which generalizes the non-relativistic one, has allowed to work out an explicit expression for the corresponding wavefunctions in configuration space given by (cf. [2, p. 383]):

$$\Psi_n(t, \xi, p; N) = e^{if/\hbar}\, e^{-inwt}\, 2^{-n/2} \alpha^{-(n+N)} H_n^{(N)}(\xi), \tag{1}$$

where the following notation has been used

$$N = \frac{mc^2}{\hbar w}, \quad \xi = \frac{w}{c}\sqrt{N}x,$$

$$\alpha(\xi; N) = \left(1 + \frac{\xi^2}{N}\right)^{1/2}, \quad P^0(\xi, p; N) = \sqrt{p^2 + m^2c^2\alpha(\xi; N)^2},$$

$$f(\xi, p; N) = \frac{2mc^2}{w}\arctan\left(\frac{\sqrt{N}(P^0 - p + mc)}{mc\xi}\right).$$

Here, n is the principal quantum number, w represents the frequency of the oscillator and p is the momentum. The functions $H_n^{(N)}(\xi)$ are polynomials which reduce to the well known classical Hermite polynomials [1,7] in the non-relativistic limit $c \to \infty$ (i.e. when $N \to \infty$) and the dimensionless real parameter N must be greater than $1/2$ because of the square integrability of the wavefunction.

From equation (1) and the Casimir operator of the representation of the relativistic algebra, Aldaya et al. [2] have shown that the RHP $y_n(\xi; N) \equiv H_n^{(N)}(\xi)$ satisfy the following second order differential equation

$$\left(1 + \frac{\xi^2}{N}\right)y_n'' - \frac{2}{N}(N + n - 1)\xi y_n' + \frac{n}{N}(2N + n - 1)y_n = 0, \tag{2}$$

and the three-term recurrence relation

$$H_{n+1}^{(N)}(\xi) = 2\left(1 + \frac{n}{N}\right)\xi H_n^{(N)}(\xi) - \frac{n(2N + n - 1)}{N}\left(1 + \frac{\xi^2}{N}\right)H_{n-1}^{(N)}(\xi).$$

From these two equations Aldaya et al. have also computed the explicit expression of these polynomials

$$\begin{aligned} H_n^{(N)}(\xi) &= \sum_{k=0}^{[n/2]} A_{n,n-2k}^{(N)}(2\xi)^{n-2k}, \\ A_{n,n-2k}^{(N)} &= \frac{(-1)^k n! N^k (N - 1/2)!(2N + n - 1)!}{k!(n - 2k)!(N + k - 1/2)!(2N)^n(2N - 1)!}. \end{aligned} \tag{3}$$

The differential equation (2) shows that the RHP are a generalization of the so-called polynomials of hypergeometric type [14]. A polynomial $P_n(\xi)$ is of hypergeometric type if it is a solution of a second order differential equation of the form

$$\sigma(\xi)P_n'' + \tau(\xi)P_n' + \lambda_n P_n = 0,$$

where σ and τ are n-independent polynomials of degree not greater than 2 and 1 respectively, and $\lambda_n = -n\tau' - (n(n-1)/2)\sigma''$ is a constant. Notice that in the RHP case $\tau \equiv \tau(\xi; n; N) = (-2/N)(N + n - 1)\xi$; then, the function τ does depend on the degree of the polynomial solution. Nevertheless, the relation $\lambda_n = -n\tau' - (n(n-1)/2)\sigma''$ is still valid.

The hypergeometric character of the RHP has been used in [19] to prove that these polynomials satisfy a varying orthogonality relation from which several asymptotic properties of the zeros have been deduced. Nagel in [13] also uses equation (2) to obtain a fruitful connection between these polynomials and the well known Gegenbauer polynomials [1,7]. By using this connection, Nagel translates some properties of the Gegenbauer polynomials to the RHP. In particular, he proves in another way (different from the one used in [19]) the aforementioned varying orthogonality as well as some other asymptotic properties. Moreover, an interesting relation between the zeros of Bessel functions and the ones of RHP is deduced.

In this paper some of the Nikiforov and Uvarov ideas [14] are extended and applied to differential equations of the RHP kind. In particular, a general method to obtain a Rodrigues-type formula for the polynomial solutions of hypergeometric-type differential equations with $\tau = \tau(\xi; n)$ is given in section 2, which also includes an integral representation and a generating function for the corresponding polynomials. In section 3, the integral representation is used in order to build up several recurrence formulas and sum rules involving the RHP.

Finally, section 4 deals with the spectrum of zeros of the RHP. Equation (1) shows that such a spectrum is the same as the spectrum of zeros of wave functions of the relativistic harmonic oscillator. In particular, after proving that all zeros are real and simple, their density (i.e. the number of zeros per unit interval) is considered and investigated via the Newton sum rules which, when conveniently normalized, represent the moments of such density functions.

2. Rodrigues formula: integral representation and generating function

Let us consider the second order differential equation

$$\sigma(\xi) y_n'' + \tau(\xi; n) y_n' + \lambda_n y_n = 0, \tag{4}$$

where σ and τ are polynomials of degree not greater than 2 and 1 respectively and λ_n is a constant.

