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SLICELY COUNTABLY DETERMINED BANACH SPACES

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ABSTRACT. We introduce the class of slicely countably determined Banach spaces which contains in particular all spaces with the RNP and all spaces without copies of ℓ_1. We present many examples and several properties of this class. We give some applications to Banach spaces with the Daugavet and the alternative Daugavet properties, lush spaces and Banach spaces with numerical index 1. In particular, we show that the dual of a real infinite-dimensional Banach with the alternative Daugavet property contains ℓ_1 and that operators which do not fix copies of ℓ_1 on a space with the alternative Daugavet property satisfy the alternative Daugavet equation.

1. INTRODUCTION

The aim of this paper is to introduce the class of slicely countably determined Banach spaces, give many examples and several properties of this class and, finally, to use this concept to give some applications to Banach spaces with the Daugavet property and to Banach spaces with numerical index 1. Let us introduce the needed notation and definitions.

Given a Banach space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$), we write S_X for its unit sphere and B_X for its closed unit ball. The dual space of X is denoted by X^* and $L(X)$ is the Banach algebra of all bounded linear operators from X to X. The space X has the *Daugavet property* [19] if every rank-one operator $T\in L(X)$ satisfies

$$\|\mathrm{Id}+T\| = 1+\|T\|. \tag{DE}$$

In this case, all operators on X which do not fix copies of ℓ_1 (in particular, weakly compact operators) also satisfy (DE) [29]. If every rank-one operator $T\in L(X)$ satisfies the norm equality

$$\max_{\theta\in\mathbb{T}} \|\mathrm{Id}+\theta\, T\| = 1+\|T\| \tag{aDE}$$

($\mathbb{T}$ being the set of modulus one scalars), X has the *alternative Daugavet property* [25] and then all weakly compact operators on X also satisfy (aDE). A Banach space is said to have *numerical index* 1 [13] if every $T\in L(X)$ satisfies that $v(T)=\|T\|$, where

$$v(T) = \big\{|x^*(Tx)| \,:\, x\in S_X,\, x^*\in S_{X^*},\, x^*(x)=1\big\}$$

is the *numerical radius* of the operator T. It is known [13] that

$$v(T)=\|T\| \qquad\Longleftrightarrow\qquad T \text{ satisfies (aDE)}.$$

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Then, X has numerical index 1 if and only if every $T \in L(X)$ satisfies (aDE). It follows from the above discussion that

$$\boxed{\text{Daugavet property}} \Longrightarrow \boxed{\text{Alternative Daugavet property}} \Longleftarrow \boxed{\text{Numerical index 1}}$$

None of the above implications reverses in general [25, Example 3.2]. For the first implication, it is even known that it is not reversible under any isomorphic property [25, Corollary 3.3]. On the other hand, it is known that the second implication reverses for Asplund spaces and for Banach spaces with the Radon-Nikodým property [23, Remark 6]. We refer the interested reader to [15, 16, 18, 24] and the already cited references for recent results, more information and background on these properties.

We will say that X is *slicely countably determined* (*SCD* in short) if every bounded convex subset A of X is an *SCD set*, i.e. there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A such that $A \subseteq \overline{\operatorname{conv}}(B)$ whenever $B \subseteq A$ intersects all the S_n's. Here a *slice* of a convex set A is the subset given by

$$S(A, x^*, \varepsilon) = \{x \in A : \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \varepsilon\}$$

and $\overline{\operatorname{conv}}(\cdot)$ stands for the closed convex hull. This isomorphic property, which clearly implies separability, is sufficient to get numerical index 1 from the alternative Daugavet property and it is weaker than both RNP and being Asplund (for separable spaces). Actually, this property is satisfied by both separable strongly regular spaces and separable Banach spaces which do not contain copies of ℓ_1. This is the main motivation of the study of SCD spaces.

In section 2 we study SCD sets, giving examples and elementary properties. We show, for instance, that the sequence of slices can be replaced by a sequence of relatively weakly open sets or by a sequence of convex combinations of slices. In section 3 we study SCD spaces and show some stability properties. For instance, it is a three space property, so it is stable for finite sums, and it is stable for some infinite unconditional sums.

Since it is not easy to deal with Banach spaces with numerical index 1, there are in the literature several geometrical sufficient conditions (see [18]), the weakest one being the so-called lushness. A Banach space X is said to be *lush* [8] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice $S = S(B_X, x^*, \varepsilon)$ with $x^* \in S_{X^*}$ such that $x \in S$ and $\operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon$ (where $\operatorname{aconv}(A)$ denotes the absolutely convex hull of the set A). Lush spaces have numerical index 1 [8, Proposition 2.2], but it has been very recently shown that the converse result is not true [17]. We refer to [7, 8] for background.

It is actually shown in section 4 that an SCD Banach space with the alternative Daugavet property is lush. This result allows us to show that ℓ_1 embeds in the dual of every real infinite-dimensional Banach space with the alternative Daugavet property. This answers in the positive [18, Problem 18].

Section 5 is devoted to SCD-operators and hereditary-SCD-operators. A bounded linear operator $T : X \longrightarrow Y$ between two Banach spaces X and Y is said to be an *SCD-operator* if $T(B_X)$ is an SCD set, and T is a *hereditary-SCD-operator* if every bounded convex subset of $T(B_X)$ is SCD. We show that SCD-operators on a Banach space with the alternative Daugavet property satisfy (aDE). Therefore, operators which do not fix copies of ℓ_1 on a Banach space with the alternative Daugavet property satisfy (aDE). For a Banach space with the Daugavet property it is shown that every SCD-operator is strong Daugavet (and so it satisfies (DE)), and every hereditary-SCD-operator is narrow.

Section 6 is devoted to the study of sets with a countable π-base of the weak topology. It is shown in section 2 that these sets are SCD, but it is not known whether the converse result is true. It is also shown in section 2 that separable sets without ℓ_1 sequences have countable π-bases of the weak topology, and in this section we show that the same is true for CPCP sets and for bounded convex subsets of both $c_0(\ell_1)$ and $\ell_1(c_0)$. We also show some characterizations of SCD sets which remind of the existence of countable π-bases of the weak topology. One of these characterizations allows us

to show that the set of extreme points of the weak*-closure (in the bidual space) of an SCD set has a countable π-base of the weak* topology, and so it is weak* separable. The set of extreme points of a convex set B will be denoted by $\mathrm{ext}(B)$.

Finally, section 7 contains several open questions.

2. Slicely countably determined sets

Definition 2.1. Let X be a Banach space and let A be a convex bounded subset of X. A countable family $\{V_n : n \in \mathbb{N}\}$ of subsets of A is called *determining* for A if $A \subseteq \overline{\mathrm{conv}}(B)$ for every $B \subseteq A$ intersecting all the sets V_n. Equivalently, $\{V_n : n \in \mathbb{N}\}$ is determining for A if for every sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in V_n$ $(n \in \mathbb{N})$, one has $A \subseteq \overline{\mathrm{conv}}\big(\{v_n : n \in \mathbb{N}\}\big)$.

We give three easy observations which will be useful later on. The first one is a consequence of the Hahn-Banach theorem. The second and third ones are straightforward.

Proposition 2.2. *Let X be a Banach space and let A be a convex bounded subset of X. A sequence $\{V_n : n \in \mathbb{N}\}$ of subsets of A is determining if and only if every slice of A contains one of the V_n.*

Proof. The "if" part is evident: if $B \subseteq A$ intersects all the V_n, then it intersects all the slices of A, and then by the Hahn-Banach theorem $\overline{\mathrm{conv}}(B) \supseteq A$. Now the "only if" part. Assume that some slice S of A does not contain any of the V_n. Then $A \setminus S$ is a convex relatively closed subset of A intersecting all the V_n. But $A \setminus S \neq A$, which means that $\{V_n : n \in \mathbb{N}\}$ is not determining. □

Remark 2.3. *Let X be a Banach space and let A be a convex bounded subset of X. Suppose that there is a sequence $\{a_n : n \in \mathbb{N}\}$ of points in A such that $A \subseteq \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$ and that for every $n \in \mathbb{N}$, there is sequence $\{V_{n,m} : m \in \mathbb{N}\}$ of subsets of A such that $a_n \in \overline{\mathrm{conv}}(B)$ whenever $B \subseteq A$ intersects $V_{n,m}$ for every $m \in \mathbb{N}$. Then, the family $\{V_{n,m} : n, m \in \mathbb{N}\}$ is determining for A.*

As an immediate consequence of the above result, we get the following.

Remark 2.4. *Let X be a Banach space and let A be a separable convex bounded subset of X. Suppose that for every $a \in A$ there is a sequence $\{V_m^a : m \in \mathbb{N}\}$ of subsets of A such that $a \in \overline{\mathrm{conv}}(B)$ whenever $B \subseteq A$ intersects V_m^a for every $m \in \mathbb{N}$. Then, taking a dense sequence $\{a_n : n \in \mathbb{N}\}$ in A, the family $\{V_m^{a_n} : n, m \in \mathbb{N}\}$ is determining for A.*

We can now give the main definition of this section.

Definition 2.5. A convex bounded subset A of a Banach space X is said to be *slicely countably determined* (*SCD set* in short) if there is a determining sequence of slices of A.

Two remarks are pertinent.

Remark 2.6. *It is clear from the definition that every SCD set is separable.*

Remark 2.7. *A convex bounded subset A of a Banach space X is SCD if and only if the closure of A is an SCD set.*

Proof. Let us show first that $\overline{A}$ is SCD when A is. Consider a determining sequence of slices $S_n = S(A, x_n^*, \varepsilon_n)$ $(n \in \mathbb{N})$ for A, and let us prove that the slices $S_n' = S(\overline{A}, x_n^*, \varepsilon_n/2)$ $(n \in \mathbb{N})$ form a determining sequence for the closure of A. Consider an arbitrary slice $S = S(\overline{A}, x^*, \varepsilon)$ of $\overline{A}$. Then, $S(\overline{A}, x^*, \varepsilon/2) \cap A = S(A, x^*, \varepsilon/2)$ is a slice of A, so there is $n \in \mathbb{N}$ such that $S(A, x^*, \varepsilon/2) \supseteq S_n$

by Proposition 2.2. Therefore, S contains the closure of S_n, which in turn contains S'_n, and again Proposition 2.2 gives us that $\{S'_n\}$ is determining for $\overline{A}$.

For the converse implication, we consider a determining sequence $\{S(\overline{A}, x_n^*, \varepsilon_n) : n \in \mathbb{N}\}$ for $\overline{A}$, and it is straightforward to show that $\{S(A, x_n^*, \varepsilon_n) : n \in \mathbb{N}\}$ is determining for A. □

Our fist goal is to present the basic examples related to Definition 2.5: Radon-Nikodým and Asplund sets are SCD, whereas the unit ball of a Banach space with the Daugavet property is not.

We start with subsets having sufficiently many denting points. Let X be a Banach space and let A be a closed convex bounded subset of X. A point of A is said to be a *denting point* if it belongs to slices of A of arbitrarily small diameter. We write $\operatorname{dent}(A)$ to denote the set of denting points of A. We say that A is *dentable* (in the sense of Ghoussoub-Godefroy-Maurey-Schachermayer [14, §III]) if $A = \overline{\operatorname{conv}}\big(\operatorname{dent}(A)\big)$ [14, Proposition III.3].

Proposition 2.8. *Let X be a Banach space and let A be a closed convex bounded subset of X. If A is separable and dentable, then A is SCD.*

Proof. Since A separable, so is the set of its denting points, so we may find a countable collection of denting points $\{a_n : n \in \mathbb{N}\}$ of A which is dense in $\operatorname{dent}(A)$. Now, for every $n, m \in \mathbb{N}$, we consider a slice $S_{n,m}$ of A containing a_n and having diameter less than $1/m$. Then, the sequence $\{S_{n,m} : n, m \in \mathbb{N}\}$ is determining for A. Indeed, if $B \subseteq A$ intersects all the $S_{n,m}$, then $a_n \in \overline{B}$ for every $n \in \mathbb{N}$, so

$$A \subseteq \overline{\operatorname{conv}}\big(\operatorname{dent}(A)\big) = \overline{\operatorname{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\operatorname{conv}}(\overline{B}) = \overline{\operatorname{conv}}(B). \qquad \square$$

We recall that there is a concept of *Radon-Nikodým set* (defined in terms of vector measures) which is equivalent to dentability of all its closed convex bounded subsets (see [3, §5] or [6, §2]).

Example 2.9. *Let X be a Banach space and let A be a closed convex bounded separable Radon-Nikodým subset of X. Then, A is an SCD set.*

The norm $\|\cdot\|$ on a Banach space X is said to be *LUR* at $x_0 \in S_X$, if $\lim \|x_n - x_0\| = 0$ whenever $(x_n)_{n\in\mathbb{N}} \subseteq B_X$ is such that $\lim \|x_n + x_0\| = 2$. If the norm is LUR at each point of S_X, we say that X (or its norm) is *LUR* (see [11, Chapter II] for background). It is clear that every point in the unit sphere of a Banach space X with a LUR norm is denting so, in this case, B_X is dentable.

