# FURTHER PROPERTIES OF THE BISHOP-PHELPS-BOLLOBÁS MODULI

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ABSTRACT. We continue the study of the Bishop-Phelps-Bollobás moduli $\Phi_X(\delta)$ and $\Phi_X^S(\delta)$ initiated in [M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido, and F. Rambla-Barreno. J. Math. Anal. Appl. 412 (2014), no. 2, 697–719]. In particular, for a uniformly non-square Banach space $X$ we present a simple proof of the previously known fact that $\Phi_X(\delta) < \sqrt{2\delta}$ for $\delta \in (0, 1/2)$ and extend this result to the whole range of $\delta \in (0, 2)$. We demonstrate also the continuity of $\Phi_X$ with respect to $X$.

## 1. Introduction

In 1970, B. Bollobás [2] gave a refinement of the classical Bishop-Phelps theorem [1] on the density of the set of norm-attaining functionals in the dual of every Banach space. Such a refinement allows to approximate at the same time a functional and a vector in which it almost attains the norm by a norm-attaining functional and a point in which it attains the norm, respectively. This paper deals with two moduli, recently introduced in [5], which measure what is the best possible Bishop-Phelps-Bollobás theorem in a given space. Let us give the necessary definitions. Given a real or complex Banach space $X$, we write $X^*$ to denote the topological dual of $X$. We write $B_X$ and $S_X$ to denote respectively the closed unit ball and the unit sphere of the space. We consider the set in $X \times X^*$ given by

$$\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : \text{Re } x^*(x) = 1\}.$$

**Definition 1.1** (Bishop-Phelps-Bollobás moduli, [5])**.**
Let $X$ be a Banach space. The *Bishop-Phelps-Bollobás modulus* of $X$ is the function $\Phi_X : (0, 2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0, 2)$, $\Phi_X(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in B_X \times B_{X^*}$ with $\text{Re } x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

The *spherical Bishop-Phelps-Bollobás modulus* of $X$ is the function $\Phi_X^S : (0, 2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0, 2)$, $\Phi_X^S(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $\text{Re } x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

Many properties of these moduli have been established in the recent papers [4, 5], to which we refer for background. It is clear that $\Phi_X(\delta) \geqslant \Phi_X^S(\delta)$ for every $\delta \in (0, 2)$ and every Banach space $X$, and this inequality can be strict (for instance, this happens when $X$ is a Hilbert space, see [5, §4]). One of the main results of [5] states that $\Phi_X(\delta) \leqslant \sqrt{2\delta}$ for every $\delta \in (0, 2)$ and every Banach space $X$ [5, Theorem 2.1] and that this inequality is sharp [5, §4]: for instance, spaces of continuous functions or spaces of integrable functions satisfy $\Phi_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 2)$. Conversely, it is shown in [5, §5] and in [4, Corollary 3.4] that if $\Phi_X(\delta) = \sqrt{2\delta}$ for some $\delta \in (0, 1/2)$, then $X$ contains almost isometric copies of the real space $\ell_\infty^{(2)}$ (i.e. the space $\mathbb{R}^2$ endowed with the maximum norm). The proofs of this fact are involved and cannot be extended to larger values of $\delta$. The first aim of this paper is to extend this result to all values of $\delta \in (0, 2)$ with a very simple proof. To do so, we provide a new sufficient geometric condition for a Banach space to contain almost isometric copies of the real space $\ell_\infty^{(2)}$. This is the content of section 2. Finally, section 3 is devoted to prove the continuity of the two Bishop-Phelps-Bollobás

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moduli in the set of all equivalent norms of a given Banach space (which can be endowed naturally with the structure of metric space).

## 2. Banach spaces with the greatest possible modulus

Our aim now is to show that a Banach space $X$ with maximum value of $\Phi_X(\delta)$ for some $\delta \in (0,2)$ contains almost isometric copies of the **real** space $\ell_\infty^{(2)}$. For $\delta < 1/2$ this was proved, with a much more complicated proof, in [5, Theorem 5.8] and [4, Corollary 3.4] (in last reference, the result is a consequence of a quantitative approach). Let us first recall the following definition.

**Definition 2.1.** Let $X$, $E$ be Banach spaces. $X$ *contains almost isometric copies of* $E$ if for every $\varepsilon > 0$ there exist a subspace $E_\varepsilon \subset X$ and a bijective linear operator $T : E \longrightarrow E_\varepsilon$ with $\|T\| < 1 + \varepsilon$ and $\|T^{-1}\| < 1 + \varepsilon$.