Following some ideas of [14], we denote by $V_{n,k} \equiv d^k y_n / d\xi^k$ the kth derivative of the nth degree polynomial solution of equation (4). It is easy to show by induction that they satisfy a second order differential equation of a similar form

$$\sigma(\xi) V_{n,k}'' + \tau_k(\xi; n) V_{n,k}' + \nu_{n,k} V_{n,k} = 0, \tag{5}$$

where

$$\tau_k(\xi; n) = \tau(\xi; n) + k\sigma'(\xi); \quad \nu_{n,k} = \lambda_n + k\tau'(\xi; n) + \frac{k(k-1)}{2}\sigma''. \tag{6}$$

So, a necessary and sufficient condition for equation (4) to have a polynomial solution of degree n is $\nu_{n,n} = 0$, that is $\lambda_n = -n\tau'(\xi; n) - (n(n-1)/2)\sigma''$. Consequently, since for equation (2) this last condition holds, one can say that each RHP $H_n^{(N)}(\xi)$ is exactly of degree n.

The selfadjoint form of equation (5) is

$$\frac{d}{d\xi}(\sigma\rho_{n,k}V'_{n,k}) + \rho_{n,k}\nu_{n,k}V_{n,k} = 0, \tag{7}$$

where $\rho_{n,k} = \rho_{n,k}(\xi)$ is the symmetrization factor of equation (5), i.e.

$$\frac{\rho'_{n,k}(\xi)}{\rho_{n,k}(\xi)} = \frac{\tau_k(\xi;n) - \sigma'(\xi)}{\sigma(\xi)} \Rightarrow \begin{cases} \rho_{n,k}(\xi) = \sigma(\xi)^k \rho_{n,0}(\xi), \\ \rho_{n,k}(\xi) = \sigma(\xi)^{(k-m)}\rho_{n,m}(\xi). \end{cases} \tag{8}$$

Here $\tau_k(\xi;n)$ is given by equation (6) and $\rho_{n,0}(\xi)$ is the symmetrization factor of equation (4). It should be noticed that this factor $\rho_{n,0}$ may depend on the degree n of the polynomial, contrary to what happens in the conventional hypergeometric-type differential equations [14]. In spite of this, the method described by these authors to obtain a Rodrigues formula, can be extended to the case that we consider here in an almost straightforward way.

Taking into account equation (8) and $V'_{n,k} = V_{n,k+1}$, from equation (7) we obtain

$$\rho_{n,k}V_{n,k} = \left(-\frac{1}{\nu_{n,k}}\right)\frac{d}{d\xi}(\rho_{n,k+1}V_{n,k+1}),$$

and iterating $m-k$ times in this last expression,

$$V_{n,k} = \frac{C_{n,k}}{C_{n,m}}\frac{1}{\rho_{n,k}}\frac{d^{m-k}}{d\xi^{m-k}}(\rho_{n,m}V_{n,m}), \tag{9}$$

where

$$C_{n,r} = (-1)^r \prod_{j=0}^{r-1} \nu_{n,j}, \quad C_{n,0} = 1, \tag{10}$$

and $\nu_{k,j}$ is defined in equation (6).

Putting $k=0$ and $m=n$ in equation (9) one has the following Rodrigues-type formula:

$$V_{n,0} = y_n = \frac{V_{n,n}}{C_{n,n}}\frac{1}{\rho_{n,0}}\frac{d^n}{d\xi^n}(\sigma^n\rho_{n,0}). \tag{11}$$

Notice that $V_{n,n}$ is a constant which depends on the normalization of the polynomial y_n. For example, if monic polynomials are considered (i.e. polynomials with leading coefficient equal to unity) then $V_{n,n} = n!$.

Moreover, by using Cauchy's integral formula for analytic functions, equation (11) provides an integral representation for the polynomial solutions of equation (4); namely,

$$y_n(\xi) = \frac{V_{n,n}}{C_{n,n}}\frac{1}{\rho_{n,0}(\xi)}\frac{n!}{2\pi i}\int_\Gamma \frac{\sigma(s)^n\rho_{n,0}(s)}{(s-\xi)^{n+1}}\,ds, \tag{12}$$

where Γ denotes a closed contour in the complex plane encircling the point $s=\xi$ and such that $\sigma^n\rho_{n,0}$ is analytic in the region inside it.

On the other hand, equation (9) also provides a Rodrigues-type formula for the

kth derivative of y_n. Just putting $m = n$ in equation (9), we obtain

$$V_{n,k} = \frac{d^k}{d\xi^k} y_n = \frac{V_{n,n} C_{n,k}}{C_{n,n}} \frac{1}{\sigma^k \rho_{n,0}} \frac{d^{n-k}}{d\xi^{n-k}} (\sigma^n \rho_{n,0}).$$

From this expression and using again Cauchy's integral formula for analytic functions, an integral representation for the kth derivative of y_n can also be given:

$$V_{n,k} = \frac{d^k}{d\xi^k} y_n = \frac{V_{n,n} C_{n,k}}{C_{n,n}} \frac{1}{\sigma(\xi)^k \rho_{n,0}(\xi)} \frac{(n-k)!}{2\pi i} \int_\Gamma \frac{\sigma(s)^n \rho_{n,0}(s)}{(s-\xi)^{n-k+1}} ds, \tag{13}$$

where the contour Γ has the same characteristics as stated above. Notice that the method we have just described provides a Rodrigues-type formula (and also an integral representation) for any polynomial solution of the differential equation (4), directly in terms of its coefficients σ and τ.

