NUMERICAL INDEX OF SOME POLYHEDRAL NORMS ON THE PLANE

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Abstract. We give explicit formulae for the numerical index of some (real) polyhedral spaces of dimension two. Concretely, we calculate the numerical index of a family of hexagonal norms, two families of octagonal norms and the family of norms whose unit balls are regular polygons with an even number of vertices.

1. Introduction

In 1918, O. Toeplitz [26] introduced the *numerical range* of a $n \times n$ matrix A, as the subset of scalars given by

$$W(A) = \{(Ax \mid x) \ : \ x \in S\},$$

where $(\cdot \mid \cdot)$ denotes the usual inner product of two vectors, and S is the unit sphere of the n-dimensional Euclidean space. This numerical range owes part of its motivation to the classical theory of quadratic forms on Hilbert spaces, and for this, it is sometimes called the *field of values* of the matrix or the *Hilbert space numerical range*. This point set in the base field has been shown to be very useful since, for instance, it contains all the eigenvalues of the matrix and (surprisingly) it is convex (Hausdorff-Toeplitz Theorem [10]). Some properties of the Hilbert space numerical range are discussed in the classical book of P. Halmos [9, §17]. Further developments can be found in a recent book of K. Gustafson and D. Rao [8].

By contrast to the long history of the Hilbert space numerical range, the birth of the general theory was long delayed. No concept of numerical range appropriate for general normed linear spaces appeared until 1961 and 1962, when distinct, somehow equivalent, concepts were introduced independently by F. Bauer [1] and G. Lumer [14] to generalize Toeplitz's numerical range. Although both ranges can be defined for bounded linear operators on arbitrary normed linear spaces, we will restrict ourselves to the finite-dimensional case. Let us recall some definitions and notation. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Given a norm $\|\cdot\|$ in $\mathbb{K}^n$, we write $X = (\mathbb{K}^n, \|\cdot\|)$ for the vector space $\mathbb{K}^n$ endowed with the norm $\|\cdot\|$, and we will write S_X and B_X to denote, respectively, the unit sphere and the closed unit ball of X. In the algebra of $n \times n$ matrices, $M_n(\mathbb{K})$, we can define the operator norm associated to the norm of X as

$$\|A\| = \max\{\|Ax\| \ : \ x \in B_X\}$$

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for every $A \in M_n(\mathbb{K})$. We write $L(X)$ to denote the algebra $M_n(\mathbb{K})$ endowed with the norm defined above, which is no more than the Banach algebra of all bounded linear operators on X. We write X^* to denote the topological dual of X, i.e., X^* is $\mathbb{K}^n$ endowed with the dual norm of $\|\cdot\|$, which we will denote by $\|\cdot\|'$. Finally, we write

$$\Pi(X) = \{(x, x^*) \in X \times X^* \ : \|x\| = \|x^*\|' = x^*(x) = 1\}.$$

The *numerical range* or *field of values* of $T \in L(X)$ is the set of scalars

$$V(T) = \{x^*(Tx) \ : \ (x, x^*) \in \Pi(X)\}.$$

This is Bauer's definition, while Lumer's definition depends on the election of a subset G of $\Pi(X)$ such that for every $x \in S_X$, there is only one $x^* \in S_{X^*}$ with $(x, x^*) \in G$, and then, the numerical range of T is

$$\{x^*(Tx) \ : \ (x, x^*) \in G\}.$$

Nevertheless, both numerical ranges have the same closed convex hull (compare [1, Theorem 4.3] and [14, Theorem 14]) and thus, the same maximum of the modulus of their elements. Therefore, if we define the *numerical radius* of T as

$$v(T) = \max\{|\lambda| \ : \ \lambda \in V(T)\},$$

this definition does not change if we replace Bauer's numerical range by Lumer's. Many results on numerical ranges and numerical radius in the finite-dimensional setting can be found in the papers [1, 20, 22, 23, 24, 25]. Very recent results can be found in [12] and references therein. A complete survey on numerical ranges and their relations to spectral theory of operators can be found in the books by F. Bonsall and J. Duncan [3, 4] and we refer the reader to these books for general information and background.

It is clear that the numerical radius is a seminorm on $L(X)$, and that $v(T) \leqslant \|T\|$ for every $T \in L(X)$. Quite often, v is actually an equivalent norm on $L(X)$. It is then natural to consider the so called *numerical index* of the space X, namely the constant $n(X)$ defined as the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. Equivalently,

$$n(X) = \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\}.$$

Note that $0 \leqslant n(X) \leqslant 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms.

