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Categorías de funtores continuos para la dinámica continua y discreta

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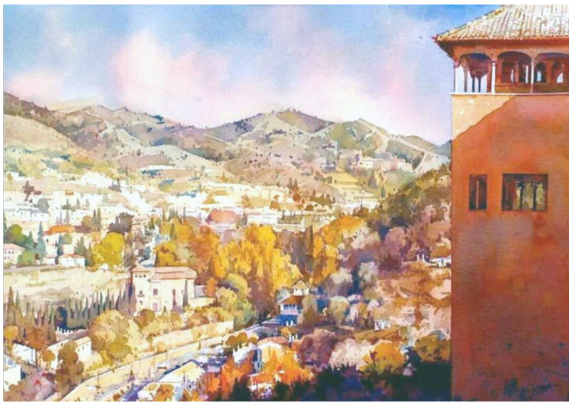
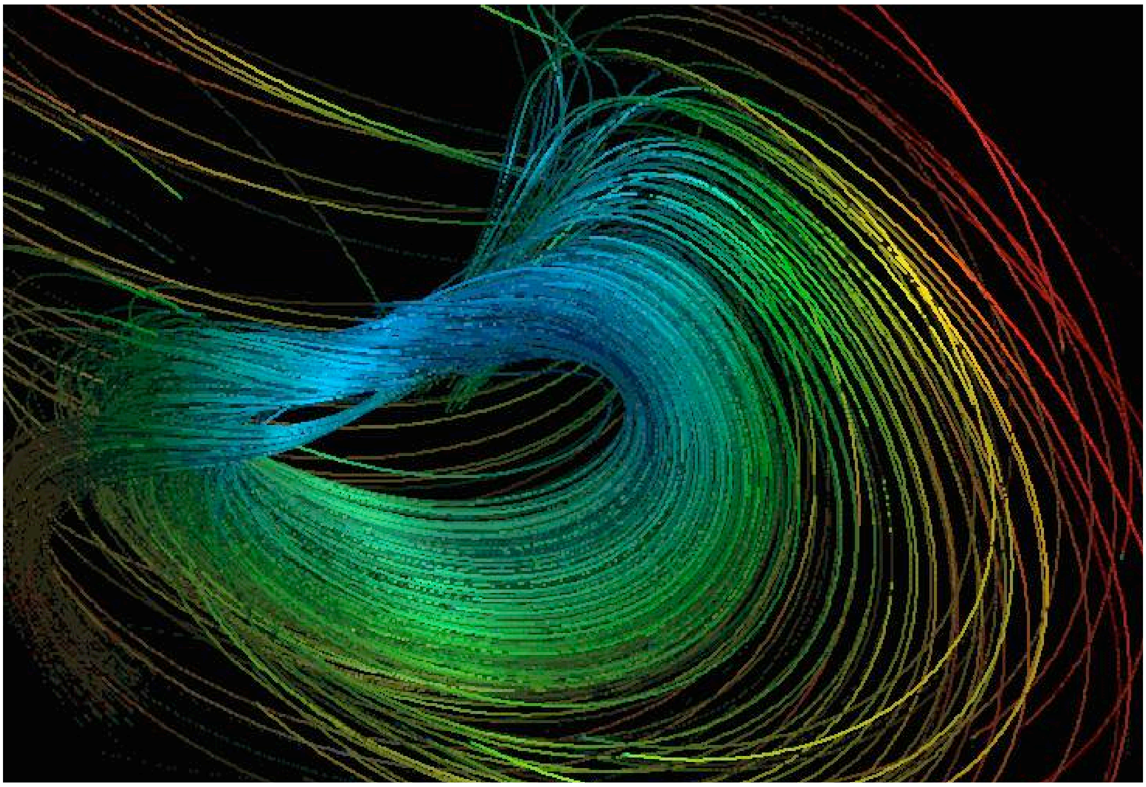


The following images and pictures suggest mathematical objects that appears in some contexts, for instance, as the phase space of all the solutions of a differential equation.

One of the objetives of this talk is to find some categorical models which can be used to represent dynamical continuous and discrete systems.