The application of equations (2), (6), (8) and (10) to the RHP gives

$$\begin{aligned} \sigma(\xi) &= \left(1 + \frac{\xi^2}{N}\right); \quad \rho_{n,0} = [\sigma(\xi)]^{-(N+n)}, \\ \nu_{n,k} &= \lambda_n + k\tau'(\xi; n) + \frac{k(k-1)}{2} \sigma'', \\ C_{n,n} &= (-1)^n \prod_{j=0}^{n-1} \nu_{n,j} = (-1)^n \frac{n!\,(2N)_n}{N^n}, \end{aligned} \tag{14}$$

where $(a)_s = a(a+1)\cdots(a+s-1)$ denotes the well known Pochhammer symbol. So, if we choose the normalization given in equation (3) also considered in [2], i.e. $V_{n,n} = n!\,(2N)_n/N^n$, then the Rodrigues-type formula and the corresponding integral representation for the RHP and their derivatives are

$$H_n^{(N)}(\xi) = (-1)^n \sigma(\xi)^{N+n} \frac{d^n}{d\xi^n} [\sigma(\xi)^{-N}], \tag{15}$$

$$H_n^{(N)}(\xi) = (-1)^n \sigma(\xi)^{N+n} \frac{n!}{2\pi i} \int_\Gamma \frac{\sigma(s)^{-N}}{(s-\xi)^{n+1}} ds, \tag{16}$$

$$\frac{d^k}{d\xi^k} [H_n^{(N)}(\xi)] = \frac{(-1)^{n+k} n!\,(2N+n-k)_k}{(n-k)!\,N^k \sigma(\xi)^{k-N-n}} \frac{d^{n-k}}{d\xi^{n-k}} [\sigma(\xi)^{-N}], \tag{17}$$

$$\frac{d^k}{d\xi^k} [H_n^{(N)}(\xi)] = \frac{(-1)^{n+k} n!\,(2N+n-k)_k}{2\pi i N^k \sigma(\xi)^{k-N-n}} \int_\Gamma \frac{\sigma(s)^{-N}}{(s-\xi)^{n-k+1}} ds, \tag{18}$$

where $\sigma(\xi)$ is given by equation (14) and Γ is a closed contour encircling the point $s = \xi$ and such that it does not contain the points $s = \pm i\sqrt{N}$, $N > 1/2$, ensuring in this way that the function $\sigma(s)$ (see equation (14)) has no zeros in the region inside it.

Now we are able to compute explicitly the generating function for the RHP, which is defined by

$$\Phi(\xi, t) = \sum_{n=0}^{\infty} \frac{H_n^{(N)}(\xi)}{n!} t^n, \tag{19}$$

for sufficiently small $|t|$. Inserting in this equation the expression of $H_n^{(N)}(\xi)$ given by equation (16), and interchanging summation and integration (this is always possible for fixed ξ and sufficiently small $|t|$), we obtain

$$\Phi(\xi, t) = \sigma(\xi)^N \frac{1}{2\pi i} \int_\Gamma \frac{\sigma(s)^{-N}}{(s-\xi)} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(s-\xi)^n} \sigma(\xi)^n \, ds.$$

Since the geometric series in the integrand can be summed, the above expression becomes

$$\Phi(\xi, t) = \sigma(\xi)^N \frac{1}{2\pi i} \int_\Gamma \frac{\sigma(s)^{-N}}{s - \xi + t\sigma(\xi)} \, ds.$$

Due to the choice of the contour Γ, the integrand has a single pole inside it as $s = \xi - t\sigma(\xi)$, close to $s = \xi$ for sufficiently small $|t|$. Its residue is

$$R = \sigma(s)^{-N}\big|_{s=\xi - t\sigma(\xi)} = (1 + s^2/N)^{-N}\big|_{s=\xi - t(1+\xi^2/N)}.$$

So, the generating function for the RHP (defined by equation (19)) is given by

$$\Phi(\xi, t) = R\left(1 + \frac{\xi^2}{N}\right)^N = \left\{\frac{N}{(1+\xi^2/N)t^2 - 2\xi t + N}\right\}^N. \tag{20}$$

It is interesting to remark that in the limit when $N \to \infty$, equations (15) and (20) reduce to

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n}\left[e^{-\xi^2}\right], \quad \lim_{N\to\infty} \Phi(\xi, t) = \exp\{2\xi t - t^2\},$$

respectively, which are the well-known Rodrigues formula and the generating function for the classical Hermite polynomials [1,6].

Finally, let us mention that an alternative way of obtaining the Rodrigues formula (15)–(16) for the RHP (it has been obtained here as a particular case of the general expression given in (9)–(11)) is to use their connection [13] with the Gaussian hypergeometric functions and hence with the Gegenbauer polynomials (cf. [13, equations (2)–(4)]).

3. Sum rules and recurrence relations

As has been shown in the previous section, several methods developed in [14] for hypergeometric-type polynomials can be extended to the polynomial solutions of (4). This is also the case when trying to compute differential-difference relations (cf. [14, p. 14]), the basic ingredient of these computations being the integral representations for the polynomials and their derivatives given by equations (16) and (18).

Here we generalize and extend the aforementioned approach to obtain several sum rules involving the RHP and their derivatives, which are obtained directly in terms of the coefficients of the differential equation that they satisfy. As an illustration, the following two kinds of sum rules are considered:

(I) Sum rules involving more than two RHP of consecutive degrees, $H_n^{(N)}(\xi)$, $n = k,\, k+1, \ldots, m$ $(m-k \geq 2)$:

$$S_{\mathrm{I}} = \sum_{n=k}^{m} A_n(\xi) H_n^{(N)}(\xi). \tag{21}$$

It is shown below that it is always possible to find coefficients $A_n(\xi)$, $n = k,\, k+1, \ldots, m$, such that $S_{\mathrm{I}} = 0$. For the explicit expression of these A-coefficients, see equation (26).