The concept of numerical index was first suggested by G. Lumer in a lecture to the North British Functional Analysis Seminar in 1968. At that time, it was known that in a complex Hilbert space (with dimension greater than 1) $\|T\| \leqslant 2v(T)$ for all $T \in L(X)$; in the real case, there exists a norm-one operator which numerical range reduces to zero. In our terminology, for a Hilbert space H with dimension greater than 1, $n(H) = 1/2$ if it is complex, and $n(H) = 0$ if it is real. Actually, real and complex general normed linear spaces behave in a very different way with regard to the numerical index, as summarized in the following equalities (see [5]):

(1) $$\{n(X) \ : \ X \text{ complex normed linear space }\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real normed linear space }\} = [0, 1].$$

The fact that $n(X) \geqslant \mathrm{e}^{-1}$ for every complex normed linear space was observed by B. Glickfeld [7] (by making use of a classical Theorem of H. Bohnenblust and S. Karlin [2]), who also gave an example where this inequality becomes an equality. In the already cited paper [5], J. Duncan, C. McGregor, J. Pryce, and A. White showed that $C(K)$, the normed linear space of all continuous scalar-valued function on a compact Hausdorff space K, has numerical index 1, and the same is true for $L_1(\mu)$, the space of all integrable functions with respect to a positive measure μ. For very recent results about numerical index in the finite-dimensional setting, we refer the reader to [16] and [21, §4]. For recent results in the infinite-dimensional context, we refer to [6, 11, 13, 15, 17, 19] and references therein.

Let us comment that, roughly speaking, when one finds an explicit computation of the numerical index of a normed linear space in the literature, only few values appear: 0 (real Hilbert spaces), e^{-1} (Glickfeld's example), 1/2 (complex Hilbert spaces) and 1 ($C(K)$, $L_1(\mu)$, and many more). Actually, when the authors of [5] prove Eq. (1), they only have examples of normed linear spaces whose numerical indices are the extremes of the intervals, and then a connectedness argument is applied.

The present paper tries to cover this gap. Namely, we give explicit formulae for the numerical index of some polyhedral spaces of dimension two which are none of the above. Here, by a *polyhedral space* we mean a finite-dimensional (real) normed linear space whose unit ball is a polyhedron (i.e., it is the convex hull of a finite set of points). We also use the name *polyhedral norm* to denote the norm of a polyhedral space. Concretely, we will calculate the numerical index of a family of hexagonal norms (Theorem 1), two families of octagonal norms (Theorem 2 and Corollary 3) and, finally, the family of those two-dimensional normed linear spaces whose unit balls are regular polygons with an even number of vertices (Theorem 5).

To finish the introduction, we recall some useful facts about numerical radius and numerical index that we will use in the paper without explicit reference.

Let X and Y be normed linear spaces, and suppose that there exists a surjective isometry $S : X \longrightarrow Y$. Then, for every operator $T \in L(Y)$, it is immediate to check that

$$v(T) = v(S^{-1}TS) \qquad \text{and} \qquad \|T\| = \|S^{-1}TS\|$$

which, in particular, implies that isometrically isomorphic Banach spaces have the same numerical index.

Given a normed linear space X, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [3, § 9]) and it clearly follows that $n(X^*) \leqslant n(X)$. So, in finite dimension, one has $n(X^*) = n(X)$.

For a convex set A, $\operatorname{ext}(A)$ will stand for the set of its extreme points, i.e., those points which are not the mid point of any segment contained in A. A nonempty compact convex subset of $\mathbb{R}^n$ is equal to the convex hull of its extreme points (Minkowski's Theorem, see [27, Corollary 1.13] for instance). This result gives the following equality for every operator T on a Banach space X:

$$\|T\| = \sup\{\|Tx\| \ : \ x \in \operatorname{ext}(B_X)\}. \tag{2}$$

It can also be deduced from Minkowski's theorem that

$$v(T) = \sup\{x^*(Tx) \ : \ x \in \operatorname{ext}(B_X),\ x^* \in \operatorname{ext}(B_{X^*}),\ x^*(x) = 1\} \tag{3}$$

(for another approach, see [18, Lemma 2.5]).

2. Hexagonal norms

For each $\gamma \in [0,1]$ let us write $X_\gamma = (\mathbb{R}^2, \|\cdot\|_\gamma)$, where the norm is given by

$$\|(x,y)\|_\gamma = \max\{|y|, |x| + (1-\gamma)|y|\} \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

First of all, we note that X_0 is an $L_1(\mu)$ space and X_1 is a $C(K)$ space, which give $n(X_0) = n(X_1) = 1$, and the unit balls of both spaces are squares. So we can restrict our study to the case $0 < \gamma < 1$, which gives hexagonal norms, i.e., the associated unit ball is an hexagon (see Figure 1). By just using the definition of $\|\cdot\|_\gamma$, it is easy to check that

$$\operatorname{ext}(B_{X_\gamma}) = \{\pm(\gamma, 1),\ \pm(1,0),\ \pm(\gamma,-1)\}.$$

Therefore, we get

$$\|(x,y)\|'_\gamma = \max\{|x|, |y| + \gamma\,|x|\} \qquad (x,y) \in X^*_\gamma,$$

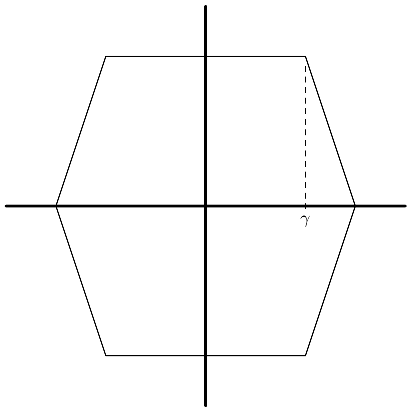

FIGURE 1. Hexagonal norms

and

$$\mathrm{ext}(B_{X_\gamma^*}) = \{\pm(1, 1-\gamma),\ \pm(0,1), \pm(-1, 1-\gamma)\}.$$

We deduce that, for each $0 < \gamma < 1$, X_γ^* is isometrically isomorphic to $X_{1-\gamma}$.