The unit ball of the real space $\ell_\infty^{(2)}$ is the square with the vertexes $u = (1,1)$, $v = (1,-1)$, $-u$ and $-v$, and the vertexes satisfy $\|u - v\| = \|u + v\| = 2$. The following easy result, which we state here for future use, is well known and follows from the above description of the shape of the unit ball of $\ell_\infty^{(2)}$.

**Lemma 2.2.** *Let $X$ be a Banach space.*

(a) *$X$ contains the real space $\ell_\infty^{(2)}$ isometrically if and only if there are elements $u, v \in S_E$ such that $\|u - v\| = \|u + v\| = 2$.*
(b) *$X$ contains almost isometric copies of the real space $\ell_\infty^{(2)}$ if and only if there are elements $u_n, v_n \in X$, $n \in \mathbb{N}$ such that $\lim \|u_n\| = \lim \|v_n\| = 1$ and $\liminf \|u_n - v_n\| \geqslant 2$, $\liminf \|u_n + v_n\| \geqslant 2$.*
(c) *$X$ contains almost isometric copies of the real space $\ell_\infty^{(2)}$ if and only if $X^*$ does (see* [7, Corollary 2], *for instance).*

Our promised result can be stated as follows.

**Theorem 2.3.** *Let $X$ be a Banach space and suppose that there is $\delta \in (0,2)$ satisfying $\Phi_X(\delta) = \sqrt{2\delta}$. Then, $X$ contains almost isometric copies of the real space $\ell_\infty^{(2)}$.*

We need a couple of preliminary results. The first one is a sufficient condition for a Banach space to contains almost-isometric copies of the real space $\ell_\infty^{(2)}$ which can be of independent interest.

**Lemma 2.4.** *Let $X$ be a Banach space. Suppose that there exist $k \in (0,1)$ and two sequences $(x_n)$ in $S_X$ and $(y_n)$ in $X \setminus \{0\}$ satisfying*

$$\limsup \|x_n - y_n\| \leqslant k \qquad \textit{and} \qquad \liminf \left\| x_n - \frac{y_n}{\|y_n\|} \right\| \geqslant 2k.$$

*Then $X$ contains almost isometric copies of the real space $\ell_\infty^{(2)}$.*

We will use the following result which is surely well known, but we include an elementary proof as we have not found any explicit reference.

**Remark 2.5.** Let $X$ be a Banach space, $k \in (0,1)$ and let $(u_n)$, $(v_n)$ sequences of elements of $X$ such that

$$\limsup \|u_n\| \leqslant 1, \quad \limsup \|v_n\| \leqslant 1 \quad \text{and} \quad \liminf \|k u_n + (1-k) v_n\| \geqslant 1.$$

Then $\liminf \|u_n + v_n\| \geqslant 2$.

*Proof.* Write $m_n = \|(1-k)u_n + k v_n\|$, take $f_n \in S_{X^*}$ such that

$$\operatorname{Re} f_n\big((1-k)u_n + k v_n\big) = \big\|(1-k)u_n + k v_n\big\| = m_n,$$

and observe that $\limsup \operatorname{Re} f_n(u_n) \leqslant 1$ and $\limsup \operatorname{Re} f_n(v_n) \leqslant 1$. As

$$m_n = (1-k) \operatorname{Re} f_n(u_n) + k \operatorname{Re} f_n(v_n),$$

we have

$$\operatorname{Re} f_n(u_n) = \frac{1}{1-k}\big(m_n - k \operatorname{Re} f_n(v_n)\big) \quad \text{and} \quad f_n(v_n) = \frac{1}{k}\big(m_n - (1-k)\operatorname{Re} f_n(u_n)\big).$$

Now,

$$\liminf \operatorname{Re} f_n(u_n) \geqslant \frac{1}{1-k}\big(\liminf m_n - k \limsup \operatorname{Re} f_n(v_n)\big) \geqslant 1$$

and

$$\liminf \operatorname{Re} f_n(v_n) \geqslant \frac{1}{k}\big(\liminf m_n - (1-k) \limsup \operatorname{Re} f_n(u_n)\big) \geqslant 1.$$