(II) Sum rules involving a RHP and its first m derivatives $(m \geq 2)$, $[H_n^{(N)}(\xi)]^{(j)}$, $j = 0, 1, \ldots, m$:

$$S_{\mathrm{II}} = \sum_{j=0}^{m} B_j(\xi) \frac{d^j}{d\xi^j} [H_n^{(N)}(\xi)]. \tag{22}$$

It is shown below that it is always possible to find coefficients $B_j(\xi)$, $j = 0, 1, \ldots, m$, such that $S_{\mathrm{II}} = 0$. For the explicit expression of these B-coefficients, see equation (27).

Let us begin with the sum rule **I**. Our aim is to find the A-coefficients of equation (21) so that $S_{\mathrm{I}} = 0$. Using the integral representation of the RHP $H_n^{(N)}(\xi)$ given by equation (16), the above sum becomes, after some manipulations:

$$S_{\mathrm{I}} = \frac{\sigma(\xi)^N}{2\pi i} \int_{\Gamma} \frac{1}{\sigma(s)^N (s-\xi)^{m+1}} P(s,\xi)\, ds, \tag{23}$$

where Γ is a closed contour in the complex plane encircling the point $s = \xi$ such that σ has no zeros inside it, and

$$P(s,\xi) = \sum_{n=0}^{m-k} (-1)^{m-n} (m-n)!\, A_{m-n}(\xi) \sigma(\xi)^{m-n} (s-\xi)^n.$$

Notice that $P(s,\xi)$ is a polynomial in the variable s of degree $m-k \geq 2$.

Obviously, if we choose $P(s,\xi) = 0$ then $S_{\mathrm{I}} = 0$, but this choice implies $A_n(\xi) = 0$, $n = k,\, k+1, \ldots, m$, which is the trivial case we are not interested in. However, let $Q(s,\xi)$ be an arbitrary function of ξ (smooth enough) such that it is a polynomial of degree $m-k-2$ $(\equiv \deg P - 2)$ in the variable s. Then, it is always possible to choose the A-coefficients and the function $Q(s,\xi)$ in such a way that the integrand in equation (23) can be expressed as

$$\frac{1}{\sigma(s)^N (s-\xi)^{m+1}} P(s,\xi) = \frac{d}{ds}\left[\frac{\sigma(s)^{1-N}}{(s-\xi)^m} Q(s,\xi)\right]. \tag{24}$$

If this relation holds, it is then clear that it implies $S_{\mathrm{I}} = 0$ since the contour Γ (see equation (23)) is a closed one.

In order to prove that $A_n(\xi)$ and Q can always be chosen so that equation (24) holds true, the only step needed is to perform the derivative in its right hand side. This yields

$$P(s,\xi) = \left\{\frac{2(1-N)}{N} s(s-\xi) - m\sigma(s)\right\} Q(s,\xi) + (s-\xi)\sigma(s)\frac{d}{ds}[Q(s,\xi)], \tag{25}$$

which shows that, in fact, Q must be a polynomial in s of degree two less than the degree of P.

On the other hand, considering equation (25) as a function of s and expanding both sides of it in powers of $(s-\xi)$ the following relation is obtained:

$$\begin{aligned}
&\sum_{n=0}^{m-k} (-1)^{m-n}(m-n)!\,A_{m-n}(\xi)\sigma(\xi)^{m-n}(s-\xi)^n \\
&\quad = \sum_{n=2}^{m-k} \frac{n-m-2N}{(n-2)!\,N} \frac{d^{n-2}}{ds^{n-2}}[Q(s,\xi)]\Big|_{s=\xi}(s-\xi)^n \\
&\quad + \sum_{n=1}^{m-k-1} \frac{2(n-m-N)}{(n-1)!\,N}\,\xi\,\frac{d^{n-1}}{ds^{n-1}}[Q(s,\xi)]\Big|_{s=\xi}(s-\xi)^n \\
&\quad + \sum_{n=0}^{m-k-2} \frac{(n-m)}{n!}\sigma(\xi)\frac{d^n}{ds^n}[Q(s,\xi)]\Big|_{s=\xi}(s-\xi)^n.
\end{aligned}$$

Then, equating the coefficients of the powers of $(s-\xi)$ in this latter expression, a linear system is obtained, which contains $m-k+1 \geq 3$ equations and $2(m-k)$ unknowns ($m-k+1$ A-coefficients and $m-k-1$ derivatives of Q). Thus, $m-k-1$ unknowns can be chosen arbitrarily (e.g. the derivatives of Q) and the remaining $m-k+1$ are determined in terms of them. Using the notation

$$Q_n \equiv Q_n(\xi) = \frac{d^n}{ds^n}[Q(s,\xi)]\Big|_{s=\xi}; \quad (Q_0 \equiv Q_0(\xi) = Q(\xi,\xi)),$$

the following expressions for $A_n(\xi)$ $(n = k, k+1, \ldots, m)$ are obtained:

$$A_m(\xi) = \frac{(-1)^{m+1}}{(m-1)!\,\sigma(\xi)^{m-1}} Q_0,$$

$$A_{m-1}(\xi) = \frac{(-1)^{m-1}}{(m-1)!\,\sigma(\xi)^{m-1}}\left\{(1-m)\sigma(\xi)Q_1 + \frac{2(1-m-N)}{N}\xi Q_0\right\},$$

$$A_{m-n}(\xi) = \frac{(-1)^{m-n}}{(m-n)!\sigma(\xi)^{m-n}} \left\{ \frac{(n-m)\sigma(\xi)}{n!} Q_n + \frac{2(n-m-N)}{N(n-1)!} \xi Q_{n-1} \right.$$

$$\left. + \frac{n-m-2N}{N(n-2)!} Q_{n-2} \right\}, \quad (2 \le n \le m-k-2),$$

$$A_{k+1}(\xi) = \frac{(-1)^{k+2}}{(k+1)!\sigma(\xi)^{k+1}} \left\{ \frac{2(N+k+1)}{N(m-k-2)!} \xi Q_{m-k-2} + \frac{2N+k+1}{N(m-k-3)!} Q_{m-k-3} \right\},$$

$$A_k(\xi) = \frac{(-1)^{k+1}}{k!\sigma(\xi)^k} \left\{ \frac{2N+k}{N(m-k-2)!} Q_{m-k-2} \right\}. \tag{26}$$