Theorem 1. *For every $\gamma \in [0,1]$, let X_γ be defined as above. Then,*

$$n(X_\gamma) = \begin{cases} \dfrac{1}{1+2\gamma} & \text{if } \ 0 \leqslant \gamma \leqslant \frac{1}{2}, \\ \\ \dfrac{1}{3-2\gamma} & \text{if } \ \frac{1}{2} \leqslant \gamma \leqslant 1 \end{cases}$$

Proof. On one hand, since X_γ^* is isometrically isomorphic to $X_{1-\gamma}$, we have

$$n(X_{1-\gamma}) = n(X_\gamma^*) = n(X_\gamma).$$

On the other hand, $n(X_0) = n(X_1) = 1$. So, it suffices to prove that $n(X_\gamma) = \dfrac{1}{3-2\gamma}$ for $\frac{1}{2} \leqslant \gamma < 1$ to finish the proof. Let us fix $\frac{1}{2} \leqslant \gamma < 1$. We start by showing that $n(X_\gamma) \geqslant \dfrac{1}{3-2\gamma}$. To this end, we take $T \in L(X_\gamma)$ and claim that $\|T\|_\gamma \leqslant (3-2\gamma) v_\gamma(T)$. Indeed, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix that represents T. By using Eq. (2) we obtain

$$\begin{aligned} \|T\|_\gamma &= \max\{\|(a\gamma+b, c\gamma+d)\|_\gamma,\ \|(a\gamma-b, c\gamma-d)\|_\gamma,\ \|(a,c)\|_\gamma\} \\ &= \max\{|c\gamma+d|,\ |a\gamma+b| + (1-\gamma)|c\gamma+d|,\ |c\gamma-d|, \\ &\qquad |a\gamma-b| + (1-\gamma)|c\gamma-d|,\ |c|,\ |a| + (1-\gamma)|c|\}. \end{aligned} \tag{4}$$

Now we compute the numerical radius of T. By using Eq. (3) we obtain

$$\begin{aligned} v_\gamma(T) = \max\{&|c\gamma+d|,\ |a\gamma+b+(1-\gamma)(c\gamma+d)|,\ |a+(1-\gamma)c|, \\ &|d-c\gamma|,\ |a\gamma-b+(1-\gamma)(d-c\gamma)|,\ |a-(1-\gamma)c|\}. \end{aligned} \tag{5}$$

From this, we get

$$\begin{aligned} |a\gamma+b| &\leqslant |a\gamma+b+(1-\gamma)(c\gamma+d)| + (1-\gamma)|c\gamma+d| \\ &\leqslant v_\gamma(T) + (1-\gamma)v_\gamma(T) \end{aligned}$$

hence,

$$|a\gamma+b| + (1-\gamma)|c\gamma+d| \leqslant (3-2\gamma)v_\gamma(T). \tag{6}$$

Analogously,

$$|a\gamma - b| + (1-\gamma)|c\gamma - d| \leqslant (3-2\gamma)v_\gamma(T). \tag{7}$$

In addition, we have

$$\gamma|c| \leqslant \gamma|c| + |d| \leqslant v_\gamma(T)$$

which gives

$$|c| \leqslant \frac{1}{\gamma}v_\gamma(T) \leqslant (3-2\gamma)v_\gamma(T). \tag{8}$$

In view of Eq. (4), (5), (6), (7), and (8), it is clear that

$$\|T\|_\gamma \leqslant (3-2\gamma)v_\gamma(T)$$

as we claimed.

For the reverse inequality, $n(X_\gamma) \leqslant \frac{1}{3-2\gamma}$, we consider the operator $T : X_\gamma \longrightarrow X_\gamma$ represented by the matrix

$$\begin{pmatrix} 0 & \frac{2-\gamma}{3-2\gamma} \\ \frac{-1}{\gamma(3-2\gamma)} & 0 \end{pmatrix}.$$

Using Eq. (4) and (5) it is easy to check that

$$\|T\|_\gamma = 1 \quad \text{and} \quad v_\gamma(T) = \frac{1}{3-2\gamma}$$

so,

$$n(X_\gamma) \leqslant \frac{1}{3-2\gamma}$$

which finishes the proof. □

3. Octagonal norms

For $\xi \in [0,1]$, we define $X_\xi = (\mathbb{R}^2, \|\cdot\|_\xi)$ where the norm $\|\cdot\|_\xi$ is given by

$$\|(x,y)\|_\xi = \max\left\{|x|,\ |y|,\ \frac{|x|+|y|}{1+\xi}\right\} \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