Example 2.10. *Let X be a separable Banach space with a LUR norm. Then, B_X is SCD.*

It is well known that every separable Banach space admits a LUR renorming (see [11, Theorem II.2.6.]). Therefore, the following result follows immediately from Proposition 2.8.

Example 2.11. *Every separable Banach space X admits an equivalent norm $|\cdot|$ such that $B_{(X,|\cdot|)}$ is an SCD set.*

Our second family of elementary examples of SCD sets deals with the so-called *Asplund property*, a concept related to differentiability of convex continuous functions, which can be equivalently reformulated in terms of separability and duality [6, §5]. A separable closed convex bounded subset A of a Banach space X has the Asplund property if and only if the semi-normed space (X^*, ρ_A) is separable, where

$$\rho_A(x^*) = \sup\{|x^*(a)| : a \in A\} \qquad (x^* \in X^*).$$

Of course, separable closed convex bounded subsets of Asplund spaces have the Asplund property.

Example 2.12. *Let X be a Banach space and let A be a closed convex bounded subset of X. If A is separable and has the Asplund property, then, A is SCD.*

Proof. We take a ρ_A-dense countable family $\{x_n^* : n \in \mathbb{N}\}$ in (X^*, ρ_A), and consider the slices

$$S_{n,m} = S(A, x_n^*, 1/m) \qquad (n, m \in \mathbb{N}).$$

We are done by just proving that if $\{v_{n,m} : n, m \in \mathbb{N}\}$ satisfies that $v_{n,m} \in S_{n,m}$ for every $n, m \in \mathbb{N}$, then

$$A \subseteq \overline{\operatorname{conv}}\left(\{v_{n,m} : n, m \in \mathbb{N}\}\right).$$

Indeed, suppose to the contrary that there are $a \in A$, $x^* \in X^*$, and $\delta > 0$ such that

$$\operatorname{Re} x^*(a) > \sup_{n,m} \operatorname{Re} x^*(v_{n,m}) + \delta.$$

Now, we may find $N \in \mathbb{N}$ such that $\rho_A(x_N^* - x^*) < \delta/2$ and so

$$\begin{aligned} \operatorname{Re} x_N^*(a) + \delta/2 > \operatorname{Re} x^*(a) &> \sup_{n,m} \operatorname{Re} x^*(v_{n,m}) + \delta \\ &\geqslant \sup_m \operatorname{Re} x^*(v_{N,m}) + \delta > \sup_m \operatorname{Re} x_N^*(v_{N,m}) + \delta/2 = \sup \operatorname{Re} x_N^*(A) + \delta/2, \end{aligned}$$

a contradiction. □

We now show that there are convex bounded subsets of separable Banach spaces which are not SCD.

Example 2.13. *Let X be a separable Banach space with the Daugavet property. Then, B_X is not an SCD set. In particular, $B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD sets.*

Proof. Fix $x_0 \in S_X$ and an arbitrary sequence of slices $(S_n)_{n\in\mathbb{N}}$. We will get the result by showing that there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in S_n$ for every $n \in \mathbb{N}$ and such that $x_0 \notin \overline{\operatorname{lin}\{x_n : n \in \mathbb{N}\}}$. To do so, we use [19, Lemma 2.8] which says, in particular, that for every finite-dimensional subspace $Y \subseteq X$, every $\varepsilon > 0$, and every slice S of B_X, there is an $x \in S$ such that

$$\|y + tx\| \geqslant (1 - \varepsilon)(\|y\| + |t|) \qquad \forall y \in Y.$$

Using this result, one can select inductively elements $x_n \in S_n$, $n \in \mathbb{N}$, in such a way, that

$$\|y + tx_n\| \geqslant \left(1 - \frac{1}{4^n}\right)(\|y\| + |t|) \qquad \left(y \in \operatorname{lin}\{x_k : k < n\}\right).$$

Then, $\{x_n : n = 0, 1, \ldots\}$ form a sequence equivalent to the unit vector basis of ℓ_1, so x_0 is not in the closure of $\operatorname{lin}\{x_n : n \in \mathbb{N}\}$, as desired. □

For the case of $C[0,1]$, it is possible to give a direct proof without using the Daugavet property, which we include here for the sake of completeness.

Example 2.14. *If K is an uncountable metrizable compact space, then the unit ball of $C(K)$ is not an SCD set.*

Proof. Let $\mathcal{M}$ be a maximal family of mutually orthogonal continuous measures in $C(K)^*$. This induces a decomposition of $C(K)^*$ as

$$C(K)^* = \left[\bigoplus_{\mu\in\mathcal{M}} L^1(\mu)\right]_{\ell_1} \oplus_1 \ell_1(K), \tag{1}$$

where $\ell_1(K)$ is the family of all discrete measures (see [1, pp. 84–85], for instance). As a consequence, we have

$$C(K)^{**} = \left[\bigoplus_{\mu\in\mathcal{M}} L^\infty(\mu)\right]_{\ell_\infty} \oplus_\infty \ell_\infty(K). \tag{2}$$

Let us write the slices of $B_{C[0,1]}$ in the form

$$U[\nu,\alpha] = \{x \in B_{C(K)} : \operatorname{Re}\nu(x) > \alpha\},$$

where $\nu \in C(K)^*$, $\|\nu\| = 1$, and $-1 < \alpha < 1$. Suppose, for the sake of contradiction, that there existed a countable family of slices $\mathcal{B}_0$ such that every other slice contains one from the family. Then, for every $\mu \in \mathcal{M}$, there exist $V_\mu = U[\nu_\mu,\alpha_\mu] \in \mathcal{B}_0$ such that $V_\mu \subseteq U[\mu,0]$.

Now, for each $\nu \in C(K)^*$ we write

$$\operatorname{Supp}_{\mathcal{M}}(\nu) = \{\mu \in \mathcal{M} : \mu \not\perp \nu\}.$$

Notice that this is a countable set which corresponds to the support of ν in the left-hand side of the decomposition (1). We claim that $\mu \in \operatorname{Supp}_{\mathcal{M}}(\nu_\mu)$ for every $\mu \in \mathcal{M}$. This leads to a contradiction with the facts that $\mathcal{B}_0$ and all the sets $\operatorname{Supp}_{\mathcal{M}}(\nu_\mu)$ are countable, while $\mathcal{M}$ is uncountable. Let us prove the claim. Suppose that $\mu \notin \operatorname{Supp}_{\mathcal{M}}(\nu_\mu)$ and let g be an element of the unit ball of $C(K)^{**}$ where ν_μ attains its norm. Consider $f \in L^\infty(\mu)$ the μ-coordinate of g when we view g as an element of the ℓ_∞-sum according to (2). Let now g' be the element of $C(K)^{**}$ obtained from g by changing the μ-coordinate from f to $-f$. This is a new element of the unit ball of $C(K)^{**}$ which satisfies that $g'(\mu) = -g(\mu)$ while $g'(\nu_\mu) = g(\nu_\mu) = 1$. Hence, for either $h = g$ or $h = g'$, we have an element h in the unit ball of $C(K)^{**}$ such that $h(\nu_\mu) = 1$ and $h(\mu) < 0$. Since the unit ball of $C(K)$ is dense in the unit ball of $C(K)^{**}$, it follows that $V_\mu \setminus U[\mu,0] \neq \emptyset$. □

Remark 2.15. *A subset of an SCD set is not necessarily SCD.* Indeed, let $X = C[0,1]$. By Example 2.11, there is an equivalent norm $|\cdot|$ on X such that $A = B_{(X,|\cdot|)}$ is SCD. Now, it is possible to find $\lambda > 0$ such that $C = \lambda\, B_{(X,\|\cdot\|_\infty)}$ is contained in A. Finally, C is not SCD by Example 2.13.

Our next goal is to extend the above preliminary examples to more intriguing ones. We will use several times the so-called *Bourgain's lemma* [4, Lemma 5.3] (it was rediscovered in [29]), so we state it for the sake of completeness. We refer the reader to [12, Lemma 7.3] for a reference easier to get. We recall that a *convex combination of slices* of a convex bounded subset A of a Banach space X is a subset of A of the form $\sum_{k=1}^{m} \lambda_i S_i$ where $\lambda_i > 0$, $\sum \lambda_i = 1$ and the S_i's are slices of A.

Lemma 2.16 (Bourgain's lemma)**.** *Let X be a Hausdorff locally convex space and let $K \subseteq X$ be closed bounded and convex. Then, every nonempty relatively weakly open subset of K contains a convex combination of slices.*

Remark 2.17. *The condition of closedness of the set in Bourgain's lemma can be omitted.* Indeed, let A be a convex bounded set and let U be a relatively weakly open subset of A. We denote by V a relatively weakly open subset of $\overline{A}$ such that $V \cap A = U$. By Bourgain's lemma, there are slices $S_1, S_2, \ldots, S_n$ of $\overline{A}$ and coefficients $\lambda_k > 0$ of a convex combination, such that $\sum_1^n \lambda_k S_k \subseteq V$. Then, $S_k \cap A$ are slices of A and $\sum_1^n \lambda_k S_k \cap A \subseteq V \cap A = U$.

The first consequence is an easy observation.

Proposition 2.18. *In the definition of SCD sets, instead of slices one can take convex combinations of slices. Hence, by Bourgain's lemma above, one can also take relatively weakly open subsets.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a determining sequence formed by convex combination of slices of A. Now, for every $n \in \mathbb{N}$, there exists a collection of slices $\{S_{n,m} : m = 1,\dots k_n\}$ and positive numbers $\{\lambda_{n,m} : m = 1,\dots k_n\}$ with $\sum_{m=1}^{k_n} \lambda_{n,m} = 1$, such that $\sum_{m=1}^{k_n} \lambda_{n,m} S_{n,m} \subseteq V_n$. Then, the collection of slices $\{S_{n,m} : n \in \mathbb{N},\ 1 \leqslant m \leqslant k_n\}$ is determining for A. Indeed, let B be a subset of A such that $B \cap S_{n,m} \neq \emptyset$ for all n, m, and consider $b_{n,m} \in B \cap S_{n,m}$ for every n, m. If we take $a_n = \sum_{m=1}^{k_n} \lambda_{n,m} b_{n,m}$, it is clear that $a_n \in \operatorname{conv}(B) \cap V_n$. So we know that $\operatorname{conv}(B) \cap V_n \neq \emptyset$ for all n, which by the assumption gives us that $\overline{\operatorname{conv}}(B) \supseteq A$.

Finally, if A has a determining sequence of relatively weakly open subsets $\{V_n : n \in \mathbb{N}\}$, Bourgain's lemma allows us to find convex combinations of slices inside the V_n's and the proof above shows that A is SCD. □

The first consequence of this result is that Proposition 2.8 can be extended from dentable sets to *huskable* sets (the same definition with relatively weakly open sets instead of slices). With not much work, we are going to extend the result to the following more general setting. A closed convex bounded subset A of a Banach space X has *small combinations of slices* [14, 26] if every slice of A contains convex combinations of slices of A with arbitrarily small diameter.

Theorem 2.19. *Let X be a Banach space and let A be a separable closed convex bounded subset of X having small combinations of slices. Then, A is an SCD set.*

Proof. By [14, Corollary III.7], for every $x \in A$ and every $\varepsilon > 0$, there is a convex combination of slices of A contained in $B(x, \varepsilon)$. Now, we take a countable dense subset $\{x_n : n \in \mathbb{N}\}$ of A and for $(n, m) \in \mathbb{N} \times \mathbb{N}$, we take $V_{n,m}$ a convex combination of slices of A contained in $B(x_n, 1/m)$. Then, if $B \subseteq A$ intersects all the $V_{n,m}$, it intersects also all the balls $B(x_n, 1/m)$. Therefore, the set $\{x_n : n \in \mathbb{N}\}$ is contained in $\overline{B}$ and so, $A = \overline{\operatorname{conv}}(B)$. Finally, Proposition 2.18 gives us that A is SCD. □

RNP sets have small combinations of slices, so the above result extends Example 2.9. Even more, strongly regular sets (in particular, huskable sets, CPCP sets) have small combinations of slices [14, Proposition III.5]. We recall that a closed convex bounded subset A of a Banach space is said to be *strongly regular* if every non-empty convex subset L of A contains a convex combination of slices of L of arbitrarily small diameter. A has the *convex point of continuity property* (*CPCP* in short) if every convex closed subset B of A contains a weak-to-norm point of continuity of the identity mapping. In this case, for every convex subset B of A and for every $\varepsilon > 0$, there is a relatively weakly open subset $C \subseteq B$ with $\operatorname{diam}(C) < \varepsilon$ [5].

Corollary 2.20. *Let X be a Banach space and let A be a closed convex bounded subset of X. If A is separable and strongly regular, then A is SCD. In particular, separable CPCP sets are SCD.*

Our next aim is to extend Example 2.12 to sets which do not contain ℓ_1 sequences. We need the following topological definition. By a *π-base* of a topological space (T, τ) we understand a family $\{O_i : i \in I\}$ of nonempty open sets such that every nonempty open subset O of T contains one of the elements of the family. The following result is another consequence of Bourgain's lemma.