Here, as stated above, $\{Q_n, n = 0, 1, \ldots, m-k-2\}$ can be chosen arbitrarily and each choice uniquely determines the function $Q(s,\xi)$, which is the corresponding Taylor interpolating polynomial in the variable s. For example, putting

$$\frac{Q_n}{n!} = (m-n)!\sigma(\xi)^{m-n-1},$$

a sum rule of the first kind (see equation (21))

$$S_{\mathrm{I}} = \sum_{n=k}^{m} A_n(\xi) H_n^{(N)}(\xi) = 0 \quad (m-k \ge 2),$$

is obtained, whose A-coefficients are polynomials in ξ of degree at most two, as can be easily deduced from equation (26). On the other hand, it should be remarked that when $m = k+2$, this sum rule gives rise to the already known (cf. [2, Eq. (7)]) three-term recurrence relation satisfied by the RHP, which is a particular case of the general result we have given here. Notice that when $m = k+2$, the degree of P is exactly two, so Q does not depend on s. It means that the only arbitrary coefficient appearing in the three-term recurrence relation is Q_0 which multiplies the whole relation. From this fact one can conclude that, except for a common factor, there exists only one three-term recurrence relation involving three RHP of consecutive degrees. Of course, many others sum rules of this kind could be obtained for different choices of the functions Q_n and the parameters m and k.

The method we have just described can be applied in a similar way to obtain the sum rules S_{II} given by equation (22). In this case we try to obtain the B-coefficients in equation (22) so that $S_{\mathrm{II}} = 0$. In doing so, we rewrite this sum by using the integral representations for the RHP and their derivatives given by equations (16), (18). The resulting expression is

$$S_{\mathrm{II}} = \frac{(-1)^n n! \sigma(\xi)^{N+n}}{2\pi i} \int_{\Gamma} \frac{1}{\sigma(s)^N (s-\xi)^{n+1}} P(s,\xi)\, ds,$$

where Γ is the same contour as in the previous case and $P(s,\xi)$ is a polynomial in s

of degree $m \geq 2$ given by

$$P(s,\xi) = \sum_{j=0}^{m} \frac{(-1)^j (2N+n-j)_j}{N^j \sigma(\xi)^j} B_j(\xi)(s-\xi)^j.$$

Here the well-known Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$ has been used.

At this point, the same argument we have just described for the sums of the first kind allows to conclude that the B-coefficients can always be chosen so that $S_{\mathrm{II}} = 0$. They are given by:

$$\begin{aligned}
B_0(\xi) &= -n\sigma(\xi)Q_0, \\
B_1(\xi) &= -\frac{N\sigma(\xi)}{(2N+n-1)}\left\{(1-n)\sigma(\xi)Q_1 + \frac{2(1-n-N)}{N}\xi Q_0\right\}, \\
B_j(\xi) &= \frac{(-1)^j N^j \sigma(\xi)^j}{(2N+n-j)_j}\left\{\frac{j-n}{j!}\sigma(\xi)Q_j + \frac{2(j-n-N)}{N(j-1)!}\xi Q_{j-1}\right. \\
&\quad \left. + \frac{j-n-2N}{N(j-2)!}Q_{j-2}\right\}, \quad (2 \leq j \leq m-2), \\
B_{m-1}(\xi) &= \frac{(-1)^{m-1}N^{m-1}\sigma(\xi)^{m-1}}{(2N+n-m+1)_{m-1}}\left\{\frac{m-1-n-2N}{N(m-3)!}Q_{m-3}\right. \\
&\quad \left. + \frac{2(m-1-n-N)}{N(m-2)!}\xi Q_{m-2}\right\}, \\
B_m(\xi) &= \frac{(-1)^m N^m \sigma(\xi)^m}{(2N+n-m)_m}\frac{(m-n-2N)}{N(m-2)!}Q_{m-2}.
\end{aligned} \tag{27}$$

Here, the coefficients Q_i $(i = 0,\ldots,m-1)$ can be chosen arbitrarily and the same notation as in equation (26) has been used.

For instance, the choice

$$Q_n = \frac{(2N+n-1)_j}{\sigma(\xi)^{j+1}},$$

gives rise to a sum rule

$$S_{\mathrm{II}} = \sum_{j=0}^{m} B_j(\xi)\frac{d^j}{d\xi^j}[H_n^{(N)}(\xi)] = 0 \quad (m \geq 2),$$

whose B-coefficients are polynomials in ξ of degree at most two, as can be deduced by taking into account equation (27). Moreover, it is interesting to remark that when $m = 2$, the above sum rule becomes the second order differential equation satisfied by RHP (already given in equation (2)). In fact, this sum rule is, for each value of m, an mth order linear differential equation satisfied by the RHP.