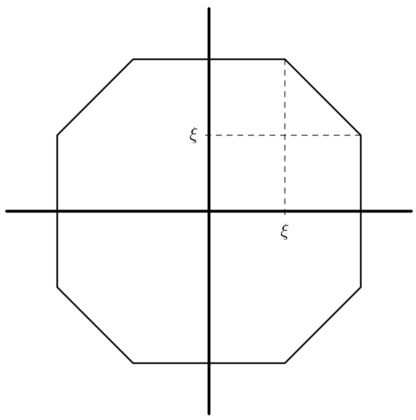

FIGURE 2. Octagonal norms

Observe that X_0 is an $L_1(\mu)$ space and X_1 is a $C(K)$ space, so $n(X_0) = 1 = n(X_1)$. Let $0 < \xi < 1$ be fixed. The unit ball of X_ξ is an octagon (see Figure 2), with (straightforward computation)

$$\mathrm{ext}(B_{X_\xi}) = \{\pm(1,\xi),\ \pm(1,-\xi),\ \pm(\xi,1),\ \pm(\xi,-1)\}.$$

Therefore, we get

$$\|(x,y)\|'_\xi = \max\{|x| + \xi|y|,\ |y| + \xi|x|\} \qquad (x,y) \in X_\xi^*,$$

and

$$\mathrm{ext}(B_{X_\xi^*}) = \left\{\pm(1,0),\ \pm(0,1),\ \pm\left(\frac{1}{1+\xi}, \frac{1}{1+\xi}\right),\ \pm\left(\frac{1}{1+\xi}, \frac{-1}{1+\xi}\right)\right\}.$$

Theorem 2. *For every $\xi \in [0,1]$, let X_ξ be defined as above. Then*

$$n(X_\xi) = \max\left\{\xi,\ \frac{1-\xi}{1+\xi}\right\}.$$

Proof. Let $T \in L(X_\xi)$ be represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By using Eq. (2) we obtain

$$\text{(9)} \qquad \|T\|_\xi = \max\left\{|a| + |b|\xi,\ |c| + |d|\xi,\ \frac{|a+b\xi| + |c+d\xi|}{1+\xi},\ \frac{|a-b\xi| + |c-d\xi|}{1+\xi},\right.$$
$$\left.|a|\xi + |b|,\ |c|\xi + |d|,\ \frac{|a\xi+b| + |c\xi+d|}{1+\xi},\ \frac{|a\xi-b| + |c\xi-d|}{1+\xi}\right\}.$$

Now we compute the numerical radius of T. By using Eq. (3) we get

$$\text{(10)} \qquad v_\xi(T) = \max\left\{|a| + |b|\xi,\ |d| + |c|\xi,\ \frac{|a\xi+d| + |b+c\xi|}{1+\xi},\ \frac{|a+d\xi| + |b\xi+c|}{1+\xi}\right\}.$$

In order to prove $n(X_\xi) \leqslant \max\left\{\xi,\ \frac{1-\xi}{1+\xi}\right\}$, we consider the operator $U \in L(X_\xi)$ represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using Eq. (9) and (10) it is clear that

$$\|U\|_\xi = 1 \quad \text{and} \quad v_\xi(U) = \max\left\{\xi,\ \frac{1-\xi}{1+\xi}\right\}.$$

Conversely, let us prove that $n(X_\xi) \geqslant \max\left\{\xi,\ \frac{1-\xi}{1+\xi}\right\}$. On one hand, in view of Eq. (10), it is immediate that

$$\xi(|a|\xi + |b|) \leqslant |a| + |b|\xi \leqslant v_\xi(T)$$
$$\xi(|c| + |d|\xi) \leqslant |c|\xi + |d| \leqslant v_\xi(T)$$

and then

$$\xi\left(\frac{|a|\xi + |b| + |c|\xi + |d|}{1+\xi}\right) \leqslant \frac{v_\xi(T) + \xi v_\xi(T)}{1+\xi} = v_\xi(T)$$
$$\xi\left(\frac{|a| + |b|\xi + |c| + |d|\xi}{1+\xi}\right) \leqslant \frac{\xi v_\xi(T) + v_\xi(T)}{1+\xi} = v_\xi(T).$$

All these inequalities yield

$$\xi\|T\|_\xi \leqslant v_\xi(T) \qquad \forall\, T \in L(X_\xi)$$

hence,

$$n(X_\xi) \geqslant \xi.$$

On the other hand, let us prove that $n(X_\xi) \geqslant \dfrac{1-\xi}{1+\xi}$. First, we observe that the operator S represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an isometry satisfying $S^2 = I$. So, we have

$$v_\xi(T) = v_\xi(STS) \quad \text{and} \quad \|T\|_\xi = \|STS\|_\xi,$$

and therefore, there is no loss of generality in assuming that

$$|a| + |c| \geqslant |b| + |d|.$$

With this in mind, it is easy to check that

$$\begin{aligned} &\left(\frac{1-\xi}{1+\xi}\right)\left(\frac{|b|+|d|+|a|\xi+|c|\xi}{1+\xi}\right) \leqslant \frac{|a|+|c|-|b|\xi-|d|\xi}{1+\xi} \leqslant v_\xi(T), \\ &\left(\frac{1-\xi}{1+\xi}\right)\left(\frac{|a|+|c|+|b|\xi+|d|\xi}{1+\xi}\right) \leqslant \frac{|a|+|c|-|b|\xi-|d|\xi}{1+\xi} \leqslant v_\xi(T). \end{aligned} \tag{11}$$