Finally,

$$\liminf \|u_n + v_n\| \geqslant \liminf \operatorname{Re} f_n(u_n + v_n) \geqslant \liminf \operatorname{Re} f_n(u_n) + \liminf \operatorname{Re} f_n(v_n) \geqslant 2. \qquad \square$$

*Proof of Lemma 2.4.* Up to subsequences, we may and do suppose that

$$\lim \|x_n - y_n\| \leqslant k, \quad \lim \left\| x_n - \frac{y_n}{\|y_n\|} \right\| \geqslant 2k, \quad \text{and} \quad \exists \lim \|y_n\|.$$

We first observe that since $(x_n)$ lies in $S_X$, using the triangle inequality we have that

$$\lim \big|1 - \|y_n\|\big| = \big|1 - \lim \|y_n\|\big| \leqslant k.$$

Now, we have

$$\begin{aligned} 2k \leqslant \lim \left\| x_n - \frac{y_n}{\|y_n\|} \right\| &= \lim \frac{1}{\|y_n\|} \big\| \|y_n\| x_n - y_n \big\| \\ &= \lim \frac{1}{\|y_n\|} \big\| \|y_n\|(x_n - y_n) + (1 - \|y_n\|) y_n \big\| \\ &\leqslant \lim \|x_n - y_n\| + \lim \big|1 - \|y_n\|\big| \leqslant k + k = 2k. \end{aligned}$$

Hence, all the inequalities above are in fact equalities and we have

$$\text{(1)} \qquad \lim \|x_n - y_n\| = k, \quad \lim \left\| x_n - \frac{y_n}{\|y_n\|} \right\| = 2k, \quad \text{and} \quad |1 - \lim \|y_n\|| = k.$$

Using Lemma 2.2, it is enough to find two sequences $(u_n)$, $(v_n)$ in $X$ such that $\lim \|u_n\| = 1$, $\lim \|v_n\| = 1$, $\liminf \|u_n + v_n\| \geqslant 2$ and $\liminf \|u_n - v_n\| \geqslant 2$. We distinguish two cases depending on the values of $\lim \|y_n\|$. Suppose first that $\lim \|y_n\| = 1 - k$ and take

$$u_n = \frac{y_n}{1-k} \quad \text{and} \quad v_n = \frac{x_n - y_n}{k} \qquad (n \in \mathbb{N}),$$

which satisfy that $\lim \|u_n\| = \lim \|v_n\| = 1$. We have $(1-k)u_n + k v_n = x_n \in S_X$, and we may apply Remark 2.5 to get that $\liminf \|u_n + v_n\| \geqslant 2$. On the other hand,

$$\begin{aligned} \|u_n - v_n\| &= \frac{1}{k}\|k u_n - (x_n - y_n)\| = \frac{1}{k}\|k u_n - x_n + (1-k) u_n\| \\ &= \frac{1}{k}\|u_n - x_n\| = \frac{1}{k}\left\| \frac{y_n}{1-k} - x_n \right\| \\ &\geqslant \frac{1}{k}\left( \left\| \frac{y_n}{\|y_n\|} - x_n \right\| - \left\| \frac{y_n}{1-k} - \frac{y_n}{\|y_n\|} \right\| \right) \longrightarrow 2. \end{aligned}$$

Therefore, $\liminf \|u_n - v_n\| \geqslant 2$. This finishes the proof in this case.

If, otherwise, $\lim \|y_n\| = 1 + k$, take

$$u_n = \frac{y_n}{1+k} \quad \text{and} \quad v_n = \frac{y_n - x_n}{k} \qquad (n \in \mathbb{N}),$$

which satisfy that $\lim \|u_n\| = \lim \|v_n\| = 1$. Observe that

$$\begin{aligned}\|u_n - v_n\| &= \frac{1}{k}\left\|k\frac{y_n}{1+k} + x_n - y_n\right\| = \frac{1}{k}\left\|x_n - \frac{y_n}{1+k}\right\| \\ &\geqslant \frac{1}{k}\left(\left\|x_n - \frac{y_n}{\|y_n\|}\right\| - \left\|\frac{y_n}{1+k} - \frac{y_n}{\|y_n\|}\right\|\right) \longrightarrow 2.\end{aligned}$$