For completeness, let us mention that other sum rules can be obtained by means of the Hansen techniques [10] or its generalizations [3] which start from three-term recurrence relations. Indeed, these two authors have been able to compute sum rules similar to the sum of the first kind (equation (21)) for sets $\{f_n\}_{n\in\mathbb{N}}$ of functions which satisfy a three-term recurrence relation of the form

$$a_n(x)f_n + b_n(x)f_{n+1} + c_n(x)f_{n+2} = 0.$$

Of course, the method used to compute the sums from this relation is completely different from the one presented here, which uses as a starting point the second order differential equation given in equation (2).

Finally, it should be mentioned that the approach which has been presented here is being generalized [21] in order to obtain general sum rules and recurrence formulas for functions of hypergeometric type (not necessarily polynomials) and their derivatives, which can be computed in terms of the coefficients of the differential equation that they satisfy. These types of functions are defined for being analytic solutions of a linear second order differential equation of the form

$$\sigma(x)y'' + \tau(x;\lambda)y' + \lambda y = 0,$$

where σ is a polynomial of degree at most two, τ is also a polynomial in x of degree at most one and λ is a constant.

4. Zeros of the relativistic Hermite polynomials

Here, we will study the distribution of zeros of the RHP. As clearly shown by equation (1), these zeros describe the nodes of the wave functions of the quantum relativistic harmonic oscillator.

First of all we prove the following result.

Lemma

All zeros of $H_n^{(N)}(\xi)$ $(N > 1/2)$ are real and simple. Moreover, no two consecutive, RHP $H_n^{(N)}(\xi)$ and $H_{n+1}^{(N)}(\xi)$ have a zero in common.

This lemma can be proved in various ways. Here we will consider that which makes use of the Rodrigues formula given by equation (15). Let us consider the function $g(\xi)$ defined by

$$g(\xi) = \sigma(\xi)^{-(N+n)}H_n^{(N)}(\xi).$$

The use of equation (15) allows us to write its derivative in the form

$$\frac{d}{d\xi}g(\xi) = -\sigma(\xi)^{-(N+n+1)}H_{n+1}^{(N)}(\xi).$$

We now assume that all zeros of the nth degree RHP (and hence of the function $g(\xi)$) are real and simple. Under this assumption we can denote them by $\xi_{1,n} < \xi_{2,n} < \cdots < \xi_{n,n}$. Then, the Rolle theorem from elementary calculus applied to $g(\xi)$ allows us to conclude that $H_{n+1}^{(N)}(\xi)$ has at least one root between two consecutive zeros of $H_n^{(N)}(\xi)$. Moreover, the function $g(\xi)$ satisfies the limiting conditions

$$\lim_{\xi\to\pm\infty} g(\xi) = 0.$$

So, the same aforementioned theorem tells us that $H_{n+1}^{(N)}(\xi)$ has two more real zeros, one of them lying in $(-\infty, \xi_{1,n})$ and the other in $(\xi_{n,n}, \infty)$. In this way we have proved that if $H_n^{(N)}(\xi)$ has n real and simple zeros then $H_{n+1}^{(N)}(\xi)$ must have $n+1$ roots which are also real and simple.

The first statement of the lemma follows from the fact that $H_1^{(N)}(\xi)$ has a simple and real root and the second is a straight consequence of the first one and the differential-difference relation

$$\frac{d}{d\xi} H_n^{(N)}(\xi) = \frac{n}{N}(2N+n-1)H_{n-1}^{(N)}(\xi),$$

which, as shown in [2, equation (8)], is satisfied by the RHP.

We are now in a position to study the distribution of zeros of the RHP, defined as

$$\Omega_n(\xi) = \frac{1}{n}\sum_{k=1}^{n} \delta(\xi - \xi_{k,n}), \tag{28}$$

where $\xi_{k,n}$, $k = 1, 2, \ldots, n$, are the zeros of $H_n^{(N)}(\xi)$. Here we will characterize this distribution by means of its moments $\mu_r^{(n)}$, $r = 0, 1, 2, \ldots$, or, equivalently, in terms of the Newton sum rules of the zeros $N_r^{(n)}$, $r = 0, 1, 2, \ldots$, i.e.

$$\mu_r^{(n)} = \frac{1}{n} N_r^{(n)} = \frac{1}{n}\sum_{k=1}^{n} \xi^r \delta(\xi - \xi_k) = \frac{1}{n}\sum_{k=1}^{n} (\xi_{k,n})^r. \tag{29}$$

In the last few years several methods (see e.g. [4,5,8,9,18,20]) have been developed to obtain these spectral quantities for an arbitrary polynomial. In particular, it is possible to compute them in an exact and recurrent way starting either from the explicit expression of the polynomials or from the differential equation satisfied by them (if any). Briefly, the two corresponding procedures are as follows:

(a) *Moments from the explicit expression*

Let

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C_{n,k} x^{n-k}; \quad C_{n,0} = 1,$$

be the monic explicit expression of an nth degree polynomial. Then [8], the Newton sum rules of their zeros can be expressed in terms of Bell polynomials Y_r [15] as follows

$$N_r^{(n)} = -\frac{1}{(r-1)!} Y_r(f_1 g_1, \ldots, f_r g_r),$$

where $f_i = (-1)^{i-1}(i-1)!$ and $g_i = (-1)^i C_{n,i}$. This expression, together with the recurrence relation satisfied by the Bell polynomials [15], allows us to write the following recurrence for the Newton sum rules [8]

$$N_r^{(n)} = (-1)^{r+1}\left\{ rC_{n,r} + \sum_{j=1}^{r-1} (-1)^j C_{n,r-j} N_j^{(n)} \right\}, \tag{30}$$