Dynamical systems, topological monoids and categories of continuous functors
A tensor product for small **Top**-categories
Prolongations, suspensions, telescopes and tubes

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Dynamical systems, topological monoids and categories of continuous functors

In this section we analyze:

- an important source of dynamical systems: Differential equations

- some discretization and anti-discretization constructions

- categorical models for the phase spaces

Differential equations and dynamical systems

Given a differentiable compact manifold M and a differentiable vector field $X\colon M \to TM$ a curve $\gamma\colon (a,b) \to M$, $s \to \gamma(s), s \in (a,b)$ is said to be an integral curve of X if it satisfies that

$$\left.\frac{d\gamma}{dt}\right|_s = T_s\gamma(\left.\frac{d}{dt}\right|_s) = X_{\gamma(s)}$$

The existence and unicity theorem for differential equations and the compactness of M ensure that for each $p \in M$, there is a unique $\gamma^p\colon \mathbb{R} \to M$ such that $\gamma^p(0) = p$ and γ^p is an integral curve of X. Consequently, we also have that the map

$$\varphi\colon \mathbb{R} \times M \to M, \quad \varphi(t,p) = \gamma^p(t)$$

is continuous (differentiable) and φ also satisfies that

$$\varphi(0,p) = p, \quad \varphi(t, \varphi(s,p)) = \varphi(t+s, p)$$

Then, (M, φ) is a continuous flow and the family of homeomorphims $\varphi_t\colon M \to M$, $t \in \mathbb{R}$ is said to be a uniparametric group of homeomorphims.

When X is only continuous we can apply the existence Peano theorem, but in general one can not ensure uniqueness under usual initial condition.

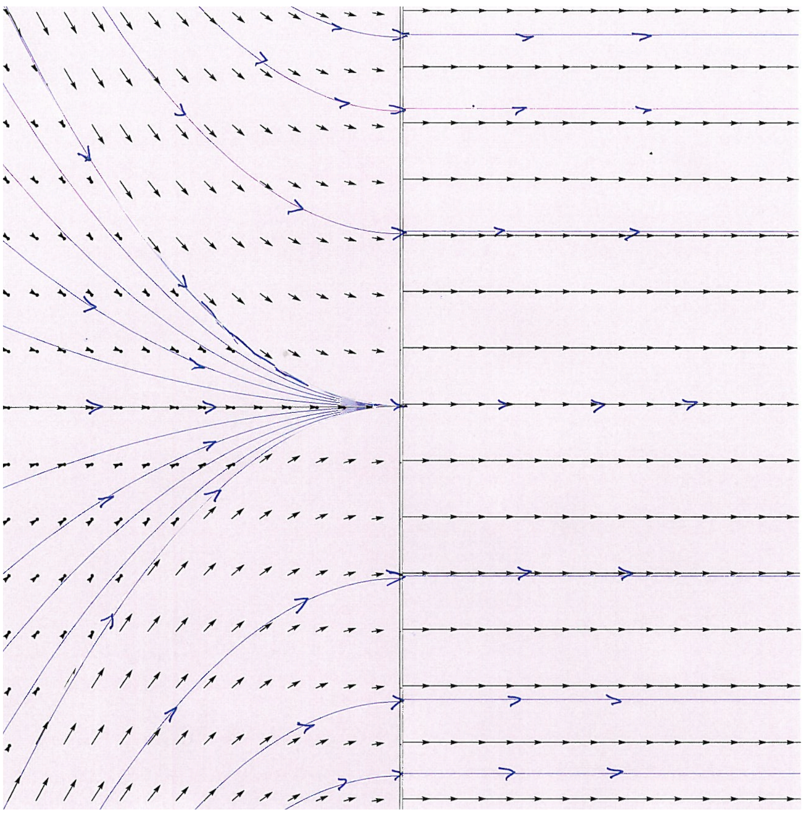
There is an important class of continuos vector fields having the uniqueness condition in forward time, but not in backward time. Some results in this direction can be seen in the monograph of Agarwal and Lakshmikantham [1] or in the book of Filippov [10].

In these cases, we only have a continuous map

$$\varphi\colon \mathbb{R}_+ \times M \to M, \quad \varphi(t,p) = \gamma^p(t), t \geq 0$$

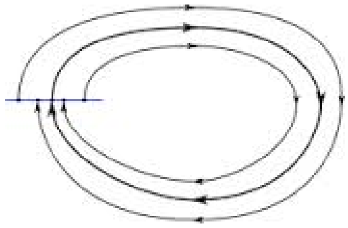
which is called a semi-flow. It is important to note that for semi-flows for each $t \in \mathbb{R}_+$, the continuous map $\varphi_t\colon M \to M$, $t \geq 0$ need not be a homeomorphism.

When the action is given by $\mathbb{Z}$ one has a discrete flow and if we consider actions of $\mathbb{N}$, we have a discrete semi-flow.