We claim that the following inequalities also hold

$$\begin{aligned} &\left(\frac{1-\xi}{1+\xi}\right)(|a|\xi+|b|) \leqslant v_\xi(T), \\ &\left(\frac{1-\xi}{1+\xi}\right)(|c|+|d|\xi) \leqslant v_\xi(T). \end{aligned} \tag{12}$$

We only prove the first inequality, since the proof of the second one is analogous. Indeed, if $|a|\xi + |b| \leqslant \dfrac{|b|+|d|+|a|\xi+|c|\xi}{1+\xi}$, using Eq. (11), we are done. Otherwise,

$$\frac{|b|+|d|+|a|\xi+|c|\xi}{1+\xi} \leqslant |a|\xi + |b|,$$

which implies

$$|c|\xi + |d| \leqslant \xi(|a|\xi + |b|).$$

Using this inequality and Eq. (10) it is clear that

$$\left(\frac{1-\xi}{1+\xi}\right)(|a|\xi+|b|) \leqslant \frac{|a|\xi+|b|-(|c|\xi+|d|)}{1+\xi} \leqslant v_\xi(T),$$

as we claimed. In view of Eq. (9), (10), (11), and (12), we get

$$\left(\frac{1-\xi}{1+\xi}\right)\|T\|_\xi \leqslant v_\xi(T),$$

and therefore

$$n(X_\xi) \geqslant \left(\frac{1-\xi}{1+\xi}\right),$$

which finishes the proof. □

Since the numerical index of a finite-dimensional normed linear space and the one of its dual coincide, the above theorem allows us to calculate the numerical indices of the elements of another family of octagonal norms: the one consisting of the duals of the above family. Concretely, for $\varrho \in [1/2, 1]$, we consider the space $Y_\varrho = (\mathbb{R}^2, \|\cdot\|_\varrho)$ where the norm is given by

$$\|(x,y)\|_\varrho = \max\left\{|x| + \frac{1-\varrho}{\varrho}|y|,\ |y| + \frac{1-\varrho}{\varrho}|x|\right\} \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

For $\varrho \in]1/2, 1[$, the unit ball of Y_ϱ is an octagon (Figure 3) with

$$\operatorname{ext}(B_{Y_\varrho}) = \{\pm(1,0),\ \pm(0,1),\ \pm(\varrho,\varrho),\ \pm(\varrho,-\varrho)\}.$$

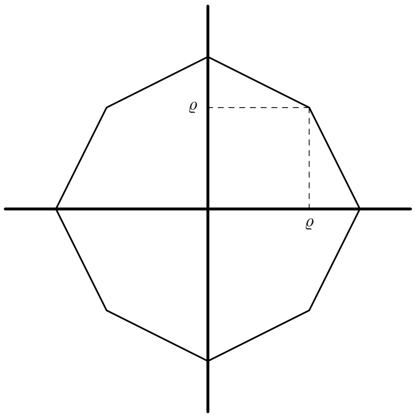

Figure 3.
More octagonal norms

By the comments before Theorem 2, it is clear that $X_\xi^* = Y_\varrho$ with $\varrho = \dfrac{1}{1+\xi}$. Equivalently, $Y_\varrho^* = X_\xi$ with $\xi = \dfrac{1-\varrho}{\varrho}$. Therefore, the following corollary is an immediate consequence of Theorem 2.

Corollary 3. *For every $\varrho \in [1/2, 1]$, let Y_ϱ be defined as above. Then*
$$n(Y_\varrho) = \max\left\{\frac{1-\varrho}{\varrho},\ 2\varrho - 1\right\}.$$

4. Regular polygons

Our next goal is to compute the numerical index of those two-dimensional real Banach spaces whose balls are regular polygons. Since the number of extreme points of the unit ball of a normed linear space has to be even, we restrict ourselves to regular polygons with an even number of vertices.

Let n be a positive integer greater or equal than 2. For $k = 1, 2, \ldots, 2n$, we write
$$x_k = \left(\cos\left(\tfrac{k\pi}{n}\right),\ \sin\left(\tfrac{k\pi}{n}\right)\right),$$
$$x_k^* = \frac{1}{\cos\left(\frac{\pi}{2n}\right)}\left(\cos\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right),\ \sin\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\right),$$
and we define X_n to be the two dimensional real Banach space such that
$$\mathrm{ext}(B_{X_n}) = \{x_k \ :\ k = 1, 2, \ldots, 2n\}.$$
In order to obtain the extreme points of $B_{X_n^*}$ we need the following easy fact.