$$N_0^{(n)} = n, \quad N_1^{(n)} = C_{n,1}.$$

(b) *Moments from the second order differential equation*

Let $P_n(x)$ be an nth degree polynomial solution of the differential equation

$$g_2(x)P_n''(x) + g_1(x)P_n'(x) + g_0(x)P_n(x) = 0, \tag{31}$$

where $g_i(x)$, $i = 0, 1, 2$, are polynomials of degree c_i defined by

$$g_i(x) = \sum_{j=0}^{c_i} a_j^{(i)} x^j, \quad i = 0, 1, 2, \tag{32}$$

with constant coefficients $a_j^{(i)}$. Assuming that the zeros of $P_n(x)$ are simple, the following relation is fulfilled [4]:

$$2 \sum_{m=-1}^{r+c_2-3} a_{m+3-r}^{(2)} J_{m+2} = -\sum_{j=0}^{c_1} a_j^{(1)} N_{r+j-1}^{(n)}, \quad r = 1, 2, 3, \ldots, \tag{33}$$

where the J-quantities are

$$J_k = \sum_{l_1 \neq l_2} \frac{x_{l_1,n}^k}{x_{l_1,n} - x_{l_2,n}} = \begin{cases} 0, & \text{if } k = 0, \\ (1/2)n(n-1), & \text{if } k = 1, \\ (n-1)N_1^{(n)}, & \text{if } k = 2, \\ \left(n - \dfrac{k}{2}\right) N_{k-1}^{(n)} + \dfrac{1}{2} \displaystyle\sum_{t=1}^{k-2} N_{k-1-t}^{(n)} N_t^{(n)}, & \text{if } k > 2, \end{cases}$$

$x_{i,n}$, $i = 1, 2, \ldots, n$, being the zeros of $P_n(x)$.

Since both the explicit expression and the second order differential equation of the RHP are known, these two procedures to obtain the spectral moments may be used for these polynomials.

In the first method, the monic polynomials $\tilde{H}_n^{(N)}(\xi)$ are needed. From equation (3) it follows that

$$\tilde{H}_n^{(N)}(\xi) = \sum_{j=0}^{n} (-1)^j \tilde{A}_{n,j}^{(N)} \xi^{n-j},$$

where

$$\tilde{A}_{n,j}^{(N)} = \begin{cases} 0, & j \text{ odd}, \\ \dfrac{(-N)^{j/2} n! (N - \frac{1}{2})!}{2^j (j/2)! (n-j)! (N + (j-1)/2)!}, & j \text{ even}. \end{cases}$$

Then equation (36) gives

$$\left.\begin{aligned} \mu_{2s-3}^{(n)} &= 0 \\ \mu_{2s}^{(n)} &= -\left\{\frac{2s}{n}\tilde{A}_{n,2s}^{(N)} + \sum_{j=1}^{s-1}\tilde{A}_{n,2(s-j)}^{(N)}\mu_{2j}^{(n)}\right\} \end{aligned}\right\} \quad s = 2,3,\ldots, \tag{34}$$

for the moments $\mu_r^{(n)} = (1/n)N_r^{(n)}$ of the distribution of zeros of the RHP, where the initial conditions are

$$\mu_0^{(n)} = 1; \quad \mu_2^{(n)} = \frac{N(n-1)}{2N+1}.$$

Notice that the odd moments vanish accordingly to the fact that the RHP of even degree are even functions and those of odd degree are odd functions. In particular, this implies that the zero distribution of the RHP (see equation (28)) has to be symmetric.

On the other hand, since the zeros of the RHP are simple, the recurrence given by equation (33) in the second procedure is also valid for them. The comparison between the RHP differential equation, equation (2) and equations (31)–(32), together with the use of equation (33), allow us to find, after some manipulations, the following recurrence relationship also satisfied by the spectral moments of an nth degree RHP

$$\mu_{2s+1} = 0, \quad s = 0,1,\ldots,$$

$$\begin{aligned} (2N+2k+1)\mu_{2(k+1)}^{(n)} &= N(2n-2k-1)\mu_{2k}^{(n)} + nN\sum_{p=1}^{k-1}\mu_{2(k-p)}^{(n)}\mu_{2p}^{(n)} \\ &\quad + n\sum_{p=1}^{k}\mu_{2(k+1-p)}^{(n)}\mu_{2p}^{(n)}, \quad k = 1,2,3,\ldots, \end{aligned} \tag{35}$$

where the initial conditions are the same as in the previous relation, and the first summation on the right hand side is taken to be zero when $k = 1$.

The solutions of the two recurrences given by equations (34)–(35) are, of course, the same. The first few even moments are

$$\mu_0^{(n)} = 1, \quad \mu_2^{(n)} = \frac{N(n-1)}{2N+1},$$

$$\mu_4^{(n)} = \frac{(n-1)N^2(n^2+n-3+2(2n-3)N)}{(1+2N)^2(3+2N)},$$

$$\begin{aligned} \mu_6^{(n)} &= \{(1+2N)^3(3+2N)(5+2N)\}^{-1}\{(n-1)N^3[15+2(n^2-4)n(n+1) \\ &\quad + (60+2n(2n-5)(3n+5))N + (60+4n(5n-17))N^2]\}, \end{aligned}$$

$$\begin{aligned}\mu_8^{(n)} = {} & \{(1+2N)^4(3+2N)^2(5+2N)(7+2N)\}^{-1} \\ & \times \{(n-1)N^4[n(n+1)(213-95n^2+17n^4) \\ & -315+(-2100+1936n+704n^2-834n^3-162n^4+146n^5+10n^6)N \\ & +(-5040+5808n-344n^2-1536n^3+312n^4+80n^5)N^2 \\ & +(-5040+6816n-2432n^2-288n^3+224n^4)N^3 \\ & +(-1680+2480n-1264n^2+224n^3)N^4]\}.\end{aligned}$$

It is easy to check that in the limit when N tends to infinity, these moments become the already known [5,18] moments of the zero distribution corresponding to the classical Hermite polynomials. The expressions of the moments show clearly the heavy calculations involved in their obtention. In this sense, it could be interesting to mention that the help of computer algebra systems has been very useful. In particular, the two algorithms considered here have allowed to construct two built-in *Mathematica* [17] programs (see [9,18,20]) which are able to calculate these moments.