Therefore, $\liminf \|u_n - v_n\| \geqslant 2$. On the other hand,

$$\begin{aligned}\big\|(1-k)u_n + kv_n\big\| &= \big\|(1-k)u_n + y_n - x_n\big\| = \big\|(1-k)u_n + (1+k)u_n - x_n\big\| \\ &= \big\|2u_n - x_n\big\| \geqslant 2\|u_n\| - \|x_n\| \longrightarrow 1,\end{aligned}$$

so $\liminf \big\|(1-k)u_n + kv_n\big\| \geqslant 1$ and we may apply Remark 2.5 to get that $\liminf \|u_n + v_n\| \geqslant 2$. □

Observe that if the sequences $(x_n)$ and $(y_n)$ in Lemma 2.4 are constant, what we get (with much easier proof) is an isometric copy of the real space $\ell_\infty^{(2)}$. Let us state this result.

**Corollary 2.6.** *Let $X$ be a Banach space. Suppose that there are $x \in S_X$, $y \in X \setminus \{0\}$ and $k \in (0,1)$ satisfying*

$$\|x - y\| = k \qquad \text{and} \qquad \left\|x - \frac{y}{\|y\|}\right\| = 2k.$$

*Then the real linear span of $\{x, y\}$ is isometrically isomorphic to the real space $\ell_\infty^{(2)}$.*

We would like to mention that both Lemma 2.4 and Corollary 2.6 are false for $k = 0$ and $k = 1$. The case of $k = 0$ is immediate, as in every Banach space we may find unit vectors $x, y$ satisfying the requirements of the corollary, and the corresponding constant sequences satisfy the requirements of the lemma. The case of $k = 1$ in the corollary cannot happen: if $X$ is a Banach space, $x \in S_X$ and $y \in X \setminus \{0\}$ satisfy $\|x - y\| = 1$ and $\left\|x - \frac{y}{\|y\|}\right\| = 2$, it follows that $|1 - \|y\|| = 1$ (see eq. (1)), so $\|y\| = 2$; but then

$$4 = \|2x - y\| \leqslant \|x\| + \|x - y\| \leqslant 2,$$

a contradiction. Finally, hypothesis of Lemma 2.4 for $k = 1$ are satisfied in every Banach space $X$. Indeed, fix $x \in S_X$ and consider $x_n = x \in S_X$ and $y_n = \frac{-1}{n}x \in X \setminus \{0\}$. Then, $\|x_n - y_n\| = 1 + \frac{1}{n}$ and $\left\|x_n - \frac{y_n}{\|y_n\|}\right\| = \|2x\| = 2$.

For the proof of Theorem 2.3 we will also need the following result, which is a particular case of [8, Corollary 2.2], which we state for the sake of clearness.

**Lemma 2.7** ([8, Corollary 2.2])**.** *Let $X$ be a Banach space. Suppose that $x^* \in S_{X^*}$, $\delta > 0$ and $x \in B_X$ are such that*

$$\operatorname{Re} x^*(x) \geqslant 1 - \delta.$$

*Then, for every $k \in (0,1)$ there exist $y^* \in X^*$ and $y \in B_X$ such that*

$$\operatorname{Re} y^*(y) = \|y^*\|, \qquad \|x - y\| \leqslant \frac{\delta}{k}, \qquad \|x^* - y^*\| \leqslant k.$$

*Proof of Theorem 2.3.* Consider a strictly increasing sequence $(\rho_n)$ of positive numbers with $\lim \rho_n = 1$ and such that $\frac{\sqrt{2\delta}}{2\rho_n} < 1$ for every $n \in \mathbb{N}$. By [5, Proposition 3.8] or [4, Theorem 2.1], we have that $\Phi_X^S(\delta) = \sqrt{2\delta}$, so for every $n \in \mathbb{N}$ there are $x_n \in S_X$ and $x_n^* \in S_{X^*}$ satisfying that

$$\operatorname{Re} x_n^*(x_n) \geqslant 1 - \delta$$

and such that

$$\max\{\|x_n - z\|, \|x_n^* - z^*\|\} \geqslant \sqrt{2\delta}\rho_{n+1} \tag{2}$$

for every $(z,z^*)\in\Pi(X)$. Next, we apply Lemma 2.7 with $x_n^*\in S_{X^*}$, $x_n\in B_X$, and $k_n=\frac{\sqrt{2\delta}}{2\rho_n}\in(0,1)$ to obtain $y_n^*\in X^*$ and $y_n\in S_X$ satisfying

$$\|y_n^*\|=\operatorname{Re} y_n^*(y_n),\qquad \|x_n-y_n\|\leqslant\frac{\delta}{k_n}=\sqrt{2\delta}\rho_n\,,\qquad\text{and}\qquad \|x_n^*-y_n^*\|\leqslant k_n=\frac{\sqrt{2\delta}}{2\rho_n}\,.$$