Fact 4. *Let n be a positive integer greater or equal than 2 and let $m \in \mathbb{Z}$. Then,*
$$\left|\cos\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right).$$
If, in addition, n is even, then
$$\left|\sin\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right).$$

Proof. We observe that the inequality
$$\left|\cos\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right)$$

is equivalent to the fact that

$$\frac{m\pi}{n} + \frac{\pi}{2n} \notin \left]-\frac{\pi}{2n}, \frac{\pi}{2n}\right[\;+\; \pi\mathbb{Z},$$

which is equivalent

$$2m + 1 \notin\,]-1, 1[\, + 2n\mathbb{Z},$$

and this last statement is obviously true. Analogously, we note that the inequality

$$\left|\sin\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right)$$

is equivalent to

$$\frac{m\pi}{n} + \frac{\pi}{2n} \notin \left]\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2} + \frac{\pi}{2n}\right[\;+\; \pi\mathbb{Z},$$

which is equivalent to

$$2m + 1 \notin\,]n - 1, n + 1[\, + 2n\mathbb{Z},$$

a statement which is true when n is even. □

For $j, k \in \{1, 2, \ldots, 2n\}$ we have

$$\begin{aligned} |x_k^*(x_j)| &= \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \left|\cos\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\cos\left(\tfrac{j\pi}{n}\right) + \sin\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\sin\left(\tfrac{j\pi}{n}\right)\right| \\ &= \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \left|\cos\left(\tfrac{(k-j)\pi}{n} + \tfrac{\pi}{2n}\right)\right| \end{aligned}$$

so, the preceding fact tells us that $|x_k^*(x_j)| \leqslant 1$. Moreover, for $k \in \{1, 2, \ldots, 2n\}$ we have $x_k^*(x_k) = 1$ and $x_k^*(x_{k+1}) = 1$ (with the identification $x_{2n+1} = x_1$). Therefore,

$$\mathrm{ext}(B_{X_n^*}) = \{x_k^* \;:\; k = 1, 2, \ldots, 2n\}.$$

Now we can state and prove the promised result.

Theorem 5. *Let $n \geqslant 2$ be a positive integer, and let X_n be defined as above. Then,*

$$n(X_n) = \begin{cases} \tan\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\[2ex] \sin\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First of all, we note that X_2 is an $L_1(\mu)$ space and X_3 is isometrically isomorphic to the hexagonal space X_γ with $\gamma = \frac{1}{2}$, so $n(X_2) = 1$ and $n(X_3) = \frac{1}{2}$. Thus, we can restrict our study to the case $n \geqslant 4$.

Let $T \in L(X_n)$ be represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using Eq. (2) and (3) we get

$$\begin{aligned} \|T\|_n &= \max\{|x_k^*(Tx_j)| \;:\; j, k = 1, 2, \ldots, 2n\}, \\ v_n(T) &= \max\{|x_k^*(Tx_j)| \;:\; k = 1, 2, \ldots, 2n;\; j = k, k+1\}. \end{aligned}$$

In order to compute the norm and the numerical radius of T we need the explicit formula of $x_k^*(Tx_j)$. So, let $j, k \in \{1, 2, \ldots, 2n\}$, then we have

$$\begin{aligned} x_k^*(Tx_j) = \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \Bigg(& a\cos\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\cos\left(\tfrac{j\pi}{n}\right) + b\cos\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\sin\left(\tfrac{j\pi}{n}\right) + \\ & + c\sin\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\cos\left(\tfrac{j\pi}{n}\right) + d\sin\left(\tfrac{k\pi}{n} + \tfrac{\pi}{2n}\right)\sin\left(\tfrac{j\pi}{n}\right) \Bigg) \end{aligned}$$

so we can deduce that

$$
\begin{aligned}
x_k^*(Tx_j) = \frac{1}{\cos\left(\frac{\pi}{2n}\right)}\Bigg(& \frac{a+d}{2}\cos\left(\tfrac{(k-j)\pi}{n}+\tfrac{\pi}{2n}\right)+\frac{a-d}{2}\cos\left(\tfrac{(k+j)\pi}{n}+\tfrac{\pi}{2n}\right)+ \\
& +\frac{b+c}{2}\sin\left(\tfrac{(k+j)\pi}{n}+\tfrac{\pi}{2n}\right)+\frac{c-b}{2}\sin\left(\tfrac{(k-j)\pi}{n}+\tfrac{\pi}{2n}\right)\Bigg).
\end{aligned} \tag{13}
$$

We are ready to show that $n(X_n) \leqslant \tan\left(\frac{\pi}{2n}\right)$ if n is even, and that $n(X_n) \leqslant \sin\left(\frac{\pi}{2n}\right)$ if n is odd. To this end, we consider the operator $U \in L(X_n)$ represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For every $k \in \{1,\dots,2n\}$ one has

$$
|x_k^*(Ux_k)| = |x_k^*(Ux_{k+1})| = \tan\left(\frac{\pi}{2n}\right), \quad \text{so} \quad v_n(U) = \tan\left(\frac{\pi}{2n}\right).
$$

Now we claim that

$$
\|U\|_n = \begin{cases} \dfrac{1}{\cos\left(\frac{\pi}{2n}\right)} & \text{if } n \text{ is odd} \\[2ex] 1 & \text{if } n \text{ is even.} \end{cases}
$$