The techniques for solving differential equations based on numerical approximations are a useful tool in many scientific contexts. These discretization methods and the corresponding interpolations give a description of a complete trajectory associated to a given initial condition, which solution of a differential equation.

One of the methods introduced by Poincaré [21, 22] was the first return map: if we consider a periodic solution (a compact curve) of a differential equation on $\mathbb{R}^2$ and its initial point x_0, we can take a tranversal curve T at x_0 and consider an induced function $T \to T$ such that a point $x \in T$ is applied (when it is possible) to the first return point in T in forward time. In this way a discrete dynamical system is associated with the compact curve of the continuous dynamical system. The suspension method is the corresponding inverse technique that reconstructs (at least locally) a continuous dynamical system from a discrete system.



Top-categories as weakly enriched categories

The category of topological spaces **Top** is a cartesian monoidal category taking as "tensor" the usual product and as identity object the terminal topological space with one point.

A category enriched over **Top** is a category $\mathcal{A}$ such that for each pair of objects A, B, $\mathcal{A}(A, B)$ can also be considered as an object in **Top** and we also have for each object A of $\mathcal{A}$ a canonical continuous map $\{*\} \to \mathcal{A}(A, A), * \to \mathrm{id}_A$ and a canonical composition low

$$\mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$

which is a continuous map (moreover, some canonical diagrams have to be commutatives.)

Remark: **Top** is not a category enriched over **Top**

In our case, we work with a structure of this type but the map

$$\mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$

in general is not continuous.

If we reduce the continuity condition to the separately continuous condition, we has the notion of **Top**-category (It also could be called quasi **Top**-category).

Remark: **Top** is a **Top**-category

If $\mathbf{A}$ is a category, we denote by $\mathbf{A}_0$ the set of objects of $\mathbf{A}$ and by $\mathbf{A}_1$ the class of morphisms. Given two objects $X, Y \in \mathbf{A}_0$ the set of morphisms from X to Y is denoted by $\mathrm{Hom}_{\mathbf{A}}(X, Y)$. Finally, $\mathbf{A}^{op}$ is the opposite category.

Definition

A category $\mathbf{A}$ is said to be a **Top**-category if, for each pair of objects $X, Y \in \mathbf{A}_0$, the set of morphisms from X to Y is also provided with a topology which determines a topological space $\mathbf{A}(X, Y)$ and such that for $X, Y, Z \in \mathbf{A}_0$:

(i) if $f \in \mathbf{A}(X, Y)$, then $f^* \colon \mathbf{A}(Y, Z) \to \mathbf{A}(X, Z)$, $f^*(g) = gf$ is continuous,

(ii) if $g \in \mathbf{A}(Y, Z)$, then $g_* \colon \mathbf{A}(X, Y) \to \mathbf{A}(X, Z)$, $g_*(f) = gf$ is continuous.

If for every $X, Y, Z \in \mathbf{A}_0$ the composition

$$\mathbf{A}(Y, Z) \times \mathbf{A}(X, Y) \to \mathbf{A}(X, Z)$$

is continuous, the category $\mathbf{A}$ is said to be a **cc-Top**-category.

Definition

Given two **Top**-categories $\mathbf{A}, \mathbf{B}$, a functor $F \colon \mathbf{A} \to \mathbf{B}$ is said to be a **Top**-functor if, for every pair of objects $X, Y \in \mathbf{A}_0$, the map $\mathbf{A}(X, Y) \to \mathbf{B}(F(X), F(Y))$ is continuous. The category of **Top**-functors from $\mathbf{A}$ to $\mathbf{B}$, is denoted by $\mathbf{B}^{\mathbf{A}}$.

Definition

Let $\mathbf{A}$ be a small **Top**-category. A **Top**-functor $X \colon \mathbf{A} \to \mathbf{Top}$ is said to be continuous if, for every $a, a' \in \mathbf{A}_0$, the canonical map $\mathbf{A}(a, a') \times X_a \to X_{a'}$, $(\beta, x) \to X(\beta)(x)$, is continuous. The category of continuous functors from $\mathbf{A}$ to **Top** will be denoted by $(\mathbf{Top}^{\mathbf{A}})_{\mathbf{c}}$.

Some small **Top**-categories and categories of continuous **Top**-functors

Given a topological monoid M one can construct the following small categories: The category $\mathbf{M}$ has one object $*$ and as set of morphisms $\mathrm{Hom}_{\mathbf{M}}(*,*) = M$ and the composition is induced by the monoid product. The small category $\mathbf{M}$ is considered as a **Top**-category taking as hom-space of morphisms the topological monoid $\mathbf{M}(*,*) = M$.