As $k_n<1$ and $\|x_n^*-y_n^*\|\leqslant k_n$, we get that $y_n^*\neq 0$ and so, $\left(y_n,\frac{y_n^*}{\|y_n^*\|}\right)\in\Pi(X)$. As we have that $\|x_n-y_n\|\leqslant\sqrt{2\delta}\rho_n<\sqrt{2\delta}\rho_{n+1}$, we get from equation (2) that

$$\left\|x_n^*-\frac{y_n^*}{\|y_n^*\|}\right\|\geqslant\sqrt{2\delta}\rho_{n+1}.$$

Summarizing, we have found two sequences $(x_n^*)$ in $S_{X^*}$ and $(y_n^*)\in X^*\setminus\{0\}$ such that

$$\limsup\|x_n^*-y_n^*\|\leqslant\frac{\sqrt{2\delta}}{2}\qquad\text{and}\qquad\liminf\left\|x_n^*-\frac{y_n^*}{\|y_n^*\|}\right\|\geqslant\sqrt{2\delta}\,.$$

Now, Lemma 2.4 gives that $X^*$ contains almost isometric copies of the real space $\ell_\infty^{(2)}$, and so does $X$ (Lemma 2.2), as desired. ☐

## 3. Continuity of the moduli

Our next goal is to show that the Bishop-Phelps-Bollobás modulus of a Banach space is continuous in the set of all equivalent norms on a given Banach space endowed with a metric introduced in [3, §18].

To do so we need to introduce some notation. Given a Banach space $X$, we denote $\mathcal{E}(X)$ the set of all equivalent norms to the original norm in $X$. $\mathcal{E}(X)$ is a metric space when endowed with the following distance:

$$d(p,q)=\log\left(\min\left\{k\geqslant 1:\frac{1}{k}p\leqslant q\leqslant kp\right\}\right)\qquad(p,q\in\mathcal{E}(X)).$$

For $p_0\in\mathcal{E}(X)$ and $k>1$ we consider the open set given by $G(p_0,k)=\{p\in\mathcal{E}(X):d(p,p_0)<\log k\}$. Given $p\in\mathcal{E}(X)$ we also use $p$ to denote the dual norm in $X^*$ and we use the notation

$$\Pi_p(X)=\{(x,x^*)\in X\times X^*:p(x)=p(x^*)=x^*(x)=1\}.$$

For $\delta\in(0,2)$, we write $\Phi_{(X,p)}$ and $\Phi^S_{(X,p)}$ to denote respectively the Bishop-Phelps-Bollobás modulus and the spherical Bishop-Phelps-Bollobás modulus of $X$ when it is endowed with the norm $p$. Besides, we consider the sets

$$A_p(\delta)=\{(x,x^*)\in X\times X^*\,:\,p(x)\leqslant 1,\,p(x^*)\leqslant 1,\,\operatorname{Re} x^*(x)>1-\delta\}$$
$$A_p^S(\delta)=\{(x,x^*)\in X\times X^*\,:\,p(x)=1,\,p(x^*)=1,\,\operatorname{Re} x^*(x)>1-\delta\}.$$

Finally, we write $d_p(A,B)$ to denote the Hausdorff distance between $A,B\subset X\times X^*$ associated to the $\ell_\infty$-distance $d_{\infty,p}$ in $X\times X^*$ when $X$ and $X^*$ are endowed with the norm $p$. That is, for $(x,x^*),(y,y^*)\in X\times X^*$, we write

$$d_{\infty,p}\big((x,x^*),(y,y^*)\big)=\max\{p(x-y),p(x^*-y^*)\}$$

and

$$d_p(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}d_{\infty,p}(a,b),\sup_{b\in B}\inf_{a\in A}d_{\infty,p}(a,b)\right\}.$$

Observe that with this notation one has that

$$\Phi_{(X,p)}(\delta)=d_p(A_p(\delta),\Pi_p(X))\qquad\text{and}\qquad\Phi^S_{(X,p)}(\delta)=d_p(A_p^S(\delta),\Pi_p(X)).$$

**Theorem 3.1.** *Let $X$ be a Banach space and $\delta\in(0,2)$. The functions*

$$\Phi_{(X,\cdot)}(\delta):\mathcal{E}(X)\longrightarrow\mathbb{R}\qquad\text{and}\qquad\Phi^S_{(X,\cdot)}(\delta):\mathcal{E}(X)\longrightarrow\mathbb{R}$$

*are continuous.*

To prove this result we need two lemmas which may be of independent interest.