Indeed, suppose first that $n = 2m+1$ for some positive integer m. Using Eq. (13), we have

$$
|x_k^*(Ux_j)| = \frac{1}{\cos\left(\frac{\pi}{2n}\right)}\left|\sin\left(\tfrac{(k-j)\pi}{n}+\tfrac{\pi}{2n}\right)\right| \leqslant \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \qquad j,k \in \{1,\dots,2n\},
$$

and the equality holds for $k = m$ and $j = 2n$. Suppose otherwise that n is even. Using Eq. (13) and Fact 4, we have

$$
|x_k^*(Ux_j)| = \frac{1}{\cos\left(\frac{\pi}{2n}\right)}\left|\sin\left(\tfrac{(k-j)\pi}{n}+\tfrac{\pi}{2n}\right)\right| \leqslant 1 \qquad j,k \in \{1,\dots,2n\},
$$

and the equality holds for $k = n/2$ and $j = 2n$.

Now, in view of $v_n(U)$ and $\|U\|_n$ we can deduce that $n(X_n) \leqslant \tan\left(\frac{\pi}{2n}\right)$ if n is even, and $n(X_n) \leqslant \sin\left(\frac{\pi}{2n}\right)$ if n is odd.

Let us prove the reverse inequality. We first note that using Eq. (13) and Fact 4, it is clear that the following holds for every operator $T \in L(X_n)$

$$
\begin{aligned}
\|T\|_n &\leqslant \frac{1}{\cos\left(\frac{\pi}{2n}\right)}\left(\frac{|a+d|}{2}+\frac{|a-d|}{2}+\frac{|b+c|}{2}+\frac{|b-c|}{2}\right) && \text{if } n \text{ is odd} \\
\|T\|_n &\leqslant \left(\frac{|a+d|}{2}+\frac{|a-d|}{2}+\frac{|b+c|}{2}+\frac{|b-c|}{2}\right) && \text{if } n \text{ is even.}
\end{aligned}
$$

Therefore, the proof of the theorem will be finished if we are able to show that

$$
v_n(T) \geqslant \tan\left(\frac{\pi}{2n}\right)\left(\frac{|a+d|}{2}+\frac{|a-d|}{2}+\frac{|b+c|}{2}+\frac{|b-c|}{2}\right). \tag{14}
$$

To do this, let us define

$$
\begin{aligned}
A_n &= \left\{\left(\cos\left(\tfrac{2k\pi}{n}+\tfrac{\pi}{2n}\right),\ \sin\left(\tfrac{2k\pi}{n}+\tfrac{\pi}{2n}\right)\right)\ :\ k = 1,2,\dots,n\right\} \\
B_n &= \left\{\left(\cos\left(\tfrac{(2k+1)\pi}{n}+\tfrac{\pi}{2n}\right),\ \sin\left(\tfrac{(2k+1)\pi}{n}+\tfrac{\pi}{2n}\right)\right)\ :\ k = 1,2,\dots,n\right\} \\
C_n^1 &= \left\{(x,y) \in \mathbb{R}^2\ :\ x \geqslant \sin\left(\tfrac{\pi}{2n}\right),\ y \geqslant \sin\left(\tfrac{\pi}{2n}\right),\ x^2+y^2 = 1\right\} \\
C_n^2 &= \left\{(x,y) \in \mathbb{R}^2\ :\ x \leqslant -\sin\left(\tfrac{\pi}{2n}\right),\ y \geqslant \sin\left(\tfrac{\pi}{2n}\right),\ x^2+y^2 = 1\right\} \\
C_n^3 &= \left\{(x,y) \in \mathbb{R}^2\ :\ x \leqslant -\sin\left(\tfrac{\pi}{2n}\right),\ y \leqslant -\sin\left(\tfrac{\pi}{2n}\right),\ x^2+y^2 = 1\right\} \\
C_n^4 &= \left\{(x,y) \in \mathbb{R}^2\ :\ x \geqslant \sin\left(\tfrac{\pi}{2n}\right),\ y \leqslant -\sin\left(\tfrac{\pi}{2n}\right),\ x^2+y^2 = 1\right\}.
\end{aligned}
$$

For $n \geqslant 4$, it is not hard to check that

$$A_n \cap C_n^i \neq \emptyset \qquad i = 1,2,3,4 \tag{15}$$

$$B_n \cap C_n^i \neq \emptyset \qquad i = 1,2,3,4. \tag{16}$$

Indeed, observe that each C_n^i is an arc of the unit circle which embraces an angle of size $\frac{\pi}{2} - \frac{\pi}{n}$. Also observe that A_n and B_n are the sets of vertices of regular n-sided polygons in the unit circle. In the case $n \geqslant 6$, we have that $\frac{\pi}{2} - \frac{\pi}{n} \geqslant \frac{2\pi}{n}$, and it follows clearly that both A_n and B_n intersect each C_n^i. Straightforward verification finishes the argument in the cases $n = 4$ and $n = 5$.