Proposition 2.21. *Let X be a Banach space and let A be a convex bounded subset of X. If $(A, \sigma(X, X^*))$ has a countable π-base, then A is an SCD set.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a countable π-base of $(A, \sigma(X, X^*))$. Since slices of A have non-empty weak interior, any of them contains some of the V_n. But then, Proposition 2.2 shows that the sequence $\{V_n\}$ is determining for A and Proposition 2.18 gives that A is SCD. □

The main consequence of the above proposition is the following. We recall that an ℓ_1-*sequence* of a Banach space is just a bounded sequence which is equivalent to the natural basis of ℓ_1

Theorem 2.22. *Let X be a Banach space and let A be a separable convex bounded subset of X which contains no ℓ_1-sequences. Then, $(A, \sigma(X, X^*))$ has a countable π-base. In particular, A is an SCD set.*

Proof. By [12, Theorem 3.11], $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on $(B_{X^*}, \sigma(X^*, X))$, and we can apply [30, Lemma 4] by Todorčević, to deduce that $(A, \sigma(X, X^*))$ has a σ-disjoint π-base (i.e. a π-base $\{V_i : i \in I\}$ such that $I = \bigcup_{n\in\mathbb{N}} I_n$ and each subfamily $\{V_i : i \in I_n\}$ is a pairwise disjoint family). Now, it is clear that a σ-disjoint family of open subsets in a separable space has to be countable. Finally, A is SCD by Proposition 2.21. □

This result obviously extends Example 2.12 since Asplund sets cannot contain ℓ_1-sequences.

3. Slicely Countably Determined spaces

Definition 3.1. A separable Banach space X is said to be *slicely countably determined* (*SCD space* in short) if every convex bounded subset of X is an SCD set.

By just using the results of the previous section on SCD sets, we get the main examples of SCD spaces.

Examples 3.2.

(a) *If X is a separable strongly regular space, then X is SCD. In particular, RNP spaces (more generally, CPCP spaces) are SCD.*
(b) *Separable spaces which do not contain copies of ℓ_1 are SCD. In particular, if X^* is separable, then X is SCD.*
(c) *Both families include reflexive separable spaces, which are then SCD spaces.*

With respect to spaces which are not SCD, we only know of the Daugavet spaces.

Examples 3.3.

(a) *If X is a separable Banach space which admits an equivalent renorming with the Daugavet property, then X is not SCD.*
(b) *In particular, there is a Banach space with the Schur property which is not an SCD space.* Indeed, in [21] the existence of a separable space having the Schur property and the Daugavet property at the same time was proved.

Let us state the following immediate observations.

Remarks 3.4.

(a) Every subspace of an SCD space is SCD.
(b) For quotients the situation is different. For instance, $C[0,1]$ is a non-SCD quotient of the SCD space ℓ_1.

Our next aim is to show some stability results for the SCD spaces. The first one is a "three space property". We need the following technical lemma which shows that in Definition 3.1 it suffices to consider sets with nonempty interior.

Lemma 3.5. *Let X be a separable Banach space. If every* open *convex bounded subset of X is SCD, then X is SCD.*

Proof. Our first observation is that our hypothesis forces that every bounded convex subset A of X with nonempty interior is SCD. Indeed, notice that since A is convex, the closure of the interior of A coincides with the closure of A, and we may apply Remark 2.7 two times to get that A is SCD.

Now, let $A \subseteq X$ be bounded and convex. Since X is separable, we may find a sequence $\{x_n : n \in \mathbb{N}\} \subseteq A$ which is dense in A. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive reals which tends to zero. For every $n, m \in \mathbb{N}$ fixed, we denote $A_{n,m} = \operatorname{conv}\big(B_{\varepsilon_m}(x_n) \cup A\big)$ which clearly contains A. Since the interior of $A_{n,m}$ is not empty, we may find a determining sequence $\{S^k_{n,m} : k \in \mathbb{N}\}$ of slices of $A_{n,m}$. Now, from the structure of $A_{n,m}$, it follows that either $S^k_{n,m} \cap B_{\varepsilon_m}(x_n) \neq \emptyset$, or $S^k_{n,m} \cap A \neq \emptyset$. Let $K_{n,m}$ be the set of all indices $k \in \mathbb{N}$ for which $S^k_{n,m}$ intersects A, and denote $\widetilde{S}^k_{n,m} = S^k_{n,m} \cap A$ for all $k \in K_{n,m}$, which are clearly slices of A. Also note that for every integer $k \notin K_{n,m}$, the slice $S^k_{n,m}$ intersects $B_{\varepsilon_m}(x_n)$. Finally, the family

$$\left\{\widetilde{S}^k_{n,m} : n, m \in \mathbb{N},\ k \in K_{n,m}\right\}$$

is determining for A. Indeed, let B be a subset of A intersecting all the $\widetilde{S}^k_{n,m}$ and fix some $\varepsilon > 0$. Since the sequence $\{x_n : n \in \mathbb{N}\}$ is dense in A, there is an integer $n_0 \in \mathbb{N}$ and $b \in B$ such that $\|b - x_{n_0}\| \leqslant \frac{\varepsilon}{2}$. Also, there is $m_0 \in \mathbb{N}$ such that $\varepsilon_{m_0} \leqslant \frac{\varepsilon}{2}$, as $\varepsilon_m \to 0$ when $m \to \infty$. We know that B intersects all $S^k_{n_0,m_0}$ with $k \in K_{n,m}$. On the other hand, we also know that the slice $S^k_{n_0,m_0}$ intersects the ball $B_{\varepsilon_{m_0}}(x_{n_0})$ for every $k \notin K_{n,m}$. Hence we can deduce that the set $B_{n_0,m_0} = B \cup B_{\varepsilon_{m_0}}(x_{n_0}) \subseteq A_{n,m}$ intersects all the $S^k_{n_0,m_0}$ which implies that

$$\overline{\operatorname{conv}}\big(B_{n_0,m_0}\big) \supseteq A_{n_0,m_0} \supseteq A.$$

Finally, notice that $B_{\varepsilon_{m_0}}(x_{n_0}) \subseteq B_{\frac{\varepsilon}{2}}(x_{n_0}) \subseteq B_\varepsilon(b)$, which implies that $B_{n_0,m_0} \subseteq B + \varepsilon B_X$. Therefore, we can state that $\overline{\operatorname{conv}}\big(B + \varepsilon B_X\big) \supseteq A$, and the arbitrariness of ε gives us that $\overline{\operatorname{conv}}(B) \supseteq A$. □

We may now state the promised stability result.

Theorem 3.6. *Let X be a Banach space with a subspace Z such that Z and $Y = X/Z$ are SCD spaces. Then, X is also an SCD space.*

Proof. We denote $q : X \longrightarrow Y = X/Z$ the quotient map. Let us show that every *open* convex bounded subset $A \subseteq X$ is SCD, and then Lemma 3.5 will imply that X is SCD. To do so, as X is separable since Y and Z are, and separability is a three-space property (see [9, Theorem 2.4.h]), we only need to find, for every point $a \in A$, a sequence of weakly open subsets such that whenever $B \subseteq A$ intersects every member of the sequence, then $a \in \overline{\operatorname{conv}}(B)$ (see Remark 2.4). We fix some $a \in A$ and denote $A_a = \{x \in A : q(x) = q(a)\}$. Then, A_a is affine isomorphic to an open convex bounded subset of Z which is an SCD space (indeed, $A_a = (Z + a) \cap A$). It follows that there is a determining sequence $\{S_n\}$ of slices of A_a. Let $\{\widetilde{S}_n\}$ be their extensions to A. For every $n \in \mathbb{N}$, consider $q(\widetilde{S}_n) \subseteq Y$, which is open bounded and convex (its openness is a consequence of the Open Mapping Theorem). Now, as long as Y is SCD, we may find a determining sequence $\{S_{n,m} : m \in \mathbb{N}\}$ of slices of $q(\widetilde{S}_n)$. Let $V_{n,m} = \widetilde{S}_n \cap q^{-1}(S_{n,m})$ for every $n, m \in \mathbb{N}$. It is easy to see that $V_{n,m}$ are relatively weakly open. We will now prove that they are the sets we need.

Let $B \subseteq A$ be convex and such that $B \cap V_{n,m} \neq \emptyset$ for all $n, m \in \mathbb{N}$. Fix some $\varepsilon > 0$, and denote $B_\varepsilon = \{x \in A : \operatorname{dist}(x, B) < \varepsilon\}$. Evidently, B_ε is an open convex set intersecting all the $V_{n,m}$. Fixed $n \in \mathbb{N}$, we have that

$$B_\varepsilon \cap V_{n,m} = B_\varepsilon \cap \widetilde{S}_n \cap q^{-1}(S_{n,m}) \neq \emptyset,$$

so

$$q\big(B_\varepsilon \cap \widetilde{S}_n\big) \cap S_{n,m} \neq \emptyset$$

and the choice of $S_{n,m}$ allows us to get that

$$\overline{\operatorname{conv}}\big(q(B_\varepsilon \cap \widetilde{S}_n)\big) = \overline{q\big(B_\varepsilon \cap \widetilde{S}_n\big)} \supseteq q(\widetilde{S}_n).$$

Notice that $B_\varepsilon \cap \widetilde{S}_n$ is open and convex, hence, so is $q(B_\varepsilon \cap \widetilde{S}_n)$. This implies that the interior of the set $\overline{q(B_\varepsilon \cap \widetilde{S}_n)}$ coincides with $q(B_\varepsilon \cap \widetilde{S}_n)$. Now, using that $q(\widetilde{S}_n)$ is open, we get that

$$q\big(B_\varepsilon \cap \widetilde{S}_n\big) \supseteq q(\widetilde{S}_n)$$

and, in particular, $q(B_\varepsilon \cap \widetilde{S}_n) \ni q(a)$. This means that there exists $x_n \in B_\varepsilon \cap \widetilde{S}_n$, such that $q(x_n) = q(a)$, i.e. that $x_n \in B_\varepsilon \cap S_n$. Since $B_\varepsilon \subseteq A$ and $\{S_n\}$ is a determining sequence for A_a, we get that $B_\varepsilon \supseteq A_a$. Finally, the arbitrariness of ε implies that $\overline{B} \supseteq A_a \ni a$. □

Let us state two immediate consequences of this result.

Corollary 3.7. *Let X be a separable Banach space which is not SCD.*

(a) *X contains copies of ℓ_1, and the quotient of X over any copy of ℓ_1 also contains ℓ_1.*
(b) *Consequently, for every ℓ_1 subspace Y_1 of X, there is another ℓ_1 subspace Y_2 such that Y_1 and Y_2 are mutually complemented in the closed linear span of $Y_1 + Y_2$ (i.e. $\overline{Y_1 + Y_2} = Y_1 + Y_2 = Y_1 \oplus Y_2$). In particular, $Y_1 \cap Y_2 = 0$.*

Proof. (a) is immediate from the above theorem and Theorem 2.22. (b) follows from (a) and the "lifting" property of ℓ_1 [22, Proposition 2.f.7]. □

One may wonder whether item (b) of the above corollary can actually be a characterization of those separable Banach spaces which are not SCD. This is not the case as the following remark shows.

Remark 3.8. *The space $X = \ell_2(\ell_1)$ (which is an SCD space, even more it has the RNP) has the following property: it contains isomorphic copies of ℓ_1 and for every ℓ_1 subspace $Y \subseteq X$, there is another ℓ_1 subspace $Z \subseteq X$, such that Z and Y are mutually complemented in the closed linear span of $Y + Z$.*

Proof. Let $\{X_n\}_{n=1}^\infty$ be a sequence of isometric copies of ℓ_1. Then, X is isometric to the ℓ_2 direct sum of the spaces X_n, $[\bigoplus_{n\in\mathbb{N}} X_n]_{\ell_2}$. Fix an ℓ_1-subspace $Y \subseteq X$ and let us prove that some of the X_n can be taken as Z. Assume to the contrary that for every $n \in \mathbb{N}$

$$\inf\{\|y - x\| : y \in S_Y, x \in X_n\} = 0.$$

Then, for every $n \in \mathbb{N}$ there are $y_n \in S_Y$ and $x_n \in X_n$ with $\|y_n - x_n\| < 10^{-n}$. Since (x_n) forms a bounded sequence of disjoint elements, $(x_n) \longrightarrow 0$ in the weak topology. But then $(y_n) \longrightarrow 0$ in the weak topology as well, which is impossible since $(y_n) \subseteq S_Y$ and Y has the Schur property. □

Corollary 3.9. *Let $X_1, \ldots, X_n$ be SCD Banach spaces. Then, $X_1 \oplus \cdots \oplus X_n$ is SCD.*

Our next goal is to deal with infinite sums. To do so, we need to recall the concept of unconditional sums. Given a sequence $\{(X_n, \|\cdot\|_n) : n \in \mathbb{N}\}$ of Banach spaces, and a Banach space E of sequences whose norm satisfies

$$\|(t_i)\|_E = \|(|t_i|)\|_E \qquad \big((t_i) \in E\big),$$

we denote by $\big[\bigoplus_{n\in\mathbb{N}} X_n\big]_E$ the Banach space of all sequences $(x_n) \in \prod_{n=1}^{\infty} X_n$, so that

$$\|(x_n)\| = \|(\|x_n\|_n)\|_E < \infty.$$

Theorem 3.10. *Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of SCD spaces and let E be a Banach space of sequences whose canonical basis is a 1-unconditional and shrinking basis (i.e. E does not contain copies of ℓ_1). Then, $X = \big[\bigoplus_{n\in\mathbb{N}} X_n\big]_E$ is also an SCD space.*

Proof. For every $m \in \mathbb{N}$, we denote

$$Y_m = \big[X_1 \oplus X_2 \oplus \ldots \oplus X_m \oplus 0 \oplus 0 \oplus \cdots\big]_E \subseteq X$$

and let $P_m : X \longrightarrow Y_m$ be the natural projection. Let A be a convex bounded subset of X. Now, for every $m \in \mathbb{N}$, $P_m(A)$ is a convex bounded subset of Y_m, which is an SCD space by Corollary 3.9. Hence, there is a determining sequence $\{S_{m,k} : k \in \mathbb{N}\}$ of slices of $P_m(A)$. Consider $\widetilde{S}_{m,k} = P_m^{-1}(S_{m,k}) \cap A$. We will prove that $\{\widetilde{S}_{m,k} : k, m \in \mathbb{N}\}$ is a determining countable collection of slices of A.