**Lemma 3.2.** *Let $X$ be a Banach space, $\delta \in (0,2)$, $p_0 \in \mathcal{E}(X)$, and $k > 1$. Let $\eta > 0$ and $p, q \in G(p_0, k)$ satisfying $d(p,q) < \log(1+\eta)$.*

Case 1: *If $\delta \in (0,1]$, then*

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}\,.$$

Case 2: *If $\delta \in (1,2)$ and $(\delta-1)(1+\eta)^2 < 1$, then*

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + 2k\frac{\eta(2+\eta)}{(1+\eta)^2}\,.$$

*Proof.* We suppose first that $\delta \in (0,1]$ and we write $\delta_0 = 1 - \frac{1-\delta}{(1+\eta)^2}$. Given $(x,x^*) \in A_p(\delta)$, define $x_0 = \frac{p(x)}{q(x)}x$ and $x_0^* = \frac{p(x^*)}{q(x^*)}x^*$ which obviously satisfy $q(x_0) \leqslant 1$ and $q(x_0^*) \leqslant 1$. Besides, it is immediate to check that

$$x_0^*(x_0) = x^*(x)\frac{p(x)p(x^*)}{q(x)q(x^*)} \geqslant \frac{x^*(x)}{(1+\eta)^2} > \frac{1-\delta}{(1+\eta)^2} = 1-\delta_0,$$

and so $(x_0, x_0^*) \in A_q(\delta_0)$. Observe that if $\delta < 1$ then $\delta_0 > \delta$ and we can use Case 1 of Lemma 3.3 in [5] for $X$ endowed with the norm $q$ to obtain $(y,y^*) \in A_q(\delta)$ satisfying

$$\max\{q(x_0-y), q(x_0^*-y^*)\} < 2\frac{\sqrt{1-\delta}-\sqrt{1-\delta_0}}{1-\sqrt{1-\delta_0}} = \frac{2\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}\,.$$

So we can estimate as follows

$$p_0(x-y) \leqslant p_0(x-x_0) + p_0(x_0-y) \leqslant p_0(x)\left|1-\frac{p(x)}{q(x)}\right| + kq(x_0-y) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}$$

and an analogous argument gives us the same inequality for the number $p_0(x^*-y^*)$. Therefore, we have that $d_{p_0}\big((x,x^*), A_q(\delta)\big) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}$ for every $(x,x^*) \in A_p(\delta)$. Exchanging the roles of $p$ and $q$ one obtains $d_{p_0}\big((z,z^*), A_p(\delta)\big) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}$ for every $(z,z^*) \in A_q(\delta)$ and hence

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}\,.$$

In the particular case in which $\delta = 1$ it suffices to observe that $x_0^*(x_0) > 0$ and so $(x_0, x_0^*)$ belongs to $A_q(\delta)$. Therefore one obtains the estimation $d_{p_0}\big((x,x^*), A_q(\delta)\big) < k\eta$.

Suppose now that $\delta \in (1,2)$ and define this time $\delta_0 = 1 + (\delta-1)(1+\eta)^2$. Given $(x,x^*) \in A_p(\delta)$ we consider as in the previous case $x_0 = \frac{p(x)}{q(x)}x$ and $x_0^* = \frac{p(x^*)}{q(x^*)}x$ which satisfy $q(x_0) \leqslant 1$ and $q(x_0^*) \leqslant 1$. Using the facts that $p(x)/q(x) < 1+\eta$, $p(x^*)/q(x^*) < 1+\eta$ and $1-\delta < 0$, we can write

$$x_0^*(x_0) = x^*(x)\frac{p(x)p(x^*)}{q(x)q(x^*)} \geqslant (1-\delta)\frac{p(x)p(x^*)}{q(x)q(x^*)} > (1-\delta)(1+\eta)^2 = 1-\delta_0,$$