We are ready to prove Eq. (14). Suppose first that $(a+d)(c-b) > 0$. By using Eq. (13) with $j = k$, we obtain the following for $k = 1, 2, \ldots, 2n$

$$\begin{aligned} v_n(T) \geqslant |x_k^*(Tx_k)| = \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \Bigg| \frac{a+d}{2} \cos\left(\tfrac{\pi}{2n}\right) + \frac{a-d}{2} \cos\left(\tfrac{2k\pi}{n} + \tfrac{\pi}{2n}\right) + \\ + \frac{b+c}{2} \sin\left(\tfrac{2k\pi}{n} + \tfrac{\pi}{2n}\right) + \frac{c-b}{2} \sin\left(\tfrac{\pi}{2n}\right) \Bigg|. \end{aligned}$$

Calling $\varepsilon = \operatorname{sign}(a+d) = \operatorname{sign}(c-b)$ and using Eq. (15), we may obtain $k_0 \in \{1, 2, \ldots, n\}$ such that

$$\begin{aligned} &\frac{a-d}{2} \cos\left(\tfrac{2k_0\pi}{n} + \tfrac{\pi}{2n}\right) + \frac{b+c}{2} \sin\left(\tfrac{2k_0\pi}{n} + \tfrac{\pi}{2n}\right) \\ &\qquad = \varepsilon \left| \frac{a-d}{2} \right| \left|\cos\left(\tfrac{2k_0\pi}{n} + \tfrac{\pi}{2n}\right)\right| + \varepsilon \left| \frac{b+c}{2} \right| \left|\sin\left(\tfrac{2k_0\pi}{n} + \tfrac{\pi}{2n}\right)\right|, \end{aligned}$$

and such that

$$\left|\cos\left(\tfrac{2k_0\pi}{n} + \tfrac{\pi}{2n}\right)\right| \geqslant \sin\left(\tfrac{\pi}{2n}\right) \qquad \text{and} \qquad \left|\sin\left(\tfrac{2k_0\pi}{n} + \tfrac{\pi}{2n}\right)\right| \geqslant \sin\left(\tfrac{\pi}{2n}\right).$$

Therefore, we get

$$v_n(T) \geqslant |x_{k_0}^*(Tx_{k_0})| \geqslant \tan\left(\frac{\pi}{2n}\right) \left(\frac{|a+d|}{2} + \frac{|a-d|}{2} + \frac{|b+c|}{2} + \frac{|b-c|}{2} \right).$$

Suppose now that $(a+d)(c-b) < 0$. By using Eq. (13) with $j = k+1$, we obtain the following for $k = 1, 2, \ldots, 2n$

$$\begin{aligned} v_n(T) \geqslant |x_k^*(Tx_{k+1})| = \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \Bigg| \frac{a+d}{2} \cos\left(\tfrac{\pi}{2n}\right) + \frac{a-d}{2} \cos\left(\tfrac{(2k+1)\pi}{n} + \tfrac{\pi}{2n}\right) + \\ + \frac{b+c}{2} \sin\left(\tfrac{(2k+1)\pi}{n} + \tfrac{\pi}{2n}\right) + \frac{b-c}{2} \sin\left(\tfrac{\pi}{2n}\right) \Bigg|. \end{aligned}$$

Calling this time $\varepsilon = \operatorname{sign}(a+d) = \operatorname{sign}(b-c)$ and using Eq. (16), we may obtain $k_0 \in \{1, 2, \ldots, n\}$ such that

$$\begin{aligned} &\frac{a-d}{2} \cos\left(\tfrac{(2k_0+1)\pi}{n} + \tfrac{\pi}{2n}\right) + \frac{b+c}{2} \sin\left(\tfrac{(2k_0+1)\pi}{n} + \tfrac{\pi}{2n}\right) \\ &\qquad = \varepsilon \left| \frac{a-d}{2} \right| \left|\cos\left(\tfrac{(2k_0+1)\pi}{n} + \tfrac{\pi}{2n}\right)\right| + \varepsilon \left| \frac{b+c}{2} \right| \left|\sin\left(\tfrac{(2k_0+1)\pi}{n} + \tfrac{\pi}{2n}\right)\right|, \end{aligned}$$

and such that

$$\left|\cos\left(\tfrac{(2k_0+1)\pi}{n} + \tfrac{\pi}{2n}\right)\right| \geqslant \sin\left(\tfrac{\pi}{2n}\right) \qquad \text{and} \qquad \left|\sin\left(\tfrac{(2k_0+1)\pi}{n} + \tfrac{\pi}{2n}\right)\right| \geqslant \sin\left(\tfrac{\pi}{2n}\right).$$

Therefore, we get

$$v_n(T) \geqslant |x_{k_0}^*(Tx_{k_0+1})| \geqslant \tan\left(\frac{\pi}{2n}\right) \left(\frac{|a+d|}{2} + \frac{|a-d|}{2} + \frac{|b+c|}{2} + \frac{|b-c|}{2} \right).$$

Finally, when $(a+d)(c-b) = 0$, either of the two arguments above work with natural simplifications. In any case, we have established the validity of Eq. (14), as desired. □

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