On the other hand, though the above moments completely characterize the corresponding distribution [16], to obtain it from them becomes impossible in this case. Anyway, the moments themselves give us valuable information about the distribution. To illustrate this point, let us consider some related statistical parameters [11]. Since the zero distribution of the RHP is symmetric with respect to the origin, the mean and the skewness are zero. The variance, defined as

$$v_n(N) = \sqrt{\mu_2^{(n)}} = \left\{\frac{N(n-1)}{2N+1}\right\}^{1/2},$$

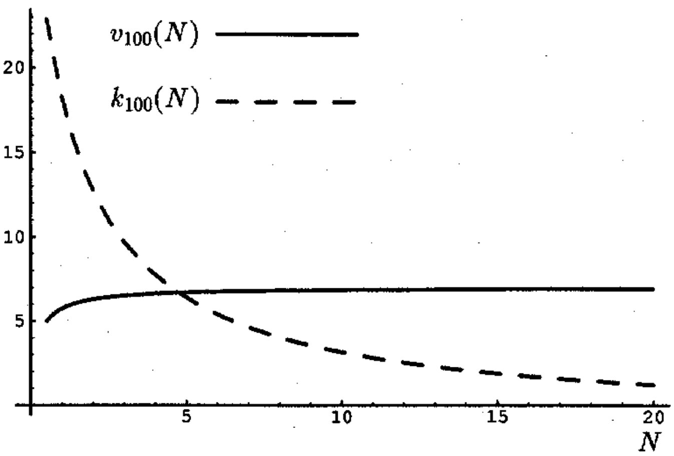

Figure 1. Variance ($v_{100}(N)$) and kurtosis ($k_{100}(N)$) of the zero distribution of the RHP of degree 100 in terms of the relativistic parameter N (varying from 1/2 to 20).

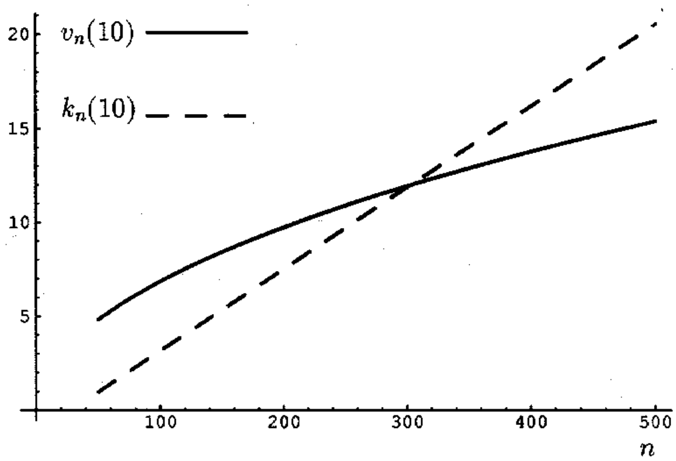

Figure 2. Variance ($v_n(10)$) and kurtosis ($k_n(10)$) of the zero distribution of the RHP with relativistic parameter $N = 10$ in terms of the principal quantum number n (varying from 50 to 500).

gives us an idea of the spread of the distribution. The kurtosis, given by

$$k_n(N) = \frac{\mu_4^{(n)}}{(\mu_2^{(n)})^2} - 3 = \frac{n^2 - 2n(N+4) + 6}{(2N+3)(n-1)},$$

gives us information about the qualitative shape of the distribution around its maximum.

Figure 1 shows the behaviour of these two parameters in the transition from the ultrarelativistic limit ($N = 1/2$) to the non-relativistic one for a fixed degree ($n = 100$), i.e. for a fixed quantum principal number. Notice that for the lowest values of N, the variance increases rapidly until it reaches an almost stationary value, while the kurtosis decreases very fast for those lowest values and tends to cross the N-axis; this happens for $N = 46.03$. From this behaviour one can conclude that the lower is N, the more peaked is the zero distribution around its maximum ($\mu_1^{(n)} = 0$). Moreover, the distribution is taller and thinner than the Gaussian one for $N < 46.03$, it tends to behave as this distribution when $N = 46.03$ and for $N > 46.03$ it becomes more flat-top than the Gaussian.

In figure 2 the behaviour of the variance and kurtosis in terms of the quantum principal number n is shown for fixed N. Since both parameters are increasing functions of n, one can conclude that the higher is the energy of a state, the more peaked is the zero distribution.

Of course, more information could be obtained by taking into account moments of higher order. Finally, let us mention that the zero distribution of the relativistic Hermite polynomials has been also studied in [19] and in [12] where a generalization of the RHP is considered. In these two works the so-called WKB method is used to obtain an approximation for the distribution of zeros defined in equation (28). Moreover, it is shown that this approximation gives the correct asymptotic limit (i.e. the limit when n tends to infinity) which is explicitly calculated.

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