and so $(x_0, x_0^*) \in A_q(\delta_0)$. Since $2 > \delta_0 > \delta$, we can use Case 2 of Lemma 3.3 in [5] for $X$ endowed with the norm $q$ to obtain $(y,y^*) \in A_q(\delta)$ satisfying

$$\begin{aligned}\max\{q(x_0-y), q(x_0^*-y^*)\} &< 2\frac{2-\delta_0}{\delta_0}\frac{\delta_0-\delta}{\delta_0-1+\sqrt{1-2\delta+\delta\delta_0}} \\ &\leqslant 2\frac{2-\delta_0}{\delta_0}\frac{\delta_0-\delta}{\delta_0-1} \leqslant 2\frac{\delta_0-\delta}{\delta_0-1} = 2\frac{\eta(2+\eta)}{(1+\eta)^2}\,.\end{aligned}$$

From this point one can proceed as in the previous case to obtain

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + 2k\frac{\eta(2+\eta)}{(1+\eta)^2}\,,$$

which finishes the proof. □

One can obtain an analogous result for the spherical modulus using the same proof.

**Lemma 3.3.** *Let $X$ be a Banach space, $\delta \in (0,2)$, $p_0 \in \mathcal{E}(X)$, and $k > 1$. Let $\eta > 0$ and $p, q \in G(p_0, k)$ satisfying $d(p,q) < \log(1+\eta)$.*

Case 1: *If $\delta \in (0,1]$, then*

$$d_{p_0}\left(A_p^S(\delta), A_q^S(\delta)\right) < k\eta + \frac{4k(1-\delta)(2\eta+\eta^2)}{\delta + 2\eta + \eta^2}.$$

Case 2: *If $\delta \in (1,2)$, suppose that $(\delta-1)(1+\eta)^2 < 1$ and $2 - \sqrt{1-(\delta-1)(1+\eta)^2} < \delta$, then*

$$d_{p_0}\left(A_p^S(\delta), A_q^S(\delta)\right) < k\eta + 2k(2\eta+\eta^2)\frac{\delta-1}{2-\delta}.$$

*Proof.* The proof follows exactly the same lines as the proof of Lemma 3.2, using Lemma 3.4 in [5] instead of Lemma 3.3 in the corresponding cases. We observe that when $\delta = 1$, Lemma 3.4 in [5] cannot be used. In this case it suffices to take into account that the element $(x_0, x_0^*)$ lies in $A_q^S(\delta)$ if $(x, x^*)$ is in $A_p^S(\delta)$ so the estimation $d_{p_0}\big((x,x^*), A_q^S(\delta)\big) < k\eta$ follows as in the proof of Lemma 3.2. □

We are ready to show that the Bishop-Phelps-Bollobás moduli are continuous in the metric space $\mathcal{E}(X)$.

*Proof of Theorem 3.1.* Fixed $p_0 \in \mathcal{E}(X)$ and $k > 1$, we consider the open set in $\mathcal{E}(X)$ given by $G(p_0,k) = \{p \in \mathcal{E}(X) : d(p,p_0) < \log k\}$. Let $\eta > 0$ be such that $(\delta-1)(1+\eta)^2 < 1$ and $p, q \in G(p_0,k)$ satisfying $d(p,q) < \log(1+\eta)$. Then we can estimate as follows

$$\begin{aligned}
\Phi_{(X,p)}(\delta) - \Phi_{(X,q)}(\delta) &= d_p\left(A_p(\delta), \Pi_p(X)\right) - d_q\left(A_q(\delta), \Pi_q(X)\right) \\
&\leqslant d_p\left(A_p(\delta), A_q(\delta)\right) + d_p\left(A_q(\delta), \Pi_p(X)\right) - d_q\left(A_q(\delta), \Pi_p(X)\right) + d_q\left(\Pi_p(X), \Pi_q(X)\right) \\
&\leqslant k d_{p_0}\left(A_p(\delta), A_q(\delta)\right) + (1+\eta) d_q\left(A_q(\delta), \Pi_p(X)\right) \\
&\qquad - d_q\left(A_q(\delta), \Pi_p(X)\right) + k d_{p_0}\left(\Pi_p(X), \Pi_q(X)\right) \\
&\leqslant k d_{p_0}\left(A_p(\delta), A_q(\delta)\right) + k\eta d_{p_0}\left(A_q(\delta), \Pi_p(X)\right) + k d_{p_0}\left(\Pi_p(X), \Pi_q(X)\right) \\
&\leqslant k d_{p_0}\left(A_p(\delta), A_q(\delta)\right) + 2k\eta + k d_{p_0}\left(\Pi_p(X), \Pi_q(X)\right).
\end{aligned}$$