Note: If the product $M \times M \to M$ is separately continuous, then $\mathbf{M}$ has also the structure of a **Top**-category.

The small **Top**-category $\tilde{\mathbf{M}}$ is the coslice category $*/\mathbf{M}$, which has as objects the set M and for two given objects $m, m' \in (*/\mathbf{M})_0$

$$\tilde{\mathbf{M}}(m,m') = \{n \in M | nm = m'\}$$

provided with the subspace topology.

Note that one has a canonical **Top**-functor

$$\tilde{\mathbf{M}} \to \mathbf{M}$$

which carries a morphism $n\colon m \to m'$ to $n : * \to *$.

It will be also interesting to consider the following modification when we have a topological submonoid $S \subset M$:

The small **Top**-category $\tilde{\tilde{\mathbf{M}}}$, which is a subcategory of the coslice category $\tilde{\mathbf{M}} = */\mathbf{M}$, has as objects the set M and for two given objects $m, m' \in (\tilde{\tilde{\mathbf{M}}})_0$

$$\tilde{\tilde{\mathbf{M}}}(m,m') = \{n \in S | nm = m'\}$$

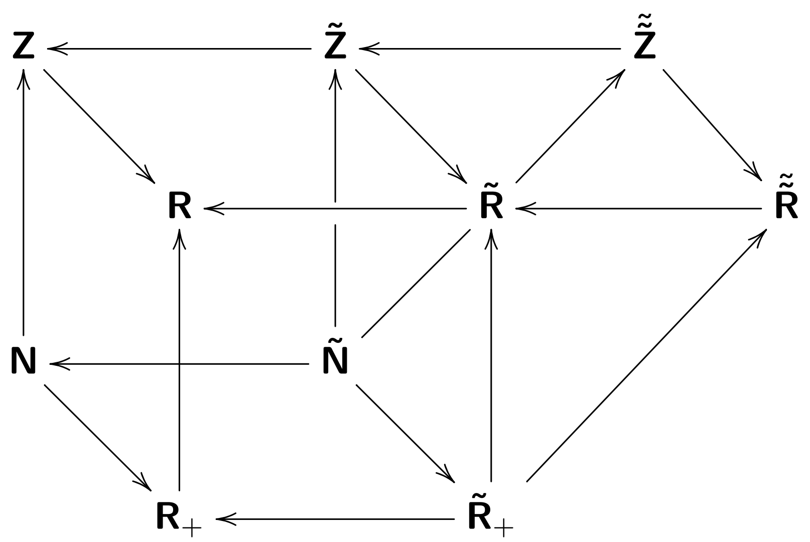
provided with the subspace topology.

Note that one has the canonical **Top**-functor given by the inclusion

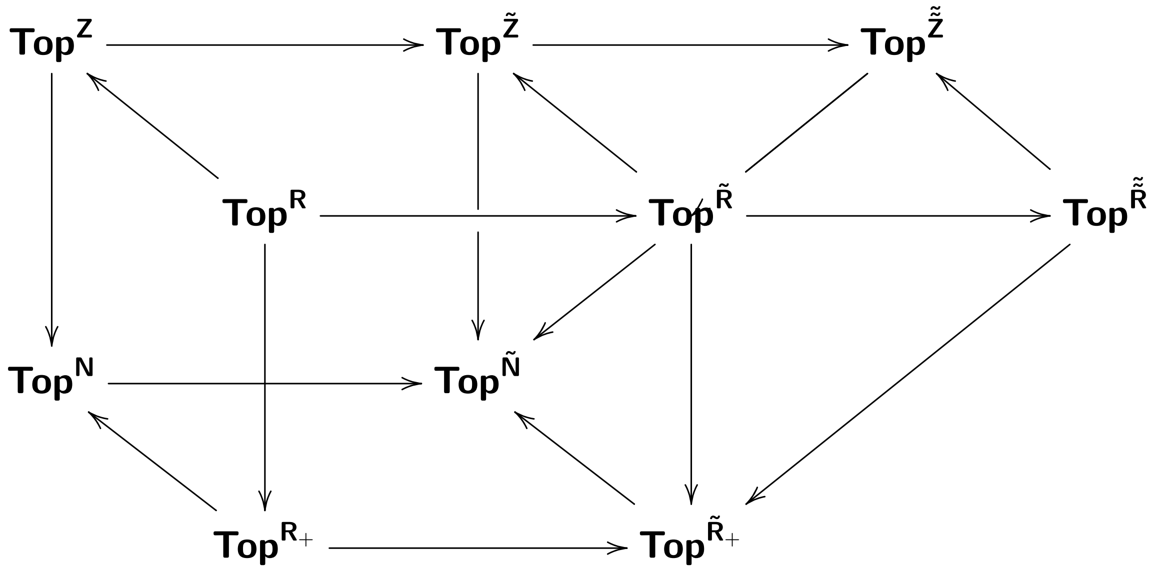
$$\tilde{\tilde{\mathbf{M}}} \to \tilde{\mathbf{M}}.$$