Let B be a subset of A intersecting all the $\widetilde{S}_{m,k}$. We fix an arbitrary point $a \in A$ and we will prove that $a \in \overline{\mathrm{conv}}(B)$. Since B intersects all the $\widetilde{S}_{m,k}$, $P_m(B)$ intersects $S_{m,k}$ for every integer k. It follows that $\overline{\mathrm{conv}}\big(P_m(B)\big) \supseteq P_m(A)$. In particular, $\overline{\mathrm{conv}}\big(P_m(B)\big) \ni P_m(a)$. That means that there exists $b_m \in \mathrm{conv}B$ such that $\|P_m(b_m - a)\| < \frac{1}{m}$. Then, it is easy to see that b_m tends to a coordinate-wise. But since the canonical basis of E is at the same time a shrinking basis, we get that b_m tends to a in the weak topology. So we can apply Mazur's theorem and get a sequence $\{b'_m\}$ with $b'_m \in \mathrm{conv}\big(\{b_k : k \geqslant m\}\big) \subseteq \mathrm{conv}(B)$ which tends to a in the norm topology. But this exactly means that $a \in \overline{\mathrm{conv}}(B)$, which was to be proved. □

The next result deals with unconditional sums when the natural basis of E is boundedly complete. Its proof, which is more bulky than the above one, needs a preliminary result which can be of independent interest.

Let X be a Banach space, A be a convex set in X and ε be a positive real. A point $a \in A$ is called an *ε-accessible point* of A if there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relatively weakly-open subsets of A, such that for every $B \subseteq A$, if B intersects all the V_n, then $\mathrm{dist}(a, \mathrm{conv}B) < \varepsilon$.

Lemma 3.11. *Let X be a Banach space and let A be a separable convex bounded subset of X. Suppose that for every convex $C \subseteq A$ and every $\varepsilon > 0$, there is an ε-accessible point in C. Then, A is an SCD set.*

Proof. Notice that, since A is separable, to prove this lemma it is enough to show that for every $\varepsilon > 0$, the set A_ε of ε-accessible points of A is dense in A. Since A_ε is convex, it is enough to show that A_ε is weakly dense in A. Fix some convex relatively weakly-open subset $V \subseteq A$. By the assumption, there is an ε-accessible point of V. But this point is also an ε-accessible point of A since V is relatively weakly-open. □

We are now able to state and prove the second result for unconditional sums.

Theorem 3.12. *Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of SCD spaces and let E be a space of sequences whose natural basis is a 1-unconditional and boundedly complete basis (i.e. E does not contain isomorphic copies of c_0). Then, $X = [\bigoplus_{n\in\mathbb{N}} X_n]_E$ is an SCD space.*

Proof. Let a convex bounded subset A of X and $\varepsilon > 0$ be fixed. Consider the subset

$$A_E = \big\{(a_n)_{n\in\mathbb{N}} \in E : \exists x = (x_n)_{n\in\mathbb{N}} \in A \text{ with } \|x_n\| = |a_n| \text{ for all } n \in \mathbb{N}\big\}.$$

Since A_E is a bounded subset of a space with the RNP, there are a functional $b = (b_n)_{n\in\mathbb{N}} \in E^*$ and a positive number α such that the slice

$$S(A_E) = \big\{(a_n)_{n\in\mathbb{N}} \in A_E : \sum_{n\in\mathbb{N}} b_n a_n > \alpha\big\}$$

has diameter smaller than $\varepsilon/4$. Taking into account that A_E is symmetric, we may assume that $b_n \geqslant 0$ (the slice of A_E defined by $|b| = (|b_n|)_{n\in\mathbb{N}}$ is isometric to $S(A_E)$). Fix an $x \in A$ with $(\|x_n\|)_{n\in\mathbb{N}} \in S(A_E)$ and pick $x_n^* \in S_{X_n^*}$ such that $x_n^*(x_n) = \|x_n\|$. Write $f_n = b_n x_n^*$, $f = (f_n)_{n\in\mathbb{N}} \in X^*$. We claim that for the slice

$$S = \left\{(x_n)_{n\in\mathbb{N}} \in A : \sum_{n\in\mathbb{N}} f_n(x_n) > \alpha\right\}$$

there is an $m \in \mathbb{N}$ with the following property

$$\big\|(0,\dots,0,y_{m+1},y_{m+2},\dots)\big\| < \frac{\varepsilon}{2} \qquad \text{for all } (y_n)_{n\in\mathbb{N}} \in S. \tag{3}$$

To show this, it is sufficient to select m in such a way that $\|(0,\dots,0,x_{m+1},x_{m+2},\dots)\| < \varepsilon/4$ and to use that $\operatorname{diam} S(A_E) < \varepsilon/4$. In fact, with such a choice of m we get

$$\begin{aligned}
\|(0,\dots,0,y_{m+1},y_{m+2},\dots)\| &= \|(0,\dots,0,\|y_{m+1}\|,\|y_{m+2}\|,\dots)\| \\
&\leqslant \big\|(0,\dots,0,\|x_{m+1}\|,\|x_{m+2}\|,\dots)\big\| + \\
&\qquad + \big\|(0,\dots,0,\big|\|x_{m+1}\| - \|y_{m+1}\|\big|,\big|\|x_{m+2}\| - \|y_{m+2}\|\big|,\dots)\big\| \\
&\leqslant \frac{\varepsilon}{4} + \big\|\big(\big|\|x_1\| - \|y_1\|\big|,\big|\|x_2\| - \|y_2\|\big|,\dots\big)\big\| \leqslant \frac{\varepsilon}{2}.
\end{aligned}$$

Let us prove that x is an ε-accessible point of A. Consider

$$Y_m = \big[X_1 \oplus X_2 \oplus \ldots \oplus X_m \oplus 0 \oplus 0 \oplus \cdots\big]_E \subseteq X$$

and $P_m : X \longrightarrow Y_m$ the natural projection. By Corollary 3.9, Y_m is an SCD space and, since $P_m(S)$ is a convex bounded set in Y_m, there exists a determining sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $P_m(S)$. Notice that Y_m^* isometrically embeds into X^*. For every integer $n \in \mathbb{N}$, we consider $\widetilde{S}_n = P_m^{-1}S_n \cap S$, which is a slice of S and, obviously, relatively weakly-open in A. Let B be a subset of A which intersects all the $\widetilde{S}_n$. We'll now prove that then $\operatorname{dist}\big(x, \operatorname{conv}(B)\big) < \varepsilon$. Since B intersects all the $\widetilde{S}_n$, we can find a sequence $\{y_n\} \subseteq B$, such that $y_n \in \widetilde{S}_n$ for every $n \in \mathbb{N}$. This implies that $P_m(y_n) \in S_n$ for all $n \in \mathbb{N}$ and so $\overline{\operatorname{conv}}\big(\{P_m(y_n) : n \in \mathbb{N}\}\big) \supseteq P_m(S)$. In particular, $P_m(x) \in \overline{\operatorname{conv}}\big(\{P_m(y_n) : n \in \mathbb{N}\}\big)$. But (3) gives us that the m-th tails of x and of all the y_n are small, that is,

$$\|x - P_m(x)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|y_n - P_m(y_n)\| < \varepsilon/2 \qquad \big(\text{for all } n \in \mathbb{N}\big).$$

This gives us that $\operatorname{dist}\big(a, \operatorname{conv}(B)\big) < \varepsilon$ and the proof is complete. □

An immediate consequence is the following.

Example 3.13. *The spaces $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.*

This result, together with those results of section 2, gives us the following examples.

Example 3.14. *The spaces $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.* Indeed, it is well known that $c_0 \otimes_\varepsilon c_0 \equiv c_0$, $c_0 \otimes_\varepsilon \ell_1 \equiv c_0(\ell_1)$, $c_0 \otimes_\pi \ell_1 \equiv \ell_1(c_0)$, and $\ell_1 \otimes_\pi \ell_1 \equiv \ell_1$ (see [27, Examples 2.19 and 3.3], for instance), so these cases are clear from the above example. For the remaining cases, just observe that $\big[c_0 \otimes_\pi c_0\big]^* \equiv \ell_1 \otimes_\varepsilon \ell_1$ (since $\big[c_0 \otimes_\pi c_0\big]^* \equiv L(c_0, \ell_1)$ [27, p. 24], $K(c_0, \ell_1) \equiv \ell_1 \otimes_\varepsilon \ell_1$ [27, Corollary 4.13] and $K(c_0, \ell_1) = L(c_0, \ell_1)$ since ℓ_1 has the Schur property and c_0^* is separable), so $c_0 \otimes_\pi c_0$ is Asplund and $\ell_1 \otimes_\varepsilon \ell_1$ has the RNP.

Since for X and Y being c_0 or ℓ_1 one has $K(X, Y) \equiv X^* \otimes_\varepsilon Y$ [27, Corollary 4.13], the following examples follow.

Example 3.15. *The spaces $K(c_0)$ and $K(c_0, \ell_1)$ are SCD. The spaces $K(\ell_1)$ and $K(\ell_1, c_0)$ contain ℓ_∞ and so they are not separable, all the more not SCD.*

Another example in this line is the following.

Example 3.16. *The spaces $\ell_2 \otimes_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$, and $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ are SCD.* Indeed, the first space has the RNP and the second is an Asplund space.

4. An application to spaces with numerical index 1

Our aim in this section is to show that SCD spaces with the alternative Daugavet property are lush. To get such a result, we need to establish a characterization of the alternative Daugavet property which can be of independent interest. We first recall a previous characterization in terms of slices.

Lemma 4.1 ([25, Proposition 2.1])**.** *A Banach space X has the alternative Daugavet property if and only if for every $x \in S_X$, every $\varepsilon > 0$ and every slice S of B_X, there is a $y \in S$ such that $\max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon$.*

We need some notation. Denote $K(X^*)$ the weak*-closure in X^* of $\operatorname{ext}(B_{X^*})$, and for every slice S of B_X and every $\varepsilon > 0$, we write

$$\begin{aligned} D(S, \varepsilon) &= \big\{y^* \in K(X^*) \,:\, S \cap \mathbb{T} S(B_X, y^*, \varepsilon) \neq \emptyset\big\} \\ &= \big\{y^* \in K(X^*) \,:\, S \cap \overline{\operatorname{aconv}}\big(S(B_X, y^*, \varepsilon)\big) \neq \emptyset\big\}, \end{aligned}$$

which is relatively weak*-open in $K(X^*)$. Here is the promised characterization of the alternative Daugavet property.

Proposition 4.2. *For a Banach space X, the following assertions are equivalent:*

(i) *X has the alternative Daugavet property.*
(ii) *For every $x \in S_X$, every $\varepsilon > 0$ and every slice $S \subseteq B_X$, there is $y^* \in K(X^*)$ such that $x \in S(B_X, y^*, \varepsilon)$ and $S \cap \mathbb{T} S(B_X, y^*, \varepsilon) \neq \emptyset$.*
(iii) *For every $x \in S_X$, every $\varepsilon > 0$ and every slice $S \subseteq B_X$, there is $y^* \in D(S, \varepsilon)$ such that $x \in S(B_X, y^*, \varepsilon)$.*
(iv) *For every $\varepsilon > 0$ and every slice $S \subseteq B_X$, the set $D(S, \varepsilon)$ is weak*-dense in $K(X^*)$.*
(v) *For every $\varepsilon > 0$ and every sequence $\{S_n \,:\, n \in \mathbb{N}\}$ of slices of B_X, the set $\bigcap_{n \in \mathbb{N}} D(S_n, \varepsilon)$ is weak*-dense in $K(X^*)$.*

Proof. The implications (i) $\iff$ (ii) $\iff$ (iii) are easy consequences of Lemma 4.1.