Exchanging the roles of $p$ and $q$ we can write

$$\left|\Phi_{(X,p)}(\delta) - \Phi_{(X,q)}(\delta)\right| \leqslant k d_{p_0}\left(A_p(\delta), A_q(\delta)\right) + 2k\eta + k d_{p_0}\left(\Pi_p(X), \Pi_q(X)\right).$$

This, together with Lemma 3.2 and the continuity of $\Pi_p(X)$ with respect to $p$ [3, Theorem 18.3], gives the continuity of $\Phi_{(X,\cdot)}(\delta)$.

A completely analogous argument allows to prove the continuity of $\Phi^S_{(X,\cdot)}$ from Lemma 3.3. □

There is a classical way to measure when two Banach spaces are close, the so-called Banach-Mazur distance, and which is related to our approach using the distance between equivalent norms. Given two isomorphic Banach spaces $X$ and $Y$, the *Banach-Mazur distance* between $X$ and $Y$ is defined by

$$d_{BM}(X,Y) = \log\inf\big\{\|T\|\|T^{-1}\| \,:\, T \text{ an isomorphism of } X \text{ onto } Y\big\}.$$

Note that $d_{BM}(X,Y) \geqslant 0$ and $d_{BM}(X,Z) \leqslant d_{BM}(X,Y) + d_{BM}(Y,Z)$. Given a Banach space $X$, we write $\mathcal{I}(X)$ to denote the set of all Banach spaces isomorphic to $X$, which is semimetric space when endowed with the Banach-Mazur distance. Then, the result above about the continuity of the Bishop-Phelps-Bollobás moduli on $\mathcal{E}(X)$ can be easily translated to the new setting.

**Corollary 3.4.** *Let $X$ be a Banach space and $\delta \in (0,2)$. The functions from $\mathcal{I}(X)$ to $\mathbb{R}$ given by*

$$Y \longmapsto \Phi_Y(\delta) \quad \text{and} \quad Y \longmapsto \Phi_Y^S(\delta) \qquad \big(Y \in \mathcal{I}(X)\big)$$

*are continuous.*

The way to deduce the above result from Theorem 3.1 is given by the next lemma, which is well-known (see [6, Exercise 1.75], for instance) and relates $\mathcal{E}(X)$ and $\mathcal{I}(X)$. We include an easy proof for the sake of completeness.

**Lemma 3.5.** *Let $X_0$, $X_1$ be Banach spaces. If $T : X_1 \longrightarrow X_0$ is an isomorphism, there exists a norm $p_1 \in \mathcal{E}(X_0)$ such that $(X_0, p_1)$ is isometrically isomorphic to $(X_1, \|\cdot\|_{X_1})$ and satisfying that*

$$\|x\|_{X_0} \leqslant p_1(x) \leqslant \|T\|\|T^{-1}\|\,\|x\|_{X_0}$$

*for all $x \in X_0$.*

*Proof.* Define $p_1(x) = \|T\|\|T^{-1}(x)\|_{X_1}$ for every $x \in X_0$. Then, it is clear that $(X_0, p_1)$ is isometrically isomorphic to $(X_1, \|\cdot\|_{X_1})$. Also, for each $x \in X_0$ we have

$$p_1(x) = \|T\|\|T^{-1}(x)\|_{X_1} \leqslant \|T\|\|T^{-1}\|\,\|x\|_{X_0}$$

and, on the other hand,

$$\|x\|_{X_0} = \|T(T^{-1}(x))\|_{X_0} \leqslant \|T\|\|T^{-1}(x)\|_{X_1} = p_1(x).$$ □

An easy consequence of the continuity of the Bishop-Phelps-Bollobás moduli is that they coincide for Banach spaces which are almost isometric.

**Corollary 3.6.** *Let $X$ and $Y$ be almost isometric Banach spaces (i.e. $d_{BM}(X,Y) = 0$). Then $\Phi_X(\delta) = \Phi_Y(\delta)$ and $\Phi_X^S(\delta) = \Phi_Y^S(\delta)$ for every $\delta \in (0,2)$.*

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