The addition gives a topological monoid structure to $\mathbb{R}$ and $\mathbb{R}_+, \mathbb{Z}, \mathbb{N}$ are topological submoids. Associated with the monoid inclusions $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{R}_+ \subset \mathbb{R}$, one has the categories $\tilde{\tilde{\mathbf{Z}}}$, $\tilde{\tilde{\mathbf{R}}}$, respectively.

Then, we have the following commutative diagram of small **Top**-categories and canonical **Top**-functors:



and the induced commutative diagram of categories:



Remark

We also have a similar diagram if we take continuous functors instead of **Top***-functors. Note that if* **A** *is one of the small* **Top***-categories:* $\mathbf{N}, \mathbf{Z}, \tilde{\mathbf{N}}, \tilde{\mathbf{Z}}, \tilde{\mathbf{R}}_+, \tilde{\mathbf{R}}, \tilde{\tilde{\mathbf{Z}}}, \tilde{\tilde{\mathbf{R}}}$, *then* $\mathbf{Top}^{\mathbf{A}} = (\mathbf{Top}^{\mathbf{A}})_{\mathbf{c}}$.

The categories of the left side of the cube

$$(\mathbf{Top}^{\mathbf{R}})_{\mathbf{c}}, (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$$

are the categories of **continuous flows and semi-flows** and

$$(\mathbf{Top}^{\mathbf{Z}})_{\mathbf{c}} = \mathbf{Top}^{\mathbf{Z}}, (\mathbf{Top})^{\mathbf{N}}_{\mathbf{c}} = \mathbf{Top}^{\mathbf{N}}$$

are the categories of **discrete flows and semi-flows**.
The objects in the categories of the right side are inverse (or direct) systems that will be used to construct anti-discretization and telescopic functors.

Remark

In diagram above, we have not considered the categories $\tilde{\tilde{\mathbf{N}}}$, $\tilde{\tilde{\mathbf{R}}}_+$ *associated to the inclusions* $\mathbb{N} \subset \mathbb{N}$, $\mathbb{R}_+ \subset \mathbb{R}_+$, *respectively, because* $\tilde{\tilde{\mathbf{N}}} = \tilde{\mathbf{N}}$ *and* $\tilde{\tilde{\mathbf{R}}}_+ = \tilde{\mathbf{R}}_+$.

Remark

It is interesting to note that functors from the front to the back side of the diagram can be considered as **discretization** *processes (for example, a continuous trajectory can be discretized by taking a unit of time). From the top to the bottom we have* **restriction** *functors and from the left to the right we have* **repetition** *functors.*

A tensor product for small **Top**-categories

In next sections we study the construction of the left adjoints of these functors which have important and interesting applications to the theory of dynamical systems.

We give a tensor construction such that all the left adjoints will be particular cases of this general construction.

Assume that we have a **Top**-functor of small **Top**-categories

$$u\colon \mathbf{M} \to \mathbf{N}$$

which induces a canonical functor

$$U\colon \mathbf{Top}^{\mathbf{N}} \to \mathbf{Top}^{\mathbf{M}}.$$

Our objective is to construct a left adjoint

$$\mathbf{N} \otimes_{\mathbf{M}} (\cdot)\colon \mathbf{Top}^{\mathbf{M}} \to \mathbf{Top}^{\mathbf{N}}$$

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For a **Top**-functor $X\colon \mathbf{M} \to \mathbf{Top}$, which carries $\alpha\colon m \to m'$ to $X(\alpha)\colon X_m \to X_{m'}$, and an object $n \in \mathbf{N}_0$, we consider the topological space sum:

$$\bigsqcup_{m\in \mathbf{M_0}} (\mathbf{N}(u(m), n) \times X_m)$$

and the following relations:
If $(\beta, x) \in \mathbf{N}(u(m), n) \times X_m$ and $(\beta', x') \in \mathbf{N}(u(m'), n) \times X_{m'}$, $(\beta, x) \sim (\beta', x')$ if there is $\alpha \in \mathbf{M}(m, m')$ such that $\beta = \beta' u(\alpha)$ and $X(\alpha)(x) = x'$.