(iii) $\implies$ (iv). To show weak*-density of $D(S, \varepsilon)$ in $K(X^*)$ it is sufficient to demonstrate that the weak* closure of $D(S, \varepsilon)$ contains every extreme point x^* of S_{X^*}. Since weak*-slices form a

base of neighborhoods of x^* in B_{X^*}, it is sufficient to prove that every weak*-slice $S(B_{X^*}, x, \delta)$ with $\delta \in (0, \varepsilon)$ intersects $D(S, \varepsilon)$, i.e. that there is a point $y^* \in D(S, \varepsilon)$, such that $y^* \in S(B_{X^*}, x, \delta)$. But we know that there is a point $y^* \in D(S, \delta) \subseteq D(S, \varepsilon)$, such that $x \in S(B_X, y^*, \delta)$, which means that $y^* \in S(B_{X^*}, x, \delta)$.

(iv) $\Longrightarrow$ (iii). If $D(S, \varepsilon)$ is weak*-dense in $K(X^*)$, then for every $x \in S_X$ there is a $y^* \in D(S, \varepsilon)$ such that $x \in S(B_X, y^*, \varepsilon)$.

The remaining equivalence (iv) $\Longleftrightarrow$ (v) follows from the fact that $D(S, \varepsilon)$ is not only weak*-dense but also weak*-open, and $K(X^*)$ is weak*-compact, so Baire's theorem is applicable. □

It is possible to give a result analogous to the above one for the Daugavet property. We need to change a little bit the notation. For every slice S of B_X and every $\varepsilon > 0$, we write

$$\begin{aligned} \widetilde{D}(S, \varepsilon) &= \{y^* \in K(X^*) \,:\, S \cap S(B_X, y^*, \varepsilon) \neq \emptyset\} \\ &= \{y^* \in K(X^*) \,:\, S \cap \overline{\operatorname{conv}}\big(S(B_X, y^*, \varepsilon)\big) \neq \emptyset\} \end{aligned}$$

which is relatively weak*-open in $K(X^*)$. The proof of the next result is almost the same as the above one, replacing Lemma 4.1 by [19, Lemma 2.2]. We include it here for future use.

Proposition 4.3. *For a Banach space X, the following assertions are equivalent:*

(i) *X has the Daugavet property.*
(ii) *For every $x \in S_X$, every $\varepsilon > 0$ and every slice $S \subseteq B_X$, there is $y^* \in K(X^*)$ such that $x \in S(B_X, y^*, \varepsilon)$ and $S \cap S(B_X, y^*, \varepsilon) \neq \emptyset$.*
(iii) *For every $x \in S_X$, every $\varepsilon > 0$ and every slice $S \subseteq B_X$, there is $y^* \in \widetilde{D}(S, \varepsilon)$ such that $x \in S(B_X, y^*, \varepsilon)$.*
(iv) *For every $\varepsilon > 0$ and every slice $S \subseteq B_X$, the set $\widetilde{D}(S, \varepsilon)$ is weak*-dense in $K(X^*)$.*
(v) *For every $\varepsilon > 0$ and every sequence $\{S_n \,:\, n \in \mathbb{N}\}$ of slices of B_X, the set $\bigcap_{n\in\mathbb{N}} \widetilde{D}(S_n, \varepsilon)$ is weak*-dense in $K(X^*)$.*

We are now ready to show the main result of this section.

Theorem 4.4. *Every Banach space X with the alternative Daugavet property whose unit ball is an SCD set is lush. In particular, every SCD space with the alternative Daugavet property is lush.*

Proof. Let $\{S_n \,:\, n \in \mathbb{N}\}$ be the sequence of slices of B_X from the definition of an SCD set. Then, by Proposition 4.2.v, for every $\varepsilon > 0$ the set $\bigcap_{n\in\mathbb{N}} D(S_n, \varepsilon)$ is weak*-dense in $K(X^*)$. So, for every $x \in S_X$ there is $y^* \in \bigcap_{n\in\mathbb{N}} D(S_n, \varepsilon)$ such that $x \in S(B_X, y^*, \varepsilon)$. According to the definition of $D(S_n, \varepsilon)$, this means that $S_n \cap \overline{\operatorname{aconv}}\big(S(B_X, y^*, \varepsilon)\big) \neq \emptyset$ for all $n \in \mathbb{N}$. Then, we obtain that $\overline{\operatorname{aconv}}\big(S(B_X, y^*, \varepsilon)\big) = B_X$, which implies lushness of X [7, Theorem 2.1]. □

Remark 4.5. Let us observe that in the above proof a (formally) weaker version of an SCD set is used. A convex bounded subset A of a Banach space X is said to be *almost slicely countably determined* (*almost-SCD* in short) if there is a sequence $\{V_n \,:\, n \in \mathbb{N}\}$ of subsets of A such that for every $B \subseteq A$ intersecting all the V_n, one has $\overline{\operatorname{aconv}}(B) \supseteq A$. The proof of the above theorem actually shows that *every Banach space X with the alternative Daugavet property whose unit ball is an almost-SCD is lush.*

Theorem 4.4 has already been known for Asplund spaces and for spaces with the RNP [23, Remark 6], regardless of the separability (necessary for the SCD and so for our result). Our next goal is to particularize Theorem 4.4 to more cases where we are able to remove the separability. The proof of the following results is a consequence of the facts that lushness and the alternative Daugavet property

are separably determined (see [7, Theorem 4.2] for the first case and the remark below for the second one).

Remark 4.6. It is shown in [20, Theorem 4.5] that the Daugavet property is separably determined. With a little effort, the proof can be adapted to the alternative Daugavet property: *A Banach space X has the alternative Daugavet property if and only if for every separable subspace $Y \subseteq X$ there is a separable subspace $Z \subseteq X$ which contains Y and has the alternative Daugavet property.*

Corollary 4.7. *Let X be a Banach space with the alternative Daugavet property. If X is strongly regular (in particular, CPCP), then X is lush.*

Corollary 4.8. *Let X be a Banach space with the alternative Daugavet property. If X does not contain ℓ_1, then X is lush.*

This latter result solves in the positive Problem 32 of [18] and it can be used to prove a necessary isomorphic condition for a real Banach space to have the alternative Daugavet property.

Corollary 4.9. *let X be an infinite-dimensional real Banach space with the alternative Daugavet property. Then, X^* contains ℓ_1.*

Proof. If X contains ℓ_1, then X^* contains a quotient isomorphic to ℓ_∞, so X^* contains ℓ_1 as a quotient and the "lifting" property of ℓ_1 [22, Proposition 2.f.7] gives us $X^* \supseteq \ell_1$. Otherwise, Corollary 4.8 gives us that X is lush. But the dual of an infinite-dimensional real lush space contains ℓ_1 [16, Corollary 4.8]. □

In particular, since Banach spaces with numerical index 1 have the alternative Daugavet property, we get the following corollary which answers in the positive Problem 18 of [18].

Corollary 4.10. *Let X be an infinite-dimensional real Banach space with $n(X) = 1$. Then, $X^* \supseteq \ell_1$.*

Let us comment that very recently it has been shown that there are Banach spaces with numerical index 1 which are not lush [17], so the above result is not covered by [16, Corollary 4.9].

5. SCD OPERATORS

Definition 5.1. Let X and Y be Banach spaces. A bounded linear operator $T : X \longrightarrow Y$ is said to be an *SCD-operator* if $T(B_X)$ is an SCD set.

By just recalling the examples of SCD sets and SCD spaces given in sections 2 and 3, we get the main examples of SCD-operators.

Examples 5.2. Let X and Y be Banach spaces and let $T : X \longrightarrow Y$ be a bounded linear operator such that $T(X)$ is separable.

(a) *If $T(B_X)$ has small combinations of slices, then T is an SCD-operator.*
(b) In particular, *if $T(B_X)$ is a Radon-Nikodým set (i.e. if T is a strong Radon-Nikodým operator), then T is an SCD-operator.*
(c) *If $T(B_X)$ does not contain ℓ_1-sequences, then T is an SCD-operator.*
(d) In particular, *if T does not fix copies of ℓ_1, then T is an SCD-operator.* Indeed, if $T(B_X)$ contains an ℓ_1-sequence $(Te_n)_{n\in\mathbb{N}}$ with $e_n \in B_X$ $(n \in \mathbb{N})$, then as in the proof of the "lifting" property of ℓ_1 [22, Proposition 2.f.7], $Y = \overline{\text{lin}}\{e_n : n \in \mathbb{N}\}$ is a copy of ℓ_1 and $T|_Y$ is an isomorphic embedding, a contradiction (see [31, Proposition 1]).

The aim of this section is to show that SCD-operators behave in a very good way with respect to the Daugavet and the alternative Daugavet equations. We start with the best result we can get for the alternative Daugavet property.

Theorem 5.3. *Let X be a Banach space with the alternative Daugavet property and let $T \in L(X)$ be an SCD-operator. Then,* $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.

Proof. Without loss of generality, we may assume that $\|T\| = 1$. We take a determining sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $T(B_X)$ and we notice that the sets $T^{-1}(S_n) \cap B_X$ are slices of B_X. Given $\varepsilon > 0$ fixed, we take $a \in S_X$ such that $\|T(a)\| > 1 - \varepsilon$. Now, Proposition 4.2.v gives us that $\bigcap_{n\in\mathbb{N}} D\big(T^{-1}(S_n), \varepsilon\big)$ is weak*-dense in $K(X^*)$ (which is norming for X), so we may find $y^* \in \bigcap_{n\in\mathbb{N}} D\big(T^{-1}(S_n), \varepsilon\big)$ such that

$$\operatorname{Re} y^*(T(a)) \geqslant \|T(a)\| - \varepsilon > 1 - 2\varepsilon. \tag{4}$$

By the definition of $D\big(T^{-1}(S_n), \varepsilon\big)$, we get that

$$\overline{\operatorname{aconv}}\big(S(B_X, y^*, \varepsilon)\big) \cap T^{-1}(S_n) \neq \varnothing \qquad (n \in \mathbb{N}).$$

Thus, $T\big(\overline{\operatorname{aconv}}\big(S(B_X, y^*, \varepsilon)\big)\big) \cap S_n \neq \varnothing$ for all $n \in \mathbb{N}$, and using the fact that $\{S_n : n \in \mathbb{N}\}$ is determining, we deduce that

$$\overline{T\big(\operatorname{aconv}\big(S(B_X, y^*, \varepsilon)\big)\big)} = \overline{\operatorname{aconv}}\big(T\big(\overline{\operatorname{aconv}}\big(S(B_X, y^*, \varepsilon)\big)\big)\big) \supseteq T(B_X).$$

In particular, $T(a) \in \overline{T\big(\operatorname{aconv}\big(S(B_X, y^*, \varepsilon)\big)\big)}$, which means that there is

$$z \in T\big(\operatorname{aconv}\big(S(B_X, y^*, \varepsilon)\big)\big) \quad \text{with} \quad \|T(a) - z\| < \varepsilon,$$

and it follows from (4) that

$$\operatorname{Re} y^*(z) > 1 - 3\varepsilon. \tag{5}$$

Notice that z can be represented in the following way

$$z = T\left(\sum_{k=1}^{m} \lambda_k \theta_k x_k\right) = \sum_{k=1}^{m} \lambda_k\, \theta_k T(x_k)$$

where $x_k \in S(B_X, y^*, \varepsilon)$, $\theta_k \in \mathbb{T}$, $\lambda_k \geqslant 0$ for $k = 1, \ldots, m$ and $\sum_{k=1}^{m} \lambda_k = 1$. Then, it follows from (5) that there exists $k_0 \in \{1, \ldots, m\}$ such that

$$\operatorname{Re} y^*\big(\theta_{k_0}\, T(x_{k_0})\big) > 1 - 3\varepsilon.$$

Now, since $x_{k_0} \in S(B_X, y^*, \varepsilon)$, we get that

$$\operatorname{Re} y^*\big(x_{k_0} + \theta_{k_0}\, T(x_{k_0})\big) > 2 - 4\varepsilon.$$

It follows that

$$\|\mathrm{Id} + \theta_{k_0} T\| \geqslant \|x_{k_0} + \theta_{k_0}\, T(x_{k_0})\| \geqslant \operatorname{Re} y^*\big(x_{k_0} + \theta_{k_0}\, T(x_{k_0})\big) > 2 - 4\varepsilon.$$

Finally, the arbitrariness of ε gives the result. □

Remark 5.4. Analogously to the situation described in Remark 4.5, in the above proof we have used a formally weaker property than being an SCD-operator. Therefore, the result proved is the following. *Let X be a Banach space with the alternative Daugavet property and let $T \in L(X)$ such that $T(B_X)$ is an almost-SCD set. Then,* $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.

We can easily obtain a version of Theorem 5.3 for operators with non separable range which is useful for applications.

Corollary 5.5. *Let X be a Banach space with the alternative Daugavet property and let $T \in L(X)$ be such that $T(B_Y)$ is an SCD set for every separable subspace Y of X. Then, $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.*

Proof. We first take a separable subspace Y_1 of X such that $\|T|_{Y_1}\| = \|T\|$. Then, Remark 4.6 provides us with a separable subspace Y_2 with the alternative Daugavet property which contains $\bigcup_{k=0}^{\infty} T^k(Y_1)$. We apply again Remark 4.6 to get a separable subspace Y_3 with the alternative Daugavet property which contains $\bigcup_{k=0}^{\infty} T^k(Y_2)$, and so on. Then, the space $Y = \overline{\bigcup_{n\in\mathbb{N}} Y_n}$ is separable, T-invariant, $\|T|_Y\| = \|T\|$, and it has the alternative Daugavet property (just use Lemma 4.1). Since $T(B_Y)$ is SCD, Theorem 5.3 gives us that

$$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| \geqslant \max_{\theta\in\mathbb{T}} \|\mathrm{Id}|_Y + \theta\, T|_Y\| = 1 + \|T|_Y\| = 1 + \|T\|. \qquad \square$$

The following particular cases are especially interesting. The first one solves [18, Problem 33].

Corollary 5.6. *Let X be a Banach space with the alternative Daugavet property and let $T \in L(X)$ be an operator which does not fix copies of ℓ_1. Then, $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.*

Corollary 5.7. *Let X be a Banach space with the alternative Daugavet property and let $T \in L(X)$ be an operator such that $T(B_X)$ is strongly regular. Then, $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.*

It is possible to show an analogous result to Theorem 5.3 for spaces with the Daugavet property and the Daugavet equation. Actually, it is possible to get a better result. We need some notation and preliminary results. A bounded linear operator $T : X \longrightarrow Y$ between two Banach spaces X and Y is said to be a *strong Daugavet operator* if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is an element $z \in S_X$ such that

$$\|x + z\| \geqslant 2 - \varepsilon \qquad \text{and} \qquad \|Ty - Tz\| < \varepsilon$$

(see [20, §3] for the definition and the following properties). If $T \in L(X)$ is a strong Daugavet operator and X has the Daugavet property, then T satisfies Daugavet equation. On the other hand, finite-rank operators from a space with the Daugavet property are strong Daugavet operators. Our next goal is to show that actually, SCD-operators are strong Daugavet operators.

Proposition 5.8. *Let X be a Banach space with Daugavet property, Y a Banach space, and let $T : X \longrightarrow Y$ be an SCD-operator. Then, T is a strong Daugavet operator.*

Proof. Since T is an SCD-operator, we may find a determining sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $T(B_X)$, and we notice that the sets $T^{-1}(S_n) \cap B_X$ are slices of B_X. We fix $\varepsilon > 0$ and $x, y \in S_X$.

Since X has the Daugavet property, Proposition 4.3.v gives us that $\bigcap_{n\in\mathbb{N}} \widetilde{D}\big(T^{-1}(S_n), \frac{\varepsilon}{2}\big)$ is weak*-dense in $K(X^*)$ (which is norming for X), so we may find $y^* \in \bigcap_{n\in\mathbb{N}} \widetilde{D}\big(T^{-1}(S_n), \frac{\varepsilon}{2}\big)$ such that

$$x \in S(B_X, y^*, \tfrac{\varepsilon}{2}). \tag{6}$$

Then, by the definition of $\widetilde{D}(T^{-1}(S_n), \frac{\varepsilon}{2})$, we have that $\overline{S(B_X, y^*, \frac{\varepsilon}{2})} \cap T^{-1}(S_n) \neq \emptyset$ for every $n \in \mathbb{N}$. Thus,

$$T\left(\overline{S(B_X, y^*, \tfrac{\varepsilon}{2})}\right) \cap S_n \neq \emptyset \qquad (n \in \mathbb{N}).$$

Now, since the sequence $\{S_n : n \in \mathbb{N}\}$ is determining, we deduce that

$$T(B_X) \subseteq \overline{\mathrm{conv}}\, T\left(\overline{S(B_X, y^*, \tfrac{\varepsilon}{2})}\right) = \overline{T\left(S(B_X, y^*, \tfrac{\varepsilon}{2})\right)}.$$

In particular, $Ty \in \overline{T\left(S(B_X, y^*, \frac{\varepsilon}{2})\right)}$, which means that there is a $z \in S(B_X, y^*, \frac{\varepsilon}{2})$ such that

$$\|Ty - Tz\| < \varepsilon.$$

Since $x \in S(B_X, y^*, \frac{\varepsilon}{2})$ by (6), we also have that

$$\|x + z\| > 2 - \varepsilon.$$

Hence, this z meets all the requirements. □

In particular, we obtain the following analogue to Theorem 5.3.

Corollary 5.9. *Let X be a Banach space with the Daugavet property. If $T \in L(X)$ is an SCD-operator, then $\|\mathrm{Id} + T\| = 1 + \|T\|$.*

Our final goal in this section is to get a better result than Proposition 5.8 for a class of operators more restrictive than the SCD-operators. We need some notation. A bounded linear operator $T : X \longrightarrow Y$ between two Banach spaces X and Y is said to be a *narrow operator* if for every $x^* \in X^*$, the operator

$$T \tilde{+} \operatorname{Re} x^* : X \longrightarrow Y \oplus_1 \mathbb{R}, \qquad x \longmapsto \left(Tx, \operatorname{Re} x^*(x)\right)$$

is a strong Daugavet operator (see [20, §3 and §4] for this definition and the following properties). Equivalently, T is narrow if and only if for every $x, y \in S_X$, every $\varepsilon > 0$, and every slice S of B_X containing y, there is an element $z \in S$ such that

$$\|x + z\| \geqslant 2 - \varepsilon \qquad \text{and} \qquad \|Ty - Tz\| < \varepsilon.$$

A narrow operator is strong Daugavet, but the converse result is not true. It is known that strong Radon-Nikodým operators and operators which do not fix copies of ℓ_1 from a Banach space with the Daugavet property are narrow. We are going to extend these results to the so-called hereditary-SCD-operators.

Definition 5.10. Let X and Y be Banach spaces. A bounded linear operator $T : X \longrightarrow Y$ is said to be a *hereditary-SCD-operator* if every convex subset of $T(B_X)$ is an SCD set.

Here is the promised result.

Theorem 5.11. *Let X be a Banach space with Daugavet property and $T : X \longrightarrow Y$ be a hereditary-SCD-operator. Then, T is narrow.*

We need the following lemma, which could be of independent interest.

Lemma 5.12. *Let $T : X \longrightarrow Y$ be a hereditary-SCD-operator. Then, for every $x^* \in X^*$ the operator $T \tilde{+} \operatorname{Re} x^* : X \longrightarrow Y \oplus_1 \mathbb{R}$ is an SCD-operator.*

Proof. Denote $P_1 : [T \tilde{+} \operatorname{Re} x^*](X) \longrightarrow T(X)$ and $P_2 : [T \tilde{+} \operatorname{Re} x^*](X) \longrightarrow \mathbb{R}$ the natural coordinate projections. What we need to show is that there is a determining sequence of relatively weakly-open subsets of the set $A = [T \tilde{+} \operatorname{Re} x^*](B_X)$. Since A is separable, it is enough to prove that for every $a \in A$ there exists a sequence of relatively weakly open sets $\{V_n : n \in \mathbb{N}\}$ such that for every $B \subseteq A$ intersecting all the V_n, $a \in \overline{\operatorname{conv}}(B)$ (see Remark 2.4).

We fix $a \in A$ and denote

$$A_a = \{b \in A : P_1(b) = P_1(a)\}.$$

It is easy to see that A_a is of the form

$$A_a = \{(P_1(a), t) : t \in \Delta_a\},$$

where Δ_a is a bounded interval in $\mathbb{R}$. We denote $\alpha_a = \inf \Delta_a$ and $\beta_a = \sup \Delta_a$ and we consider

$$S_{n,1} = \left\{b \in A \,:\, P_2(b) < \alpha_a + \tfrac{1}{n}\right\} \quad \text{and} \quad S_{n,2} = \left\{b \in A \,:\, P_2(b) > \beta_a - \tfrac{1}{n}\right\},$$

which are non-empty slices of A (since they intersect A_a) for all $n \in \mathbb{N}$ and $i = 1,2$. Now, since T is a hereditary-SCD-operator and $P_1(S_{n,i}) \subseteq T(B_X)$ is convex, for every $n \in \mathbb{N}$ and $i = 1,2$ we may find a determining sequence $\{S_{n,i}^m \,:\, m \in \mathbb{N}\}$ of slices of $P_1(S_{n,i})$. We write

$$V_{n,i}^m = S_{n,i} \cap P_1^{-1}(S_{n,i}^m) \qquad (n,m \in \mathbb{N},\ i = 1,2)$$

which are relatively weakly open subsets of A. We will prove that they are the sets we need. Indeed, let $B \subseteq A$ be such that

$$\emptyset \neq B \cap V_{n,i}^m = B \cap S_{n,i} \cap P_1^{-1}(S_{n,i}^m) \qquad (n,m \in \mathbb{N},\ i = 1,2).$$

For every $n \in \mathbb{N}$, we observe that

$$P_1(B \cap S_{n,i}) \cap S_{n,i}^m \neq \emptyset \qquad (m \in \mathbb{N},\ i = 1,2),$$

so, since the sequences $\{S_{n,i}^m \,:\, m \in \mathbb{N}\}$ are determining, we get that

$$\overline{P_1(B \cap S_{n,i})} = \overline{\operatorname{conv}}\big(P_1(B \cap S_{n,i})\big) \supseteq P_1(S_{n,i}) \qquad (i = 1,2).$$

In particular, $\overline{P_1(B \cap S_{n,i})} \ni P_1(a)$, meaning that for every $n \in \mathbb{N}$, every $i = 1,2$, and every $\varepsilon > 0$, there exists $x_{n,i}^\varepsilon \in B \cap S_{n,i}$ such that

$$\left\|P_1\left(x_{n,i}^\varepsilon\right) - P_1(a)\right\| \leqslant \varepsilon. \tag{7}$$

Now, we fix some $\varepsilon > 0$ and, since obviously $a \in A_a$, we may take $n \in \mathbb{N}$ such that

$$\alpha_a + \frac{1}{n} - \varepsilon < P_2(a) < \beta_a - \frac{1}{n} + \varepsilon.$$

So, for the corresponding $x_{n,1}^\varepsilon$ and $x_{n,2}^\varepsilon$, we have

$$P_2\big(x_{n,1}^\varepsilon\big) - \varepsilon < P_2(a) < P_2\big(x_{n,2}^\varepsilon\big) + \varepsilon.$$

Then, there is a convex combination

$$x_n^\varepsilon = \lambda_1\, x_{n,1}^\varepsilon + \lambda_2\, x_{n,2}^\varepsilon \qquad (\lambda_1 + \lambda_2 = 1)$$

(so $x_n^\varepsilon \in \operatorname{conv}(B)$) such that

$$\left|P_2\big(x_n^\varepsilon\big) - P_2(a)\right| < \varepsilon.$$

This, together with (7), implies that $\left\|x_n^\varepsilon - a\right\| < 2\varepsilon$, and the arbitrariness of $\varepsilon > 0$ gives us that $a \in \overline{\operatorname{conv}}(B)$. □

Proof of Theorem 5.11. To prove that T is narrow, it is enough to show that for every $x^* \in X^*$, the operator $T \tilde{+} \operatorname{Re} x^*$ is a strong Daugavet operator. But this fact follows from Lemma 5.12 and Proposition 5.8. □

As we did for the alternative Daugavet property in Corollary 5.5, we can extend Theorem 5.11 to the non separable case.

Corollary 5.13. *Let X be a Banach space with the Daugavet property and let $T \in L(X)$ be such that $T|_Y$ is an hereditary-SCD-operator for every separable subspace Y of X. Then, T is narrow and, in particular, $\|\mathrm{Id} + T\| = 1 + \|T\|$.*

Proof. We fix $x, y \in S_X$, a slice S of B_X and $\varepsilon > 0$. We take a separable subspace Y_1 of X such that $x, y \in Y_1$ and such that $S \cap Y_1 \neq \emptyset$ and we follow the proof of Corollary 5.5, using [20, Theorem 4.5] instead of Remark 4.6, to get a separable subspace Y of X, T-invariant, with the Daugavet property and such that $x, y \in S_X$ and $S \cap Y \neq \emptyset$. Now, $T|_Y$ is a hereditary-SCD-operator, so Theorem 5.11 gives us that $T|_Y$ is narrow. Then, we may find $z \in S \cap Y \subseteq S$ such that $\|x + z\| \geqslant 2 - \varepsilon$ and

$$\|Ty - Tz\| = \|T_Y(y) - T|_Y(z)\| < \varepsilon. \qquad \square$$

The following particular cases are especially interesting. The first one was proved in [20, Theorem 4.13] with a different argument.

Corollary 5.14. *Let X be a Banach space with the Daugavet property and let $T \in L(X)$ be an operator which does not fix copies of ℓ_1. Then, T is narrow.*

Corollary 5.15. *Let X be a Banach space with the Daugavet property and let $T \in L(X)$ be an operator such that $T(B_X)$ is strongly regular. Then, T is narrow.*

Remarks 5.16.