Now we take the quotient space

$$(\mathbf{N} \otimes_{\mathbf{M}} X)_n = \left(\bigsqcup_{m \in \mathbf{M}_0} (\mathbf{N}(u(m), n) \times X_m) \right) / \sim,$$

and $\beta \otimes x$ will denote the equivalence class represented by (β, x). The induced quotient map, $(\beta, x) \to \beta \otimes x$, will be denoted by

$$\pi_n \colon \bigsqcup_{m \in \mathbf{M}_0} (\mathbf{N}(u(m), n) \times X_m) \to (\mathbf{N} \otimes_{\mathbf{M}} X)_n.$$

If $\mathbf{N}(n, n')$ is locally compact, and $\gamma \in \mathbf{N}(n, n')$, then we also have an induced continuous map $(\mathbf{N} \otimes_{\mathbf{M}} X)(\gamma) \colon (\mathbf{N} \otimes_{\mathbf{M}} X)_n \to (\mathbf{N} \otimes_{\mathbf{M}} X)_{n'}$. Therefore, for a given $X \in \mathbf{Top}^{\mathbf{M}}$, we have constructed the **Top**-functor $\mathbf{N} \otimes_{\mathbf{M}} X \ \in \mathbf{Top}^{\mathbf{N}}$ and one also has an induced functor:

$$\mathbf{N} \otimes_{\mathbf{M}} (\cdot) \colon \mathbf{Top}^{\mathbf{M}} \to \mathbf{Top}^{\mathbf{N}}.$$

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In the construction above we have that the functors

$$U \colon \mathbf{Top}^{\mathbf{N}} \to \mathbf{Top}^{\mathbf{M}}, \quad \mathbf{N} \otimes_{\mathbf{M}} (\cdot) \colon \mathbf{Top}^{\mathbf{M}} \to \mathbf{Top}^{\mathbf{N}}$$

induce new functors

$$U \colon (\mathbf{Top}^{\mathbf{N}})_{\mathbf{c}} \to (\mathbf{Top}^{\mathbf{M}})_{\mathbf{c}}, \quad \mathbf{N} \otimes_{\mathbf{M}} (\cdot) \colon (\mathbf{Top}^{\mathbf{M}})_{\mathbf{c}} \to (\mathbf{Top}^{\mathbf{N}})_{\mathbf{c}}.$$

When no confusion is possible these restrictions will be denoted with the same symbols that the initial functors.

Theorem

Suppose that we have a **Top***-functor* $u \colon \mathbf{M} \to \mathbf{N}$ *of small* **Top***-categories,* $\mathbf{N}$ *is a* **cc-Top***-category and for every* $n, n' \in \mathbf{N}_0$, $\mathbf{N}(n, n')$ *is locally compact. Then, the functor*

$$\mathbf{N} \otimes_{\mathbf{M}} (\cdot) \colon (\mathbf{Top}^{\mathbf{M}})_{\mathbf{c}} \to (\mathbf{Top}^{\mathbf{N}})_{\mathbf{c}}$$

is left adjoint to the functor $U \colon (\mathbf{Top}^{\mathbf{N}})_{\mathbf{c}} \to (\mathbf{Top}^{\mathbf{M}})_{\mathbf{c}}$.

Prolongations, suspensions, telescopes and tubes

In this section, $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{N}$ are taking with their canonical topological monoid structures and $\mathbf{R}, \mathbf{R}_+, \mathbf{Z}, \mathbf{N}$ denote the corresponding small **Top**-categories.

Prolongations

M. Megrelishvili and L. Schröder (Globalization of Confluent Partial Actions on Topological and Metric Spaces) under some conditions extend a space X to a more large space, which is called the globalization of X where the local action are extended to global actions.

PROLONGATIONS: Given an object X in $\mathbf{Top}^{\mathbf{R}_+}$, we can consider the following **prolongation space**

$$P(X) = (\mathbb{R}_- \times X)/\sim$$

where the relation is given by

$$(r, x) \sim (r - \epsilon, \epsilon \cdot x), \quad x \in X, \quad r \in \mathbb{R}_-, \quad \epsilon \in \mathbb{R}_+.$$

The action of $\mathbb{R}$ is given by

$$t \cdot [(r, x)] = [(r + t, x)], \text{if } t \leq 0, \text{ and } [(r, t \cdot x)], \text{if } t \geq 0.$$

$P(X) \in \mathbf{Top}^{\mathbf{R}}$ and if $X \in (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$, then $P(X) \in (\mathbf{Top}^{\mathbf{R}})_{\mathbf{c}}$, and it is said to be the **prolongation continuous flow** of X.