(a) *The class of hereditary-SCD-operators is a right operator ideal.* Indeed, if $T : X_1 \longrightarrow X_2$ is an arbitrary operator and $S : X_2 \longrightarrow X_3$ is a hereditary-SCD-operator, then $[ST](B_{X_1}) \subseteq S(\|T\|B_{X_2})$, so ST is an hereditary-SCD-operator.

(b) *The class of hereditary-SCD-operators is not a left operator ideal.* Indeed, we consider a norm-one projection $T : L_1[0,1] \longrightarrow X \equiv \ell_1$ which is a hereditary-SCD-operator since ℓ_1 is RNP. We also consider a quotient map $S : \ell_1 \longrightarrow \ell_1/Y \equiv L_1[0,1]$ (by just using the factor universality of ℓ_1). Then, $ST(B_{L_1[0,1]}) = B_{L_1[0,1]}$ so ST is not even an SCD-operator.

(c) As a consequence, *there are narrow operators which are not SCD-operators.* Indeed, since the set of narrow operators is clearly a left operator ideal, the operator ST above is narrow.

6. Countable π-bases of the weak topology

It was shown in Proposition 2.21 that a convex bounded subset A of a Banach space X is SCD if it has a countable π-base of the weak topology. But we do not know whether these two properties are equivalent. The aim of this section is to discuss this possible equivalence. In a first subsection we will show that the class of sets having countable π-bases of the weak topology contains separable CPCP sets. We already know that it contains those sets which do not have ℓ_1-sequences (Theorem 2.22), so this class covers most of the examples of SCD sets presented in this paper. In the second subsection we will show that convex bounded subsets of both $\ell_1(c_0)$ and $c_0(\ell_1)$ also have countable π-bases of the weak topology. Finally, the third subsection contains several characterizations of SCD sets which remind of the property we are dealing with.

6.1. CPCP sets.

We start with a sufficient condition to have a countable π-base of the weak topology.

Proposition 6.1. *Let X be a Banach space and let A be a separable closed convex bounded subset of X such that there is a weakly dense subset B of A consisting of points of continuity of* Id $: (A, \sigma(X, X^*)) \longrightarrow (A, \|\cdot\|)$. *Then, $(A, \sigma(X, X^*))$ has a countable π-base.*

Proof. Let D be a countable norm dense subset of B, and for every $d \in D$ and every $n \in \mathbb{N}$ let U_d^n be a weak open neighborhood of d in A of diameter less than $\frac{1}{n}$. We claim that the countable family $\{U_d^n : n \in \mathbb{N}, d \in D\}$ is a π-base of A. Indeed, let W be a weakly open subset of A. Since B is weakly dense in A, $W \cap B$ is non-empty and relatively norm open in B so, since D is norm dense in B, there is $d \in D \cap W$. Now, W is a norm open neighborhood of d relative to A, so it contains $B(d, 1/n) \cap A$ for some $n \in \mathbb{N}$ and so $U_d^n \subseteq W$. We are done. $\square$

A first consequence of the above result deals with LUR renorming. It is clear from the definition that denting points are points of weak-norm continuity of the identity map and so, as it was commented before Example 2.10, the unit ball of a Banach space with a LUR norm fulfills the above condition. It was also commented there that every separable Banach space can be equivalently renormed with a LUR norm.

Example 6.2.

(a) *Let X be a separable Banach space with a LUR norm. Then, B_X has a countable π-base of the weak topology.*
(b) As a consequence, *every separable Banach space X admits an equivalent norm $|\cdot|$ such that $B_{(X,|\cdot|)}$ has a countable π-base of the weak topology.*

We are going to show that CPCP sets have countable π-bases for the weak topology. We recall that a closed convex bounded subset A of a Banach space X has the CPCP if every convex closed subset B of A contains a weak-to-norm point of continuity of the identity mapping. In this case, for every convex subset B of A and for every $\varepsilon > 0$, there is a relatively weakly open subset $C \subseteq B$ with $\operatorname{diam}(C) < \varepsilon$ [5]. We need the following result which follows from [14, Lemma I.0]; we haven't found a direct reference, so we include a proof for the sake of completeness.

Lemma 6.3. *Let X be a Banach space and let A be a closed convex bounded subset of X with the CPCP. Then, there is a weakly dense subset D of A consisting of points of weak-norm continuity of* $\mathrm{Id} : (A, \sigma(X, X^*)) \longrightarrow (A, \|\cdot\|)$.

Proof. We fix a sequence of positive ε_n tending to zero and write

$$D_n = \bigcup \{C \,:\, C \text{ is weakly open in } A \text{ and } \operatorname{diam}(C) < \varepsilon_n\}.$$

Let us prove that $D = \bigcap_{n\in\mathbb{N}} D_n$ is weakly dense in A. Indeed, let $U \subseteq A$ be relatively weakly open. We pick $U_1 \subseteq U$ convex closed with non-empty interior. Then, there is a relatively weakly open subset C_1 of A of diameter less than ε_1 such that C_1 is contained in the weak interior of U_1. We repeat the process to find a decreasing sequence C_n of weakly open sets with non-empty interior such that $\operatorname{diam}(C_n) < \varepsilon_n$ and $\overline{C_{n+1}} \subseteq C_n$. Then, the Cantor theorem tells us that there is $x \in \bigcap_{n\in\mathbb{N}} \overline{C_n}$. Now, we have in particular that $x \in C_1 \subseteq U_1 \subseteq U$. On the other hand, for every $n \in \mathbb{N}$, $x \in C_n$ and $\operatorname{diam}(C_n) < \varepsilon_n$, so $x \in D_n$. Therefore, $x \in D$. Finally, every point of D has weak neighborhoods of arbitrarily small diameter, showing that it is a point of continuity. □

This result, together with Proposition 6.1 gives the main result of the subsection.

Corollary 6.4. *Let X be a Banach space and let A be a separable closed convex bounded subset of X with the CPCP. Then, A has a countable π-base for the weak topology.*

With the above result, most of the types of SCD sets presented in the section 2 have a countable π-base of the weak topology. The only exception is the family of strongly regular sets which are not CPCP. There are two main examples of sets of this kind, but in both cases, the sets have a countable π-base of the weak topology.

Examples 6.5.

(a) *The set constructed by S. Argyros, E. Odell, and H. Rosenthal [2] which is strongly regular but does not have the CPCP is a subset of c_0, so it has a countable π-base of the weak topology since it does not have ℓ_1-sequences.*

(b) *The set constructed by W. Schachermayer* [28] *which is a subset C of a Banach space Z which does not have the CPCP but Z^{**} is strongly regular (so Z is strongly regular). But then, $(C, \sigma(X, X^*))$ has a countable π-base of the weak topology since Z does not contain ℓ_1.*

6.2. **$c_0(\ell_1)$ and $\ell_1(c_0)$.** Our goal in this subsection is to show that convex bounded subsets of the spaces $c_0(\ell_1)$ and $\ell_1(c_0)$ have a countable π-base of the weak topology. The first case is easier to demonstrate.

Example 6.6. *Every bounded convex subset A of the space $c_0(\ell_1)$ has a countable π-base of the weak topology.*

Proof. Let X denote $c_0(\ell_1)$. For every $m \in \mathbb{N}$, we denote

$$Y_m = \left[\ell_1 \oplus \ell_1 \oplus \overset{m}{\cdots} \oplus \ell_1 \oplus 0 \oplus 0 \oplus \cdots\right]_\infty \subseteq c_0(\ell_1)$$

and $P_m : X \longrightarrow Y_m$ for the natural projection. Since $P_m(A)$ is a convex bounded subset of Y_m and Y_m is isomorphic to ℓ_1, there is a countable π-base $\{S_{m,k} : k \in \mathbb{N}\}$ of $(P_m(A), \sigma(Y_m, Y_m^*))$. We are going to prove that the collection

$$\widetilde{S}_{m,k} = \left[P_m^{-1}(S_{m,k})\right] \cap A \qquad (m,k \in \mathbb{N})$$

forms a countable π-base of $(A, \sigma(X, X^*))$. Indeed, let U, V be weak neighborhoods of 0, $V + V \subseteq U$, $a \in A$, and denote $B = (a + U) \cap A$. Every relatively weakly open subset of A is of the same form as B, so we have to prove that $\widetilde{S}_{m,k} \subseteq B$ for some choice of m and k. Assume to the contrary that none of $\widetilde{S}_{m,k}$ is contained in B. For $m \in \mathbb{N}$ big enough, all the $P_m(A)$ intersect $(a + V)$. Fix $m \in \mathbb{N}$ with $C_m = (a + V) \cap P_m(A) \neq \emptyset$. Then there is $k(m) \in \mathbb{N}$ with $S_{m,k(m)} \subseteq C_m$. According to our assumption $\widetilde{S}_{m,k(m)}$ is not contained in B, so there is an $x_m \in \widetilde{S}_{m,k(m)} \setminus B$. This x_m can be written as $x_m = y_m + z_m$, where $y_m \in S_{m,k(m)} \subseteq C_m$ and $z_m \in \mathrm{Ker} P_m$. Since $x_m \in A$ and $y_m \in P_m(A)$, we have that z_m is a bounded sequence, and since by our construction (z_m) tend to 0 coordinate-wise as $m \to \infty$, we can deduce that $(z_m) \longrightarrow 0$ in the weak topology. Therefore, for some m big enough $z_m \in V$ and consequently $x_m = y_m + z_m \in (a + V) + V \subseteq a + U$. Since $x_m \in A$, this means that $x_m \in (a + U) \cap A = B$, which contradicts the selection of x_m. □

Remark 6.7. *The argument above also works for c_0-sums of RNP spaces.* Indeed, this follows from the fact that a finite-sum of RNP spaces is again a RNP space (see [9, Theorem 6.5.b], for instance).

Let us remark with an example that to have a countable π-base of the weak topology does not imply that any point has a countable base of weak neighborhoods.

Example 6.8. *The unit ball of $X = c_0(\ell_1)$ has no point with a countable base of relative weak neighborhoods.* Indeed, we consider an arbitrary $x = (x_n)_{n\in\mathbb{N}} \in B_X$, where $x_n \in \ell_1$, $\|x_n\| \longrightarrow 0$ and $\max_{n\in\mathbb{N}} \|x_n\| \leqslant 1$. We fix $n_0 \in \mathbb{N}$ such that $\|x_{n_0}\| < 1/2$ and we consider the subset

$$A = \{(y_n)_{n\in\mathbb{N}} \in B_X : y_n = x_n \text{ if } n \neq n_0,\ \|x_{n_0} - y_{n_0}\| \leqslant 1/2\}.$$

Then, A is a closed subset of B_X containing x, so if x has a countable base of relative weak neighborhoods in B_X, then x has also a countable base of relative weak neighborhoods in A. But the latter is impossible, because A is affinely homeomorphic to B_{ℓ_1}, with x being the image of $0 \in B_{\ell_1}$.

To get the second example we need a technical result.

Lemma 6.9. *Let X be a separable Banach space. Then, the following are equivalent.*

(i) *Every convex bounded subset of X has a countable π-base of the weak topology.*

(ii) *Every closed convex bounded subset A of X has a point with a countable local π-base of relatively weakly open subsets (i.e. there is $x \in A$ and a sequence $\{U_n : n \in \mathbb{N}\}$ of relatively weakly open subsets of A such that for every relative weak neighborhood V of x there is some $U_n \subseteq V$.)*

(iii) *For every $\varepsilon > 0$, every closed convex bounded subset A of X has a point with a countable local ε-base of relatively weakly open subsets (i.e. there is $x \in A$ and a sequence $\{U_n : n \in \mathbb{N}\}$ of relatively weakly open subsets of A such that for every weakly open neighborhood $V \subseteq X$ of x* in the whole space *there is $n \in \mathbb{N}$ with $U_n \subseteq V + \varepsilon B_X$.)*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear since a π-base is a local π-base, and a local π-base is an ε-base for every $\varepsilon > 0$.

(iii) $\Rightarrow$ (i). It is straightforward to show that it is enough to deal with *closed* convex bounded subsets of X. Just observe that if $\{U_n : n \in \mathbb{N}\}$ is a π-base for the weak topology of the closure of a bounded convex subset A of X, then $\{U_n \cap A : n \in \mathbb{N}\}$ is a π-base of the weak topology of A itself.

We then fix a *closed* convex bounded subset $A \subseteq X$. We first remark that, for every $\varepsilon > 0$, the subset $B_\varepsilon \subseteq A$ of points having a countable local ε-base is weakly dense in A. Indeed, we consider an arbitrary weakly open subset U of X intersecting A and we fix another weakly open subset $V \subseteq U \subseteq X$ intersecting A with $\overline{V}^{\sigma(X,X^*)} \subseteq U$. According to our assumption, there is $x \in \overline{V}^{\sigma(X,X^*)} \cap A$ with a local ε-base $\{U_n : n \in \mathbb{N}\}$. But then, $V_n = U_n \cap V$ form a countable local ε-base of relatively weakly open subsets of A for x, i.e. $x \in B_\varepsilon \cap U$, so $B \cap U \neq \emptyset$.

Now, for every $k \in \mathbb{N}$ we take a countable norm dense subset $\{b_{k,m} : m \in \mathbb{N}\}$ in $B_{1/k}$, and for every $b_{k,m}$, we select a $1/k$-base $\{U_{k,m,n} : n \in \mathbb{N}\}$. Let us show that $\{U_{k,m,n} : k,n,m \in \mathbb{N}\}$ forms a π-base for $(A, \sigma(X,X^*))$. Indeed, let U, V be weak neighborhoods of 0, $V + V \subseteq U$, $a \in A$, and denote $G = (a + U) \cap A$. We have to prove that $U_{k,m,n} \subseteq G$ for some choice of $k,m,n \in \mathbb{N}$. To do this, we take $k \in \mathbb{N}$ big enough that $\frac{1}{k} B_X \subseteq V$. According to our construction, there is $m \in \mathbb{N}$ with $b_{k,m} \in (a + V) \cap A$. Then, there is $n \in \mathbb{N}$ with $U_{k,m,n} \subseteq (a + V) + \frac{1}{k} B_X$. Therefore

$$U_{k,m,n} \subseteq \left(a + V + \frac{1}{k} B_X\right) \cap A \subseteq (a + V + V) \cap A \subseteq (a + U) \cap A = G. \qquad \square$$

We are now able to present the second example.

Example 6.10. *Every bounded convex subset A of the space $\ell_1(c_0)$ has a countable π-base of the weak topology.*

Proof. Let X denote $\ell_1(c_0)$. For $\varepsilon > 0$ fixed, arguing the same way as in the beginning of the proof of Theorem 3.12, we select an open slice $S \subseteq A$ and an $m \in \mathbb{N}$ with the following property

$$\left\|(0, \dots, 0, y_{m+1}, y_{m+2}, \dots)\right\| < \frac{\varepsilon}{2} \qquad \left((y_n)_{n\in\mathbb{N}} \in S\right). \tag{8}$$

Let us prove that every $x_0 \in S$ has a countable ε-base of relatively weakly open subsets and Lemma 6.9 will give the result.

We denote

$$Y_m = \left[c_0 \oplus c_0 \oplus \overset{m}{\dots} \oplus c_0 \oplus 0 \oplus 0 \oplus \cdots\right]_{\ell_1} \subseteq \ell_1(c_0)$$

and let $P_m : X \longrightarrow Y_m$ be the natural projection. Since Y_m is isomorphic to c_0, there is a countable local π-base $\{U_n : n \in \mathbb{N}\}$ of $P_m(x_0)$ in $(P_m(S), \sigma(X,X^*))$. Consider

$$\widetilde{U_n} = P_m^{-1}(U_n) \cap S \qquad (n \in \mathbb{N})$$

which are weakly open subsets of S, and hence they are weakly open in A. Let us show that $\{\widetilde{U_n} : n \in \mathbb{N}\}$ forms an ε-base for x_0 in A. Consider a weakly open neighborhood $V \subseteq X$ of x_0. By (8), we have that

$$\|P_m(y) - y\| < \varepsilon/2 \qquad (y \in S). \tag{9}$$

So $\left(V + \frac{\varepsilon}{2}B_X\right) \cap P_m(S)$ is a weak neighborhood of $P_m(x_0)$ in $P_m(S)$. So there is an $n \in \mathbb{N}$ such that $U_n \subseteq V + \frac{\varepsilon}{2}B_X$. Applying (9) once more, we obtain that

$$\widetilde{U_n} = P_m^{-1}(U_n) \cap S \subseteq U_n + \frac{\varepsilon}{2}B_X \subseteq V + \varepsilon B_X. \qquad \square$$

6.3. **Two characterizations of SCD sets.** The aim of this part of the section is to establish some characterizations of SCD sets which remind of countable π-bases of the weak topology. The first one deals with convex combinations of slices.

Theorem 6.11. *A bounded convex subset A of a Banach space X is an SCD set if and only if there is a sequence $\{V_n : n \in \mathbb{N}\}$ of convex combinations of slices of A such that every relatively weakly open subset of A contains some of the V_n.*

Proof. The "if" part is direct consequence of Propositions 2.2 and 2.18.

Conversely, asume that A is an SCD set and suppose without loss of generality that $A \subseteq B_X$. Let $S_n = S(A, x_n^*, \varepsilon_n)$ for $n \in \mathbb{N}$, be a determining sequence of slices for A. Let us show that the convex combinations of the S_n's with rational coefficients form the countable collection of convex combinations of slices that we need. Indeed, let U be a relatively weakly open subset of A. Select another relatively weakly open subset $V \subseteq U$ such that $\alpha = \operatorname{dist}(V, A \setminus U) > 0$. Due to Bourgain's lemma (Lemma 2.16), there is a convex combination of slices $\sum_{j=1}^m \lambda_j G_j \subseteq V$. According to Proposition 2.2, for every $j = 1, 2, \ldots, m$ there is $n(j) \in \mathbb{N}$ such that $S_{n(j)} \subseteq G_j$. Then, $\sum_{j=1}^m \lambda_j S_{n(j)} \subseteq V$. What remains is to find rationals $\mu_j > 0$ with $\sum_{j=1}^m \mu_j = 1$ and $|\mu_j - \lambda_j| < \alpha$. Then, the Hausdorff distance between $\sum_{j=1}^m \mu_j S_{n(j)}$ and $\sum_{j=1}^m \lambda_j S_{n(j)}$ is smaller than α, so $\sum_{j=1}^m \mu_j S_{n(j)} \subseteq V + \alpha\, B_X \subseteq U$. $\square$

The second result gives a reformulation of SCD in terms of topological properties of the set of extreme points of its weak* closure in the bidual. For a convex bounded subset A of a Banach space X, denote $\overline{A}^{**}$ the weak-star closure of A in X^{**}.

Theorem 6.12. *Let X be a Banach space and let A be a convex bounded subset of X. Put $W = \left(\operatorname{ext}\left(\overline{A}^{**}\right), \sigma(X^{**}, X^*)\right)$. Then, the following are equivalent:*

(i) *A is an SCD set.*
(ii) *W has a countable π-base.*

Proof. (i) $\Longrightarrow$ (ii). We take a sequence of slices $S_n = S(A, x_n^*, \varepsilon_n)$ for $n \in \mathbb{N}$ which is determining for A and we write

$$S_n^{**} = S\left(\overline{A}^{**}, x_n^*, \varepsilon_n\right) \subseteq \overline{A}^{**}$$

for the natural extensions of S_n to slices of $\overline{A}^{**}$. Then, the family $U_n = S_n^{**} \cap W$ for $n \in \mathbb{N}$ forms a π-base of W. Indeed, we consider a relatively weak*-open subset U of W. Due to Choquet's lemma (that for any locally convex topology, slices containing an extreme point of a compact convex set make up a neighborhood base of the extreme point, see [10, Definition 25.3 and Proposition 25.13]), there is a slice $S^{**} = S\left(\overline{A}^{**}, x^*, \varepsilon\right)$ of $\overline{A}^{**}$ generated by some $x^* \in X^*$ and $\varepsilon > 0$ such that $U \supseteq S^{**} \cap W \neq \emptyset$. Now, according to Proposition 2.2, there is an $n \in \mathbb{N}$ such that

$$S_n \subseteq S(A, x^*, \varepsilon/2) \subseteq S\left(\overline{A}^{**}, x^*, \varepsilon/2\right).$$

Then, S_n^{**} is contained in the relative weak*-closure of $S\big(\overline{A}^{**}, x^*, \varepsilon/2\big)$ in $\overline{A}^{**}$, so $S_n^{**} \subseteq S^{**}$ and

$$U_n = S_n^{**} \cap W \subseteq S^{**} \cap W \subseteq U.$$

(ii) $\Longrightarrow$ (i). We consider a countable π-base $\{U_n : n \in \mathbb{N}\}$ of W consisting of relatively weak*-star open subsets. Again by Choquet's lemma, there are $x_n^* \in X^*$ and $\varepsilon_n > 0$ such that

$$U_n \supseteq \widetilde{U_n} = S\big(\overline{A}^{**}, x_n^*, \varepsilon_n\big) \cap W \neq \emptyset.$$

Let us prove that the slices $S_{n,m} = S(A, x_n^*, 1/m)$ with $n, m \in \mathbb{N}$, form a determining sequence for A. Indeed, we denote $S_{n,m}^{**}$ the *closed* slices of $\overline{A}^{**}$ generated by x_n^* and $1/m$. For every slice $S = S(A, x^*, \varepsilon)$ of A, since $\{\widetilde{U_n} : n \in \mathbb{N}\}$ is a π-base of W, there is $n \in \mathbb{N}$ such that

$$S^{**} \cap W \supseteq \widetilde{U_n} \qquad \text{where } S^{**} = S\big(\overline{A}^{**}, x^*, \varepsilon_n\big),$$

so for $m \in \mathbb{N}$ big enough we have

$$S^{**} \cap W \supseteq S_{n,m}^{**} \cap W.$$

Then, taking into account that, for every $n \in \mathbb{N}$,

$$G_n = \bigcap_{m \in \mathbb{N}} S_{n,m}^{**}$$

is a closed face of $\overline{A}^{**}$, the Krein-Milman theorem gives us that

$$G_n = \overline{\operatorname{conv}\big(G_n \cap W\big)}^{\sigma(X^{**}, X^*)}.$$

Therefore,

$$S^{**} \supseteq \overline{\operatorname{conv}\big(S^{**} \cap W\big)}^{\sigma(X^{**}, X^*)} \supseteq \overline{\operatorname{conv}\left(\bigcap_{m \in \mathbb{N}} S_{n,m}^{**} \cap W\right)}^{\sigma(X^{**}, X^*)} = G_n.$$

This means that the intersection of the decreasing sequence of $\sigma(X^{**}, X^*)$ compact sets $\{S_{n,m}^{**} : m \in \mathbb{N}\}$ is contained in S^{**}. But S^{**} is a relatively $\sigma(X^{**}, X^*)$ open set in $\overline{A}^{**}$, so for sufficiently big $m \in \mathbb{N}$, all the $S_{n,m}^{**}$ are subsets of S^{**}. For these m, we have

$$S = S^{**} \cap A \supseteq S_{n,m}^{**} \cap A \supseteq S_{n,m}.$$

Finally, we use the characterization of SCD sets from Proposition 2.2. □

The following is an easy consequence of the above result.

Corollary 6.13. *Let X be a Banach space and let A be a bounded convex subset of X. If A is SCD, then $\Big(\operatorname{ext}\big(\overline{A}^{**}\big), \sigma(X^{**}, X^*)\Big)$ is separable.*

The particular case of the above corollary for subsets of separable Banach spaces without copies of ℓ_1 should be previously known. Anyway, we include an easy direct proof of this fact.

Remark 6.14. *Let X be a separable Banach space without copies of ℓ_1 and let A be a convex bounded subset of X. Then, $\Big(\operatorname{ext}\big(\overline{A}^{**}\big), \sigma(X^{**}, X^*)\Big)$ is separable.* Indeed, we write

$$C = \operatorname{conv}\big(\operatorname{ext}\big(\overline{A}^{**}\big)\big)$$

and we observe that C is $\sigma(X^{**}, X^*)$-sequentially dense in its weak*-closure $\overline{A}^{**}$ (see [12, Theorem 4.1]). Then, we take a sequence $\{y_n : n \in \mathbb{N}\}$ dense in A and we consider those extreme points of $\overline{A}^{**}$ needed to approximate each y_n by a sequence of convex combinations. The union of

all these extreme points (while countable) is weak*-dense in the set of all extreme points of $\overline{A}^{**}$ by the reversed Krein-Milman theorem.

7. Open questions

Question 7.1. Let X be a Banach space and let A be a convex bounded subset of X. If A is SCD, does A have a countable π-base for the weak topology?

Question 7.2. Let X be an SCD space. Does every convex bounded subset of X have a countable π-base for the weak topology?

Related to these questions is the following one.

Question 7.3. Let L be a compact subset of a locally convex space and let K be its closed convex hull. If L has a countable π-base, does it imply that K also has a countable π-base? What if $L = \operatorname{ext}(K)$?

Let us explain why this question is related to the above two. Observe that if D is a dense subspace of a topological space E and $\mathcal{B}$ is a π-base for E, then $\{B \cap D : B \in \mathcal{B}\}$ is a π-base for D. In particular, if $(\overline{A}^{**}, \sigma(X^{**}, X^*))$ has a countable π-base, then so does $(A, \sigma(X, X^*))$. Thus, a positive answer to the preceding question combined with Theorem 6.12 would imply a positive answer to Questions 7.1 and 7.2.

Questions 7.4.

(a) Is every Banach space with unconditional basis SCD?
(b) A simpler case: let X be a Banach space with 1-symmetric basis. Is B_X an SCD set?

Question 7.5. Are the concepts of SCD sets and almost-SCD sets equivalent (see Remark 4.5 for the definition)?

Questions 7.6. Let X be a separable Banach space such that no subspace of it can be renormed with the Daugavet property. Is X SCD?

Questions 7.7.

(a) Is the sum of two SCD-operators an SCD-operator?
(b) Is the sum of two hereditary-SCD-operators a hereditary-SCD-operator?

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