Theorem

For every $X \in (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$, the canonical map

$$P(X) \to \mathbf{R} \otimes_{\mathbf{R}_+} X, \quad [(r, x)] \to r \otimes x$$

is a natural isomorphism in $(\mathbf{Top}^{\mathbf{R}})_{\mathbf{c}}$.

Remark

Similarly, one can define the prolongation of a discrete semi-flow that can also be interpreted as a tensor product using the functor $\mathbf{Z} \otimes_{\mathbf{N}} (\cdot)$.

Suspensions

SUSPENSIONS: For a given object $X \in \mathbf{Top}^{\mathbf{N}}$, we can consider the following **suspension space**:

$$S(X) \cong ([0,1] \times X)/\sim$$

with $(1,x) \sim (0, 1 \cdot x)$, $x \in X$, and the action of $\mathbb{R}_+$ is given by

$$r' \cdot [(r,x)] = [(D(r+r'), E(r+r') \cdot x)], r' \in \mathbb{R}_+,\ r \in [0,1],\ x \in X,$$

where $D(t)$, $E(t)$ denote the decimal and integer part of a real number t, respectively.
$S(X) \in (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$, and it is said to be the **suspension continuous semi-flow** of the discrete semi-flow X.

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Theorem

For every X in $\mathbf{Top}^{\mathbf{N}}$, the canonical map

$$S(X) \to \mathbf{R}_+ \otimes_{\mathbf{N}} X, \quad [(r,x)] \to r \otimes x$$

is a natural isomorphism in $(\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$.

Remark

Similarly, one can define the suspension of a discrete flow that can also be interpreted as a tensor product using the functor $\mathbf{R} \otimes_{\mathbf{Z}} (\cdot)$.

Telescopes and tubes

TELESCOPES.

An object $X = \{X_n\}$ in $\mathbf{Top}^{\tilde{\mathbf{N}}}$ can be represented by an inverse (direct) system

$$X = \; X_0 \xrightarrow{f_1^0} X_1 \xrightarrow{f_2^1} X_2 \xrightarrow{f_2^1} \cdots$$

and for $x \in X_n$ we write $k \cdot x = f_{n+k}^n(x)$.

One can consider the **telescopic** construction $T^{\tilde{\mathbf{N}}} \colon \mathbf{Top}^{\tilde{\mathbf{N}}} \to (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$ as follows:

For $X \in \mathbf{Top}^{\tilde{\mathbf{N}}}$, $X = \{X_n\}$, we take

$$T^{\tilde{\mathbf{N}}}(X) = T^{\tilde{\mathbf{N}}}(\{X_n\}) = \Big(\bigcup_{n \in \mathbb{N}} (\{n\} \times [n, n+1] \times X_n)\Big)/\sim$$

where $(n, n+1, x) \sim (n+1, n+1, f_{n+1}^n(x))$, $n \in \mathbb{N}, x \in X_n$.

The action $\mathbb{R}_+ \times T^{\tilde{\mathbf{N}}}(\{X_n\}) \to T^{\tilde{\mathbf{N}}}(\{X_n\})$ is given by

$$\varepsilon \cdot [(n, t, x)] = [(E(\varepsilon + t), \varepsilon + t, (E(\varepsilon + t) - n) \cdot x]$$

for $\varepsilon \in \mathbb{R}_+$, $t \in [n, n+1)$, $x \in X_n$.

The continuous semi-flow $T^{\tilde{\mathbf{N}}}(\{X_n\}) \in (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$ is said to be the **telescope** of $\{X_n\}$.

Theorem

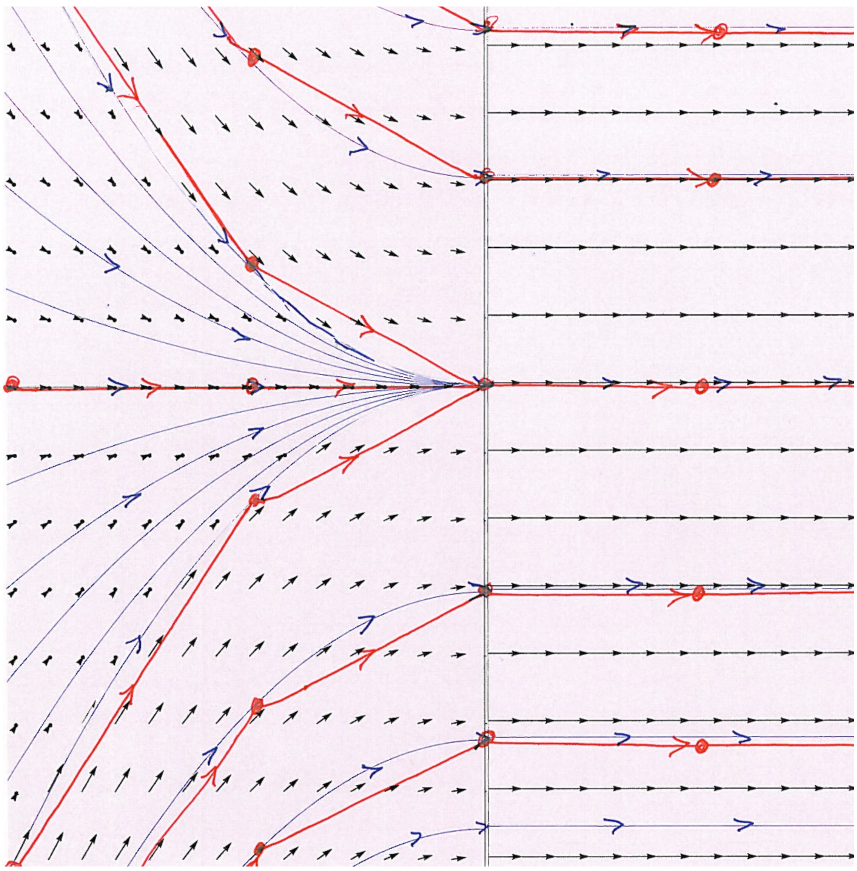
For every $X = \{X_n\} \in \mathbf{Top}^{\tilde{\mathbf{N}}}$, *we have canonical isomorphisms in* $(\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$ *:*

(i) $T^{\tilde{\mathbf{N}}}(X) \cong \mathbf{R}_+ \otimes_{\tilde{\mathbf{N}}} X$,

(ii) $T^{\tilde{\mathbf{N}}}(X) \cong S(\sqcup_{n\in\mathbb{N}} X_n)$

In a similar way, we have also a slightly different **telescopic construction**:

$$T^{\tilde{\tilde{\mathbf{Z}}}} \colon \mathbf{Top}^{\tilde{\tilde{\mathbf{Z}}}} \to (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}.$$



TUBES:

We can consider the following **tube** functors as the composites

$$T^+ = T^{\tilde{\mathbf{N}}} U \colon \mathbf{Top}^{\mathbf{N}} \xrightarrow{U} \mathbf{Top}^{\tilde{\mathbf{N}}} \xrightarrow{T^{\tilde{\mathbf{N}}}} (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$$

$$T = T^{\tilde{\tilde{\mathbf{Z}}}} U \colon \mathbf{Top}^{\mathbf{N}} \xrightarrow{U} \mathbf{Top}^{\tilde{\tilde{\mathbf{Z}}}} \xrightarrow{T^{\tilde{\tilde{\mathbf{Z}}}}} (\mathbf{Top}^{\mathbf{R}_+})_{\mathbf{c}}$$

Example

Let S^1 be the set of complex numbers with module 1. Given a prime p, taking the induced map $\tilde{p}\colon S^1 \to S^1, z \to z^p$, and suspensions, one has the canonical map $\tilde{p}\colon S^n \to S^n$ of degree p ($n \geq 1$) which gives to S^n the structure of a discrete semi-flow $(S^n, \tilde{p}) \in \mathbf{Top}^{\mathbf{N}}$. It is interesting to note that the tube $T((S^n, \tilde{p}))$, which is a continuous semi-flow, is a "uniquelly p-divisible n-sphere"; that is, the n-th homotopy group $\pi_n(T((S^n, \tilde{p}))) \cong \{z/p^k | z \in \mathbb{Z}, \quad k \in \mathbb{N}\} \subset \mathbb{Q}$.

SOME OPEN RESEARCH LINES

-to explore another categorical models for continuous and discrete dynamical systems

-to give categorical characterizations of the sub flows of periodic points, Poisson stable points, non-wandering points, attractors, omega-limits, et cetera.

-to apply the properties of theory of categories to the study of properties and classification of flows

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